Journal of Mathematical Inequalities
Volume 15, Number 4 (2021), 1425–1434 doi:10.7153/jmi-2021-15-97

ON A REVERSE OF THE TAN–XIE INEQUALITY FOR SECTOR MATRICES AND ITS APPLICATIONS

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(*Communicated by M. Niezgoda*)

Abstract. In this short paper, we establish a reverse of the derived inequalities for sector matrices by Tan and Xie, with Kantorovich constant. Then, as application of our main theorem, some inequalities for determinant and unitarily invariant norm are presented.

1. Introduction

Let $\mathbb{M}_n$ and $\mathbb{M}_n^+$ denote the set of all $n\times n$ matrices and the set of all $n\times n$ positive semidefinite matrices with entries in $\mathbb{C}$, respectivey. For $A\in\mathbb{M}_n$, the cartesian decomposition of A is presented as

$$A=\Re A+i\Im A,$$

where $\Re A=\frac{A+A^*}{2}$ and $\Im A=\frac{A-A^*}{2i}$ are the real and imaginary parts of A, respectively. The matrix $A\in\mathbb{M}_n$ is called accretive, if $\Re A$ is positive definite. Also, the matrix $A\in\mathbb{M}_n$ is called accretive-disipative, if both $\Re A$ and $\Im A$ are positive definite. For $\alpha\in\left[0,\frac{\pi}{2}\right)$, define a sector as follows:

$$S_\alpha=\{z\in\mathbb{C}:\Re z>0,|\Im z|\leqslant(\Re z)\tan\alpha\}.$$

Here, we recall that the numerical range of $A\in\mathbb{M}_n$ is defined by

$$W(A)=\{x^*Ax:x\in\mathbb{C}^n,x^*x=1\}.$$

The matrix $A\in\mathbb{M}_n$ is called sector, if whose numerical range is contained in sector S_α. In other words, $W(A)\subset S_\alpha$. Clearly, any sector matrix is accretive with extra information about the angle α. Since $W(A)\subset S_\alpha$ implies that $W(X^*AX)\subset S_\alpha$ for any nonsingular matrix $X\in\mathbb{M}_n$, also $W(A^{-1})\subset S_\alpha$, that is, inverse of every sector matrix is a sector matrix. Indeed, by defintion, $W(A)\subset S_\alpha$ is equivalent to $\pm\Im A\leqslant(\tan\alpha)\Re A$. This inequality means the Löewner partial order. Therefore, $\pm X\Im AX^*\leqslant$

Mathematics subject classification (2020): Primary 47A63; Secondary 46L05, 47A60, 26D15.
Keywords and phrases: Sector and accretive matrices, the Kantorovich constant, numerical range, determinant and norm inequality.
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$(\tan\alpha)X\Re AX^*$ which is equivalent to $W(X^*AX)\subset S_\alpha$. In addition, if we take $X=A^{-1}$, then we have

$$\pm A^{-1}\frac{A-A^*}{2i}\left(A^{-1}\right)^*\leqslant(\tan\alpha)A^{-1}\frac{A+A^*}{2}\left(A^{-1}\right)^*.$$

Thus we have

$$\mp\frac{A^{-1}-\left(A^{-1}\right)^*}{2i}\leqslant(\tan\alpha)\frac{\left(A^{-1}\right)^*+A^{-1}}{2}$$

which means $\pm\Im A^{-1}\leqslant(\tan\alpha)\Re A^{-1}$. This is equivalent to $W(A^{-1})\subset S_\alpha$.

For $A,B\in\mathbb{M}_n^+$, the weighted geometric mean, the weighted arithmetic mean and the weighted harmonic mean are defined, respectively, as follows:

$$A\sharp_v B=A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^v A^{\frac{1}{2}}, A\nabla_v B=(1-v)A+vB, A!_v B=\left((1-v)A^{-1}+vB^{-1}\right)^{-1}.$$

