Journal of
Mathematical
Inequalities
Volume 15, Number 4 (2021), 1435–1445 doi:10.7153/jmi-2021-15-98

PARTIAL DETERMINANT INEQUALITIES FOR POSITIVE SEMIDEFINITE BLOCK MATRICES

YONGTAO LI, XIQIN LIN AND LIHUA FENG*

(*Communicated by J. Mićić Hot*)

Abstract. We present some inequalities related to partial determinants for positive semidefinite block matrices. Firstly, we introduce the definition of partial matrix functions corresponding to partial traces and partial determinants, and then we provide a unified extension of a recent result of Lin [10], Chang-Paksoy-Zhang [4] and Lin-Sra [12]. Secondly, we give a new generalization of a result of Paksoy-Turkmen-Zhang [15]. Finally, we conclude with an interesting conjecture involving partial determinants.

1. Introduction

Let A and B be $n\times n$ positive semidefinite matrices. It is easy to prove by simultaneous diagonalization argument (see [20, p. 210] or [8, p. 485]) that

$$\det(A+B)\geqslant \det(A)+\det(B). \tag{1}$$

There are various extensions and improvements on (1) in the literature. The earliest improvements could be tracked back to Haynsworth [7] and later Hartfiel [6]; see [9] and [13] for recent refinements; and see, e.g., [3, 4, 10, 12, 15] for more extensions. An attractive extension of (1) is the following (assuming that C is also an n-square positive semidefinite matrix)

$$\det(A+B+C)+\det C\geqslant \det(A+C)+\det(B+C). \tag{2}$$

Inequality (2) can be found e.g., in [20, p. 215, Problem 36] and [11, Lemma 2.5].

Setting $C=0$, (2) reduces to (1). However, this is not the merely relation between (1) and (2). Recently, Lin [10] investigated one more connection between them. That is, the difference in (1) is dominated by the difference in (2). More precisely, Lin proved the following result.

$$\begin{aligned}&\det(A+B+C)+\det C-\big(\det(A+C)+\det(B+C)\big)\\&\quad\geqslant \det(A+B)-(\det A+\det B).\end{aligned} \tag{3}$$

Mathematics subject classification (2020): 15A45, 15A60, 47B65.
Keywords and phrases: Block matrices, generalized matrix function, partial traces, partial determinants.

* Corresponding author.

After years, the above mentioned results (1), (2) and (3) had been extended to the generalized matrix function. Let G be a subgroup of the symmetric group S_n on n letters and let χ be an irreducible character of G. For any $n\times n$ complex matrix $A=[a_{ij}]_{i,j=1}^n$, the *generalized matrix function* of A (also known as *immanant*) afforded by G and χ is defined as

$$\mathrm{d}_\chi^G(A) := \sum_{\sigma\in G}\chi(\sigma)\prod_{i=1}^n a_{i\sigma(i)}.$$

Generally speaking, some specific representations of finite groups lead to some acquainted functionals on the matrix space. For instance, if $G=S_n$ and χ is the signum function with value ± 1, then the generalized matrix function becomes the usual matrix determinant; By setting $\chi(\sigma)\equiv 1$ for each $\sigma\in G=S_n$, we get the permanent of the matrix; Setting $G=\{e\}\subset S_n$, one defines the product of the main diagonal entries of the matrix, which is also known as the Hadamard matrix function.

Now, we list briefly some extensions of (1), (2) and (3) to the generalized matrix function. For example, a remarkable extension (e.g., [14, p. 228]) says that

$$\mathrm{d}_\chi^G(A+B)\geqslant \mathrm{d}_\chi^G(A)+\mathrm{d}_\chi^G(B). \tag{4}$$

In 2014, Paksoy, Turkmen and Zhang [15] provided a natural extension of (2) and (4) for triple matrices, using tensor products, their treatments are embedding the vectors of Gram matrices into a "sufficiently large" inner product space. More precisely, if A,B and C are positive semidefinite, they obtained

$$\mathrm{d}_\chi^G(A+B+C)+\mathrm{d}_\chi^G(C)\geqslant \mathrm{d}_\chi^G(A+C)+\mathrm{d}_\chi^G(B+C). \tag{5}$$