It is clear that the following inequality holds

$$A!_v B\leqslant A\sharp_v B\leqslant A\nabla_v B. \tag{1.1}$$

In [11, Theorem 2.1], the authors obtained a reverse of the second inequality in (1.1) using the Kantorovich constant for every positive unital linear map Φ as follows:

$$\Phi^2(A\nabla_v B)\leqslant K^2(h)\Phi^2(A\sharp_v B). \tag{1.2}$$

From the operator monotonicity of the function $f(t)=t^{1/2}$ on $[0,\infty)$, it implies that

$$\Phi(A\nabla_v B)\leqslant K(h)\Phi(A\sharp_v B).$$

For $\Phi=id$, it is obvious that

$$A\nabla_v B\leqslant K(h)(A\sharp_v B). \tag{1.3}$$

The authors [14] defined the weighted geometric mean for two accretive matrices $A,B\in\mathbb{M}_n$ and $v\in[0,1]$ as follows:

$$A\sharp_v B=\frac{\sin v\pi}{\pi}\int_0^\infty s^{v-1}(A^{-1}+sB^{-1})^{-1}ds.$$

Tan and Xie [15] studied the inequality (1.1) for sector matrices $A,B\in\mathbb{M}_n$, $v\in[0,1]$ and $\alpha\in\left[0,\frac{\pi}{2}\right)$ and obtained the following result:

$$\cos^2(\alpha)\Re(A!_v B)\leqslant\Re(A\sharp_v B)\leqslant\sec^2(\alpha)\Re(A\nabla_v B). \tag{1.4}$$

Inspired by the nice results (1.4), we are going to present a reverse of the double inequality (1.4) for two sector matrices $A,B\in\mathbb{M}_n$ and $v\in[0,1]$ in this short paper. Moreover, we establish some new determinant and norm inequalities using the deduced inequality.

2. A reverse of the double inequality (1.4)

Our aim of this section is to establish a reverse of the double inequality (1.4) which both generalizes and extends the obtained results in recent years. To do this work, we use the Kantorovich constant $K(h) := \frac{(h+1)^2}{4h} \geqslant 1$ for $h := \frac{M}{m} \geqslant 1$ with $0 < m \leqslant M$ throughout the paper and several lemmas which we list them as follows:

LEMMA 2.1. ([12]) *Let $A \in \mathbb{M}_n$ be accretive. Then*

$$\Re(A^{-1}) \leqslant \Re^{-1}(A). \tag{2.1}$$

The next lemma is a reverse of (2.1).

LEMMA 2.2. ([13]) *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then the following inequality holds:*

$$\Re^{-1}(A) \leqslant \sec^2(\alpha)\Re(A^{-1}). \tag{2.2}$$

LEMMA 2.3. ([4]) *Let $A, B \in \mathbb{M}_n$ be positive. Then*

$$\|AB\| \leqslant \frac{1}{4}\|A+B\|^2. \tag{2.3}$$

LEMMA 2.4. (Choi inequality [3, p. 41]) *Let $A \in \mathbb{M}_n$ be positive and let Φ be a positive unital linear map. Then we have*

$$\Phi^{-1}(A) \leqslant \Phi(A^{-1}). \tag{2.4}$$

LEMMA 2.5. ([5]) *Let $A, B \in \mathbb{M}_n$ be positive and let r be a positive number. Then $A \leqslant rB$ is equivalent to $\|A^{1/2}B^{-1/2}\| \leqslant r^{1/2}$.*

THEOREM 2.1. *Let $A, B \in \mathbb{M}_n$ be sector, that is, $W(A), W(B) \subset S_\alpha$ for some $\alpha \in \left[0, \frac{\pi}{2}\right)$ and $0 \leqslant v \leqslant 1$. Then for every positive unital linear map Φ, we have the following.*

(i) If $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$. Then,

$$\Phi^2\left(\Re(A\sharp_v B)\right) \leqslant \sec^8(\alpha)K^2(h)\Phi^2\left(\Re(A!_v B)\right). \tag{2.5}$$

(ii) If $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$. Then,

$$K^{-2}(h)\cos^8(\alpha)\Phi^2\left(\Re(A\nabla_v B)\right) \leqslant \Phi^2\left(\Re(A\sharp_v B)\right). \tag{2.6}$$

Proof.
(i) From $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$, we get

$$\Re(A^{-1}) + Mm\Re(A^{-1})^{-1} \leqslant M + m.$$

$$\Re(B^{-1}) + Mm\Re(B^{-1})^{-1} \leqslant M + m.$$

If we multiply both sides of the first inequality and the second inequality, respectively, by $1-v$ and v, we obtain

$$(1-v)\Re(A^{-1}) + (1-v)Mm\Re(A^{-1})^{-1} \leqslant (1-v)(M+m).$$

$$v\Re(B^{-1}) + vMm\Re(B^{-1})^{-1} \leqslant v(M+m).$$

As the inverse of every sector matrix is sector again and every sector matrix is accretive as explained in Introduction, it follows that

$$\begin{aligned} &Mm\Re((1-v)A+vB) + \Re((1-v)A^{-1}+vB^{-1}) \\ &\leqslant Mm((1-v)\Re^{-1}(A^{-1}) + v\Re^{-1}(B^{-1})) + \Re((1-v)A^{-1}+vB^{-1}) \quad (\text{by } 2.1) \\ &\leqslant M+m. \end{aligned} \tag{2.7}$$

Thus we have,

$$\begin{aligned} &\|\Phi\left(\Re(A\sharp_v B)\right) Mm\Phi^{-1}\left(\Re(A!_v B)\right)\| \\ &\leqslant \frac{1}{4}\|Mm\Phi\left(\Re(A\sharp_v B)\right) + \Phi^{-1}\left(\Re(A!_v B)\right)\|^2 \quad (\text{by } (2.3)) \\ &\leqslant \frac{1}{4}\|Mm\Phi\left(\Re(A\sharp_v B)\right) + \Phi\left(\Re^{-1}(A!_v B)\right)\|^2 (\text{by } (2.4)) \\ &\leqslant \frac{1}{4}\|Mm\Phi\left(\Re(A\sharp_v B)\right) + \sec^2(\alpha)\Phi\left(\Re((1-v)A^{-1}+vB^{-1})\right)\|^2 \quad (\text{by } (2.2)) \\ &\leqslant \frac{1}{4}\|\sec^2(\alpha)Mm\Phi\left(\Re((1-v)A+vB)\right) + \sec^2(\alpha)\Phi\left(\Re((1-v)A^{-1}+vB^{-1})\right)\|^2 \\ &\qquad\qquad (\text{by } (1.4)) \\ &= \frac{1}{4}\sec^4(\alpha)\|\Phi\left(Mm\Re((1-v)A+vB) + \Re((1-v)A^{-1}+vB^{-1})\right)\|^2 \\ &\leqslant \frac{\sec^4(\alpha)}{4}(M+m)^2 \quad (\text{by } (2.7)\ ). \end{aligned}$$

(ii) In a similar way, we have

$$Mm\left((1-v)\Re^{-1}(A) + v\Re^{-1}(B)\right) + (1-v)\Re(A) + v\Re(B) \leqslant M+m \tag{2.8}$$

from the conditions on $\Re(A)$ and $\Re(B)$ in (ii). Thus we have

$$\begin{aligned} &\|\sec^4(\alpha)\Phi^{-1}\left(\Re(A\sharp_v B)\right) Mm\Phi\left(\Re(A\nabla_v B)\right)\| \\ &\leqslant \frac{1}{4}\|Mm\Phi^{-1}\left(\Re(A\sharp_v B)\right) + \sec^4(\alpha)\Phi\left(\Re(A\nabla_v B)\right)\|^2 \quad (\text{by } (2.3)) \\ &\leqslant \frac{1}{4}\|Mm\Phi\left(\Re^{-1}(A\sharp_v B)\right) + \sec^4(\alpha)\Phi\left(\Re(A\nabla_v B)\right)\|^2 \quad (\text{by } (2.4)) \\ &\leqslant \frac{1}{4}\|\sec^2(\alpha)Mm\Phi\left(\Re\left((A\sharp_v B)^{-1}\right)\right) + \sec^4(\alpha)\Phi\left(\Re(A\nabla_v B)\right)\|^2 \quad (\text{by } (2.2)) \\ &= \frac{1}{4}\|\sec^2(\alpha)Mm\Phi\left(\Re(A^{-1}\sharp_v B^{-1})\right) + \sec^4(\alpha)\Phi\left(\Re(A\nabla_v B)\right)\|^2 \end{aligned}$$