Soon later, Chang, Paksoy and Zhang [4, Theorem 3] (Berndt and Sra [3] independently) presented a further improvement on (5) by considering the tensor products of operators as words on certain alphabets. They obtained an analogue of (3) and confirmed a conjecture of Lin [10], which states that

$$\begin{aligned}&\mathrm{d}_\chi^G(A+B+C)+\mathrm{d}_\chi^G(A)+\mathrm{d}_\chi^G(B)+\mathrm{d}_\chi^G(C)\\ &\qquad\geqslant \mathrm{d}_\chi^G(A+B)+\mathrm{d}_\chi^G(A+C)+\mathrm{d}_\chi^G(B+C).\end{aligned} \tag{6}$$

We remark here that (6) is indeed an improvement on (5) since

$$\begin{aligned}&\mathrm{d}_\chi^G(A+B+C)+\mathrm{d}_\chi^G(C)-\Big(\mathrm{d}_\chi^G(A+C)+\mathrm{d}_\chi^G(B+C)\Big)\\ &\qquad\geqslant \mathrm{d}_\chi^G(A+B)-\Big(\mathrm{d}_\chi^G(A)+\mathrm{d}_\chi^G(B)\Big)\geqslant 0.\end{aligned}$$

On the other hand, Lin and Sra [12] presented a further extension of (1) for positive semidefinite block matrices. Before starting their result, we first fix the following standard notation. The set of $m\times n$ complex matrices is denoted by $\mathbb{M}_{m\times n}$. If $m=n$, we use $\mathbb{M}_n$ instead of $\mathbb{M}_{n\times n}$, and if $n=1$, we use $\mathbb{C}^m$ instead of $\mathbb{M}_{m\times 1}$. The identity matrix of $\mathbb{M}_n$ is denoted by I_n, or simply by I if no confusion is possible. We use $\mathbb{M}_m(\mathbb{M}_n)$ for the set of $m\times m$ block matrices with each block being an $n\times n$ matrix.

The element of $\mathbb{M}_m(\mathbb{M}_n)$ is usually written as $A=[A_{ij}]_{i,j=1}^m$, where $A_{ij}\in\mathbb{M}_n$ for all i,j. If $A=[A_{ij}]_{i,j=1}^m\in\mathbb{M}_m(\mathbb{M}_n)$, we denote

$$\det{}_2(A):=[\det A_{ij}]_{i,j=1}^m\in\mathbb{M}_m. \tag{7}$$

By convention, if $X\in\mathbb{M}_n$ is positive semidefinite, then we write $X\geqslant 0$. For two Hermitian matrices A and B of the same size, $A\geqslant B$ stands for $A-B\geqslant 0$. It is easy to verify that "$\geqslant$" is a partial ordering on the set of Hermitian matrices, referred as Löwner ordering; see, e.g., [18, Chapter 1] and [19] for related topics.

Under the above definition (7), Lin-Sra's extension of (1) can be listed below. If $A,B\in\mathbb{M}_m(\mathbb{M}_n)$ are positive semidefinite block matrices, then the following inequality in the Löwner partial order holds.

$$\det{}_2(A+B)\geqslant\det{}_2(A)+\det{}_2(B). \tag{8}$$

Clearly, when $m=1$, (8) reduces to (1).

The paper is organized as follows. In Section 2, we briefly review some basic definitions and properties of the tensor product in multilinear algebra theory. In Section 3, we will extend the above-cited results to the block positive semidefinte matrices, our result is a unified extension of (6) and (8) (Theorem 3.7 and Corollary 3.8). As an application, some new inequalities related to the trace, determinant and permanent are also included. In Section 4, we present a new extension of (5) which involves both determinants and partial determinants (Theorem 4.2 and Corollary 4.3). At the end of the paper, we conclude with an interesting conjecture.