$$\leqslant \frac{1}{4}\| \sec^4(\alpha)Mm\Phi\left(\Re((1-v)A^{-1}+vB^{-1})\right)+\sec^4(\alpha)\Phi\left(\Re(A\nabla_v B)\right)\|^2 \quad \text{(by (1.4))}$$

$$\leqslant \frac{1}{4}\| \sec^4(\alpha)Mm\Phi\left(((1-v)\Re^{-1}(A)+v\Re^{-1}(B))\right)+\sec^4(\alpha)\Phi\left(\Re((1-v)A+vB)\right)\|^2 \quad \text{(by (2.1))}$$

$$\leqslant \frac{\sec^8(\alpha)}{4}(M+m)^2. \quad \text{(by (2.8))}$$

Thus we have the desired results (i) and (ii) by Lemma 2.5. □

REMARK 2.1. The inequalities given in Theorem 2.1 give reverses for the inequalities (1.4) when Φ is an identity map. In addition, our inequality (2.6) recovers the inequality (1.3) for $\alpha = 0$ and Φ is an identity map.

REMARK 2.2. For $v = \frac{1}{2}$, the inequalities (2.5) and (2.6) recover [17, Theorem 2.18] and [17, Theorem 2.10] , respectively. This shows that our results contain the wide class of inequalities.

3. Applications

Making use of the inequalities (2.5) and (2.6), we prove some determinant inequalities. For proving the results of this section, we need to state the following useful lemmas which the first lemma is known as the Ostrowski-Taussky inequality and second lemma is a its reverse.

LEMMA 3.1. ([9]) *Let $A \in \mathbb{M}_n$ be accretive. Then*

$$\det(\Re A) \leqslant |\det A|. \tag{3.1}$$

LEMMA 3.2. ([12]) *Let $A \in \mathbb{M}_n$ such that $W(A) \subset S_\alpha$. Then*

$$|\det A| \leqslant \sec^n(\alpha)\det(\Re A). \tag{3.2}$$

COROLLARY 3.1. *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$ and $0 \leqslant v \leqslant 1$.*

(i) If $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$, then we have

$$|\det(A\sharp_v B)| \leqslant \sec^{5n}(\alpha)K^n(h)|\det(A!_v B)|. \tag{3.3}$$

(ii) If $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$, then we have,

$$|\det(A\sharp_v B)| \geqslant \cos^{5n}(\alpha)K^{-n}(h)|\det(A\nabla_v B)|. \tag{3.4}$$

Proof. First, we prove (3.3). Since $\det(cA) = c^n \det A$ for scalar $c > 0$ and $A \in \mathbb{M}_n$ in general, we have

$$\begin{aligned}|\det(A\sharp_v B)| &\leqslant \sec^n(\alpha)\det(\Re(A\sharp_v B)) \quad (\text{by } (3.2))\\ &\leqslant \sec^{5n}(\alpha)K^n(h)\det(\Re(A!_v B)) \quad (\text{by } (2.5))\\ &\leqslant \sec^{5n}(\alpha)K^n(h)|\det(A!_v B)| \quad (\text{by } (3.1)).\end{aligned}$$

The inequality (3.4) can be proven similarly

$$\begin{aligned}|\det(A\sharp_v B)| &\geqslant \det(\Re(A\sharp_v B)) \quad (\text{by } (3.1))\\ &\geqslant \cos^{4n}(\alpha)K^{-n}(h)\det(\Re(A\nabla_v B)) \quad (\text{by } (2.6))\\ &\geqslant \cos^{5n}(\alpha)K^{-n}(h)|\det(A\nabla_v B)| \quad (\text{by } (3.2)).\end{aligned}$$

This proves the results as desired. □

PROPOSITION 3.1. *Let $A, B \in \mathbb{M}_n$ with $W(A), W(B) \subset S_\alpha$. Then*

$$|\det(A\sharp B)| \leqslant \frac{\sec^{4n}(\alpha)}{2^n}|\det(I_n + A)| \cdot |\det(I_n + B)|.$$

Proof. To prove the assertion, compute

$$\begin{aligned}&|\det(A\sharp B)| \leqslant \sec^n(\alpha)\det(\Re(A\sharp B)) \quad (\text{by } (3.2))\\ &\leqslant \frac{\sec^{3n}(\alpha)}{2^n}\det(\Re(A+B)) \quad (\text{by [13, Eq. (10)]})\\ &\leqslant \frac{\sec^{3n}(\alpha)}{2^n}|\det(A+B)| \quad (\text{by } (3.1))\\ &\leqslant \frac{\sec^{4n}(\alpha)}{2^n}|\det(I_n + A)| \cdot |\det(I_n + B)| \quad (\text{by [16, Eq. (13)]}).\end{aligned}$$ □

Note that we have the following inequality for the weighted means

$$|\det(A\sharp_v B)| \leqslant \sec^{3n}(\alpha)|\det(A\nabla_v B)|$$

from (3.2), (1.4) and (3.1).

A norm $\|\cdot\|_u$ is called an unitarily invariant norm if $\|X\|_u = \|UXV\|_u$ for any unitary matrices U, V and any $X \in \mathbb{M}_n$. We use the symbols $v_j(X)$ and $s_j(X)$ as the j-th largest eigenvalue and singular value of X, respectively. The following lemmas are known.

LEMMA 3.3. (Fan-Hoffman [2, Proposition III.5.1]) *Let $A \in \mathbb{M}_n$. Then*

$$v_j(\Re A) \leqslant s_j(A), \quad (j = 1, \cdots, n). \tag{3.5}$$

LEMMA 3.4. ([6]) *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then*

$$s_j(A) \leqslant \sec^2(\alpha)v_j(\Re A), \quad (j = 1, \cdots, n). \tag{3.6}$$

LEMMA 3.5. ([19]) *Let $A \in \mathbb{M}_n$ with $W(A) \subset S_\alpha$. Then*

$$\|A\|_u \leqslant \sec(\alpha)\|\Re(A)\|_u. \tag{3.7}$$

COROLLARY 3.2. *Let $A, B \in \mathbb{M}_n$ be sector, that is, $W(A), W(B) \subset S_\alpha$ for some $\alpha \in \left[0, \frac{\pi}{2}\right)$ and $0 \leqslant v \leqslant 1$.*

(i) If $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$. Then,

$$s_j(A\sharp_v B) \leqslant \sec^6(\alpha)K(h)s_j(A!_v B),$$

(ii) If $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$. Then,

$$\cos^6(\alpha)K^{-1}(h)s_j(A\nabla_v B) \leqslant s_j(A\sharp_v B).$$

Proof. A simple computation shows that

$$\begin{aligned} s_j(A\sharp_v B) &\leqslant \sec^2(\alpha)s_j(\Re(A\sharp_v B)) \quad \text{(by (3.6))} \\ &\leqslant \sec^6(\alpha)K(h)s_j(\Re(A!_v B)) \quad \text{(by (2.5))} \\ &\leqslant \sec^6(\alpha)K(h)s_j(A!_v B) \quad \text{(by (3.5))}. \end{aligned}$$

It is easy to observe that

$$\begin{aligned} s_j(A\sharp_v B) &\geqslant s_j(\Re(A\sharp_v B)) \quad \text{(by (3.5))} \\ &\geqslant \cos^4(\alpha)K^{-1}(h)s_j(\Re(A\nabla_v B)) \quad \text{(by (2.6))} \\ &\geqslant \cos^6(\alpha)K^{-1}(h)s_j(A\nabla_v B) \quad \text{(by (3.6))}. \qquad \square \end{aligned}$$

REMARK 3.1. In the special case such that $\alpha = \frac{\pi}{4}$, we have the following inequalities for accretive–disipative matrices $A, B \in \mathbb{M}_n$ and $0 \leqslant v \leqslant 1$.