2. Preliminaries

In this section, we first review some basic definitions and notations of multilinear algebra theory [14]. Let $X\otimes Y$ denote the *Kronecker product* (also called *tensor product*) of X with Y, that is, if $X=[x_{ij}]_{i,j=1}^m\in\mathbb{M}_m$ and $Y\in\mathbb{M}_n$, then $X\otimes Y\in\mathbb{M}_m(\mathbb{M}_n)$ whose (i,j)-block is $x_{ij}Y$. Let $\otimes^r A:=A\otimes\cdots\otimes A$ denote the r-fold tensor power of A. We denote by $\wedge^r A$ the rth antisymmetric tensor power (or rth Grassmann power) of A, which is the same as the rth multiplicative compound matrix of A, and denote by $\vee^r A$ the rth symmetric tensor power of A; see [1, p. 18] for more details. We denote by $e_r(A)$ and $s_r(A)$ the rth elementary symmetric and rth complete symmetric function of the eigenvalues of A (see [8, p. 54]), respectively. Trivially, $e_1(A)=s_1(A)=\operatorname{tr}(A)$ and $e_n(A)=\det(A)$ for $A\in\mathbb{M}_n$.

Let V be an n-dimensional Hilbert space and $\otimes^n V$ be the tensor product space of n copies of V. Let G be a subgroup of the symmetric group S_n and χ be an irreducible character of G. The *symmetrizer* induced by χ on the tensor product space $\otimes^n V$ is defined by its action as

$$S(v_1\otimes\cdots\otimes v_n):=\frac{1}{|G|}\sum_{\sigma\in G}\chi(\sigma)v_{\sigma^{-1}(1)}\otimes\cdots\otimes v_{\sigma^{-1}(n)}. \tag{9}$$

All elements of the form (9) span a vector space, denoted by $V_\chi^n(G)\subset\otimes^n V$, which is called the *space of the symmetry class of tensors* associated with G and χ (see [14, p.

154, 235]). It is easy to verified that $V_\chi^n(G)$ is an invariant subspace of $\otimes^n V$ under the tensor operator $\otimes^n A$. For a linear operator A on V, the induced operator $K(A)$ of A with respect to G and χ is defined to be $K(A) = (\otimes^n A)\big|_{V_\chi^n(G)}$, the restriction of $\otimes^n A$ on $V_\chi^n(G)$.

The induced operator $K(A)$ is closely related to the generalized matrix function. Let $e_1, e_2, \ldots, e_n$ be an orthonormal basis of V and P be a matrix representation of the linear operator A on V with respect to the basis $e_1, \ldots, e_n$. Then

$$\mathrm{d}_\chi^G\left(P^T\right) = \frac{|G|}{\deg(\chi)} \langle K(A)e^*, e^* \rangle, \tag{10}$$

where $\deg(\chi)$ is the degree of χ and $e^* := e_1 * e_2 * \cdots * e_n$ is the decomposable symmetrized tensor of $e_1, \ldots, e_n$ (see [14, p. 227, 155]).

Now, we list some basic properties of tensor product for our latter use.

PROPOSITION 2.1. (see [1, pp. 16–20]) *Let A, B and C be $n \times n$ matrices. Then*

(1) $\otimes^r(AB) = (\otimes^r A)(\otimes^r B), \wedge^r(AB) = (\wedge^r A)(\wedge^r B)$ *and* $\vee^r(AB) = (\vee^r A)(\vee^r B)$.

(2) $\mathrm{tr}(\otimes^r A) = (\mathrm{tr} A)^r := p_r(A), \mathrm{tr}(\wedge^r A) = e_r(A)$ *and* $\mathrm{tr}(\vee^r A) = s_r(A)$.

(3) $\det(\otimes^r A) = (\det A)^{rn^{r-1}}, \det(\wedge^r A) = (\det A)^{\binom{n-r}{r-1}}$ *and* $\det(\vee^r A) = (\det A)^{\frac{r}{n}\binom{n+r-1}{r}}$.

Furthermore, if A, B and C are positive semidefinite matrices, then

(4) $A \otimes B, A \wedge B$ *and* $A \vee B$ *are positive semidefinite.*

(5) $\otimes^r(A+B) \geqslant \otimes^r A + \otimes^r B, \wedge^r(A+B) \geqslant \wedge^r A + \wedge^r B$ *and* $\vee^r(A+B) \geqslant \vee^r A + \vee^r B$.

Finally, we introduce the definition and notation of the partial traces, which comes from Quantum Information Theory [16, p. 12]. Given $A \in \mathbb{M}_m(\mathbb{M}_n)$, the *first partial trace map* $A \mapsto \mathrm{tr}_1(A) \in \mathbb{M}_n$ is defined as the adjoint map of the imbedding map $X \mapsto I_m \otimes X \in \mathbb{M}_m \otimes \mathbb{M}_n$. Correspondingly, the *second partial trace map* $A \mapsto \mathrm{tr}_2(A) \in \mathbb{M}_m$ is similarly given as the adjoint map of the imbedding map $Y \mapsto Y \otimes I_n \in \mathbb{M}_m \otimes \mathbb{M}_n$. Therefore, we have

$$\langle I_m \otimes X, A \rangle = \langle X, \mathrm{tr}_1(A) \rangle, \quad \forall\, X \in \mathbb{M}_n,$$

and

$$\langle Y \otimes I_n, A \rangle = \langle Y, \mathrm{tr}_2(A) \rangle, \quad \forall\, Y \in \mathbb{M}_m.$$

Assume that $A = [A_{ij}]_{i,j=1}^m$ with each $A_{ij} \in \mathbb{M}_n$, the visualized equivalent forms of the first and second partial trace are actually given in [2, Proposition 4.3.10] as

$$\mathrm{tr}_1(A) = \sum_{i=1}^m A_{ii} \quad \text{and} \quad \mathrm{tr}_2(A) = \left[\mathrm{tr} A_{ij}\right]_{i,j=1}^m.$$

Under the above definition, it follows that both $\mathrm{tr}_1(A)$ and $\mathrm{tr}_2(A)$ are positive semidefinite whenever A is positive semidefinite; see, e.g., [20, p. 237].

3. Partial matrix functions

We now define the first and second partial matrix function, which first appeared in the form of determinant function in [5]. We here introduce a slightly more general setting. Before giving our definition, we first demonstrate our motivation.

For $A=[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$, suppose that $A_{ij}=\left[a_{rs}^{ij}\right]_{r,s=1}^n$. Setting

$$G_{rs} := \left[a_{rs}^{ij}\right]_{i,j=1}^m \in \mathbb{M}_m.$$

Then we can verify that

$$\operatorname{tr}_1(A)=\sum_{i=1}^m A_{ii}=\sum_{i=1}^m \left[a_{rs}^{ii}\right]_{r,s=1}^n=\left[\sum_{i=1}^m a_{rs}^{ii}\right]_{r,s=1}^n=\left[\operatorname{tr} G_{rs}\right]_{r,s=1}^n.$$

Motivated by this relation, we now introduce the following definition.

DEFINITION 3.1. Let $\Phi : \mathbb{M}_p \to \mathbb{M}_q$ be a matrix function. The *first and second partial matrix functions* of Φ on $A=[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ are defined by

$$\Phi_1(A) := \left[\Phi(G_{rs})\right]_{r,s=1}^n \quad \text{and} \quad \Phi_2(A) := \left[\Phi(A_{ij})\right]_{i,j=1}^m.$$

Clearly, when $\Phi=\operatorname{tr}$, this definition coincides with that of the partial traces; when $\Phi=\det$, it identifies with the partial determinants, which were first introduced by Choi in [5] recently. We denote by $\widetilde{A} := [G_{rs}]_{r,s=1}^n \in \mathbb{M}_n(\mathbb{M}_m)$, and then it is easy to see that

$$\widetilde{\widetilde{A}}=A \quad \text{and} \quad \Phi_1(A)=[\Phi(G_{rs})]_{r,s=1}^n=\Phi_2(\widetilde{A}). \tag{11}$$

Moreover, one could observe the following result.