(i) If $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$. Then,

$$s_j(A\sharp_v B) \leqslant 8K(h)s_j(A!_v B).$$

(ii) If $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$. Then

$$\frac{1}{8}K^{-1}(h)s_j(A\nabla_v B) \leqslant s_j(A\sharp_v B).$$

We should emphasise that a matix A is called an accretive–disipative matrix when both $\Re A$ and $\Im A$ are positive definite. An accretive–disipative matrix never includes the information on angle α, whereas a sector matrix has an information on angle $\alpha \in \left[0, \frac{\pi}{2}\right)$ and an imaginary part $\Im A$ of a sector matrix is not necessary positive definite. Considering the complex plane, $\{z \in \mathbb{C} : \Re z > 0, \Im z > 0\} \subset \lim\limits_{\alpha \to \pi/2} S_\alpha$. Thus one may regard that an accretive–disipative matrix is a special case of a sector matrix.

COROLLARY 3.3. *Let $A,B \in \mathbb{M}_n$ with $W(A),W(B) \subset S_\alpha$. Then for any unitarily invariant norm $\|\cdot\|_u$ on $\mathbb{M}_n$, we have the following inequalities.*

(i) If $0 < mI_n \leqslant \Re(A^{-1}), \Re(B^{-1}) \leqslant MI_n$, then we have

$$\|A\sharp_v B\|_u \leqslant \sec^5(\alpha)K(h)\|A!_v B\|_u.$$

(ii) If $0 < mI_n \leqslant \Re(A), \Re(B) \leqslant MI_n$, then we have

$$\|A\sharp_v B\|_u \geqslant \cos^5(\alpha)K^{-1}(h)\|A\nabla_v B\|_u$$

Proof. We can show that the following chain of inequalities for a unitarily invariant norm:

$$\begin{aligned}\|A\sharp_v B\|_u &\leqslant \sec(\alpha)\|\Re(A\sharp_v B)\|_u \quad \text{(by (3.7))}\\ &\leqslant \sec^5(\alpha)K(h)\|\Re(A!_v B)\|_u \quad \text{(by (2.5))}\\ &\leqslant \sec^5(\alpha)K(h)\|A!_v B\|_u.\end{aligned}$$

This proves the first inequality. The second inequality can be proven similarly

$$\begin{aligned}\|A\sharp_v B\|_u \geqslant \|\Re(A\sharp_v B)\|_u &\geqslant \cos^4(\alpha)K^{-1}(h)\|\Re(A\nabla_v B)\|_u \quad \text{(by (2.6))}\\ &\geqslant \cos^5(\alpha)K^{-1}(h)\|A\nabla_v B\|_u. \quad \text{(by (3.7))} \quad \square\end{aligned}$$

REMARK 3.2. In the special case such that $\alpha = \frac{\pi}{4}$, we have the following inequalities for accretive-disipative matrices $A,B \in \mathbb{M}_n$ and any unitarily invariant norm $\|\cdot\|_u$ on $\mathbb{M}_n$,

$$4\sqrt{2}K^{-1}(h)\|A\nabla_v B\|_u \leqslant \|A\sharp_v B\|_u \leqslant \frac{1}{4\sqrt{2}}K(h)\|A!_v B\|_u.$$

PROPOSITION 3.2. *Let $A,B \in \mathbb{M}_n$ such that $W(A),W(B) \subset S_\alpha$. Then*

$$\|A\sharp B\|_u \leqslant \frac{\sec^5(\alpha)}{2}\|I_n + A\|_u \cdot \|I_n + B\|_u.$$

Proof.

$$\begin{aligned}\|A\sharp B\|_u &\leqslant \frac{\sec^3(\alpha)}{2}\|A+B\|_u \quad \text{(by [13, Eq. (14)])}\\ &\leqslant \frac{\sec^5(\alpha)}{2}\|I_n + A\|_u \cdot \|I_n + B\|_u \quad \text{(by [16, Corollary 2.8]).} \quad \square\end{aligned}$$

4. Conclusion

As we have seen, we obtained some mean inequalities for sector matrices. As for recent advanced studies on this subject, new inequalities of the Heinz mean (which interpolates an arithmetic mean and a geometric mean) for sector matrices was established in [18]. It is known that there exists other parameter extended means such as Stolarsky mean, binomial mean and Heron mean and so on. The studies on inequalities for such means of sector matrices will be an interesting future works.