LEMMA 3.2. (see [5, Theorem 7]) *For $A\in\mathbb{M}_m(\mathbb{M}_n)$, $\widetilde{A}$ and A are unitarily similar.*

Let $A=[A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$ be a positive semidefinite block matrix. It is well known that both $\det_2(A)=[\det A_{ij}]_{i,j=1}^m$ and $\operatorname{tr}_2(A)=[\operatorname{tr} A_{ij}]_{i,j=1}^m$ are positive semidefinite matrices; see, e.g., [20, p. 221, 237]. Whereafter, Zhang [21, Theorem 3.1] extends the positivity to the generalized matrix function via the generalized Cauchy-Binet formula, more precisely, he obtained that $\mathrm{d}_\chi^G{}_2(A)=[\mathrm{d}_\chi^G(A_{ij})]_{i,j=1}^m$ is also positive semidefinite.

Next, we extend the positivity to more matrix functionals.

PROPOSITION 3.3. *Let $A\in\mathbb{M}_m(\mathbb{M}_n)$ be positive semidefinite. If Φ is one of the functionals* $\operatorname{tr},\det,\operatorname{per},\mathrm{d}_\chi^G,p_r,e_r$ *and* s_r, *then $\Phi_1(A)$ and $\Phi_2(A)$ are positive semidefinite.*

Proof. By making full use of the relation (11) and Lemma 3.2, we only need to show that $\Phi_2(A)$ is positive semidefinite. This is similar as the lines in [21, Theorem 3.1], we omit the details. $\square$

The following Lemma 3.4 plays a prominent role in our extension (Theorem 3.7), it could be found in [3] or [4], we here provide a proof for convenience.

LEMMA 3.4. *Let A,B,C be positive semidefinite matrices of the same size. Then for every positive integer r, we have*

$$\begin{aligned}&\otimes^r (A+B+C)+\otimes^r A+\otimes^r B+\otimes^r C\\&\quad\geqslant \otimes^r(A+B)+\otimes^r(A+C)+\otimes^r(B+C).\end{aligned}$$

The similar result is true for $\wedge^r$ and $\vee^r$.

Proof. The proof is by induction on r. The base case $r=1$ holds with equality, and the case $r=2$ is easy to verify. Assume the required result holds for $r=m\geqslant 2$, that is

$$\begin{aligned}&\otimes^m (A+B+C)+\otimes^m A+\otimes^m B+\otimes^m C\\&\quad\geqslant \otimes^m(A+B)+\otimes^m(A+C)+\otimes^m(B+C).\end{aligned}$$

For $r=m+1$, we get from Proposition 2.1 that

$$\begin{aligned}&\otimes^{m+1} (A+B+C)\\&\quad=\big(\otimes^m(A+B+C)\big)\otimes(A+B+C)\\&\quad\geqslant\big(\otimes^m(A+B)+\otimes^m(A+C)+\otimes^m(B+C)-\otimes^m A-\otimes^m B-\otimes^m C\big)\\&\qquad\otimes(A+B+C)\\&\quad=\otimes^{m+1}(A+B)+\otimes^{m+1}(A+C)+\otimes^{m+1}(B+C)\\&\qquad-\otimes^{m+1}A-\otimes^{m+1}B-\otimes^{m+1}C\\&\qquad+\big(\otimes^m(A+B)\big)\otimes C+\big(\otimes^m(A+C)\big)\otimes B+\big(\otimes^m(B+C)\big)\otimes A\\&\qquad-\big(\otimes^m A\big)\otimes(B+C)-\big(\otimes^m B\big)\otimes(A+C)-\big(\otimes^m C\big)\otimes(A+B).\end{aligned}$$