In addition, we have some constants appearing in refined and reverse Young inequalties [7, Chapter 2]. For example, we have

$$S\left(\frac{b}{a}\right)a^{1-v}b^{v} \geqslant (1-v)a+vb \geqslant S\left(\left(\frac{b}{a}\right)^{r}\right)a^{1-v}b^{v}$$

for $a,b>0$ where $S(h):=\dfrac{h^{1/h-1}}{e\log h^{1/h-1}}$ is Specht ratio and $r:=\min\{v,1-v\}$ for $0\leqslant v\leqslant 1$. It is also known that we have the relation $S(h)\leqslant K(h)$ for $h>0$. Therefore it is not so easy to replace Kantorovich constant $K(h)$ in Theorem 2.1 by Specht ratio $S(h)$. To obtain the inequalties in Theorem 2.1 with Specht ratio $S(h)$, we will have to establish a new method. We also leave it to our future work.

Acknowledgements. The authors would like to thank the referees for their careful and insightful comments to improve our manuscript. This work was partially supported by JPSP KAKENHI Grant Number 16K05257 and 21K03341.

REFERENCES

[1] M. ALAKHRASS, *A note on sectorial matrices*, Linear Multilinear Algebra **68** (11)(2020), 2228–2238.
[2] R. BHATIA, *Matrix Analysis*, Springer–Verlag, New York, 1997.
[3] R. BHATIA, *Positive definite matrices*, Princeton University Press, Princeton, 2007.
[4] R. BHATIA AND F. KITTANEH, *Notes on matrix arithmetic–geometric mean inequalities*, Linear Algebra Appl. **308** (1–3) (2000), 203–211.
[5] M. BAKHERAD, *Refinements of a reversed AM–GM operator inequality*, Linear Multilinear Algebra **64** (9) (2016), 1687–1695.
[6] S. DRURY AND M. LIN, *Singular value inequalities for matrices with numerical ranges in a sector*, Oper. Matrices **8** (2014), 1143–1148.
[7] S. FURUICHI AND H. R. MORADI, *Advances in mathematical inequalities*, De Gruyter, 2020.
[8] T. FURUTA, J. MIĆIĆ, J. PEČARIĆ AND Y. SEO, *Mond–Pečarić method in operator inequalities*, Element, Zagreb, 2005.
[9] R. A. HORN AND C. R. JOHNSON, *Matrix Analysis*, Cambridge University Press, 2013.
[10] W. LIOA AND J. WU, *Improved operator Kantorovich and Wielandt inequalities for positive linear maps*, Filomat **31** (3) (2017), 871–876.
[11] M. LIN, *Squaring a reverse AM-GM inequality*, Studia Math. **215** (2013), 189–194.
[12] M. LIN, *Extension of a result of Hanynsworth and Hartfiel*, Arch. Math. **1** (2015), 93–100.
[13] M. LIN, *Some inequalities for sector matrices*, Oper. Matrices **10** (4) (2016), 915–921.
[14] M. RAISSOULI, M. S. MOSLEHIAN AND S. FURUICHI, *Relative entropy and Tsallis entropy of two accretive operators*, C. R. Acad. Sci. Paris Ser. I, **355** (2017), 687–693.
[15] F. TAN AND A. XIE, *An extension of the AM–GM–HM inequality*, Bull. Iran. Math. Soc. **46** (2020), 245–251.

[16] C. YANG AND F. LU, *Some generalizations of inequalities for sector matrices*, J. Inequal. Appl. **2018** (2018), Art. 183.

[17] C. YANG AND F. LU, *Some reverse mean inequalities for operators and matrices*, J. Inequal. Appl. **2019** (2019), Art. 115.

[18] C. YANG AND F. LU, *Inequalities for the Heinz mean of sector matrices involving positive linear maps*, Ann. Funct. Anal. **11** (2020), 866–878.

[19] F. ZHANG, *A matrix decomposition and its applications*, Linear Multilinear Algebra **63** (10) (2015), 2033–2042.

[20] D. ZHANG, L. HOU AND L. MA, *Properties of matrices with numerical ranges in a sector*, Bull. Iran. Math. Soc. **43** (6) (2017), 1699–1707.

(Received June 28, 2020)

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