It remains to show that

$$\begin{aligned}&\big(\otimes^m(A+B)\big)\otimes C+\big(\otimes^m(A+C)\big)\otimes B+\big(\otimes^m(B+C)\big)\otimes A\\&\quad\geqslant\big(\otimes^m A\big)\otimes(B+C)+\big(\otimes^m B\big)\otimes(A+C)+\big(\otimes^m C\big)\otimes(A+B).\end{aligned}$$

This follows immediately by the superadditivity (5) in Proposition 2.1. □

COROLLARY 3.5. *Let A,B,C be positive semidefinite matrices of the same size. Then for each positive integer r, we have*

$$\otimes^r(A+B+C)+\otimes^r C\geqslant\otimes^r(A+C)+\otimes^r(B+C).$$

Proof. By Lemma 3.4, we have

$$\begin{aligned}&\otimes^r (A+B+C)+\otimes^r C-\otimes^r(A+C)-\otimes^r(B+C)\\&\quad\geqslant\otimes^r(A+B)-\otimes^r A-\otimes^r B.\end{aligned}$$

By Proposition 2.1, the desired inequality immediately follows. □

To show our main result (Theorem 3.7), we require one more lemma.

LEMMA 3.6. ([2, p. 93]) *Let* $A = [A_{ij}]_{i,j=1}^m \in \mathbb{M}_m(\mathbb{M}_n)$. *Then* $[\otimes^r A_{ij}]_{i,j=1}^m$ *is a principal submatrix of* $\otimes^r A$ *for every positive integer* r.

Now, we present our main result, which is a unified extension of (6) and (8).

THEOREM 3.7. *Let* $A,B,C \in \mathbb{M}_m(\mathbb{M}_n)$ *be positive semidefinite. If* Φ *is one of the functionals* $\mathrm{tr}, \det, \mathrm{per}, \mathrm{d}_\chi^G, p_r, e_r$ *and* s_r, *then*

$$\begin{aligned}&\Phi_1(A+B+C)+\Phi_1(A)+\Phi_1(B)+\Phi_1(C)\\ &\quad \geqslant \Phi_1(A+B)+\Phi_1(A+C)+\Phi_1(B+C),\end{aligned}$$

and

$$\begin{aligned}&\Phi_2(A+B+C)+\Phi_2(A)+\Phi_2(B)+\Phi_2(C)\\ &\quad \geqslant \Phi_2(A+B)+\Phi_2(A+C)+\Phi_2(B+C).\end{aligned}$$

In particular, we have

$$\begin{aligned}&\Phi(A+B+C)+\Phi(A)+\Phi(B)+\Phi(C)\\ &\quad \geqslant \Phi(A+B)+\Phi(A+C)+\Phi(B+C).\end{aligned}$$

Proof. We only show that the desired result holds for $\Phi = \mathrm{d}_\chi^G$ and $\Phi = e_r$ since the rest cases of functionals can be proved similarly. Applying the relation (11) and Lemma 3.2, it suffices to show the second desired result by exchanging the role of A and $\widetilde{A}$. By Lemma 3.4, we have

$$\begin{aligned}&\otimes^r(A+B+C)+\otimes^r A+\otimes^r B+\otimes^r C\\ &\quad \geqslant \otimes^r(A+B)+\otimes^r(A+C)+\otimes^r(B+C),\end{aligned}$$

which, together with Lemma 3.6, leads to

$$\begin{aligned}&[\otimes^r(A_{ij}+B_{ij}+C_{ij})]_{i,j=1}^m+[\otimes^r A_{ij}]_{i,j=1}^m+[\otimes^r B_{ij}]_{i,j=1}^m+[\otimes^r C_{ij}]_{i,j=1}^m\\ &\quad \geqslant [\otimes^r(A_{ij}+B_{ij})]_{i,j=1}^m+[\otimes^r(A_{ij}+C_{ij})]_{i,j=1}^m+[\otimes^r(B_{ij}+C_{ij})]_{i,j=1}^m.\end{aligned}$$

By restricting the above inequality to the symmetry class $V_\chi^G(V)$, we get

$$\begin{aligned}&[K(A_{ij}+B_{ij}+C_{ij})]_{i,j=1}^m+[K(A_{ij})]_{i,j=1}^m+[K(B_{ij})]_{i,j=1}^m+[K(C_{ij})]_{i,j=1}^m\\ &\quad \geqslant [K(A_{ij}+B_{ij})]_{i,j=1}^m+[K(A_{ij}+C_{ij})]_{i,j=1}^m+[K(B_{ij}+C_{ij})]_{i,j=1}^m.\end{aligned}$$

Combining (10), the second desired result in the case of $\Phi = \mathrm{d}_\chi^G$ follows.

Along the same lines, it follows that

$$\begin{aligned}&[\wedge^r(A_{ij}+B_{ij}+C_{ij})]_{i,j=1}^m+[\wedge^r A_{ij}]_{i,j=1}^m+[\wedge^r B_{ij}]_{i,j=1}^m+[\wedge^r C_{ij}]_{i,j=1}^m\\ &\quad \geqslant [\wedge^r(A_{ij}+B_{ij})]_{i,j=1}^m+[\wedge^r(A_{ij}+C_{ij})]_{i,j=1}^m+[\wedge^r(B_{ij}+C_{ij})]_{i,j=1}^m.\end{aligned}$$

By taking trace blockwise and using Proposition 2.1, it yields the second desired result in the case of $\Phi = e_r$. $\square$

From Corollary 3.5 or Theorem 3.7, one may get the following corollary, which can be viewed as an extension of (2) and (5).

COROLLARY 3.8. *Let* $A,B,C \in \mathbb{M}_m(\mathbb{M}_n)$ *be positive semidefinite. If* Φ *is one of the functionals* $\mathrm{tr}, \det, \mathrm{per}, \mathrm{d}_\chi^G, p_r, e_r$ *and* s_r*, then*

$$\Phi_1(A+B+C)+\Phi_1(C) \geqslant \Phi_1(A+C)+\Phi_1(B+C),$$

and

$$\Phi_2(A+B+C)+\Phi_2(C) \geqslant \Phi_2(A+C)+\Phi_2(B+C).$$

In particular, we have

$$\Phi(A+B+C)+\Phi(C) \geqslant \Phi(A+C)+\Phi(B+C).$$

REMARK. By setting $\Phi = \det$ in Theorem 3.7 and Corollary 3.8, one could get the renowned determinantal inequalities (3) and (2), respectively. We remark here that these two inequalities could be proved by using a majorization approach of eigenvalues. It is more elementary and totally different from our method; see, e.g., [10, Theorem 1.1] and [20, p. 215] for more details.

4. A new extension of inequality (5)

The remaining of this section is devoted to some inequalities concerning both determinant and partial determinant. Firstly, we give two inequalities (12) and (13), which can be viewed as our starting point. It is easy to see by applying (1) to Lin-Sra's result (8) that

$$\det\big(\det_2(A+B)\big) \geqslant \det\big(\det_2 A + \det_2 B\big) \geqslant \det\big(\det_2 A\big) + \det\big(\det_2 B\big). \tag{12}$$

On the other hand, we can get that

$$\det\big(\mathrm{tr}_2(A+B)\big) = \det\big(\mathrm{tr}_2 A + \mathrm{tr}_2 B\big) \geqslant \det\big(\mathrm{tr}_2 A\big) + \det\big(\mathrm{tr}_2 B\big). \tag{13}$$

Motivated by this observation, we will present an extension of both (12) and (13) in the following Theorem 4.2. To preceed our result, we need to extend Corollary 3.8 to more general setting, which plays an essential role in the proof of Theorem 4.2.

PROPOSITION 4.1. *Let* $X,Y,W,Z \in \mathbb{M}_m(\mathbb{M}_n)$ *be positive semidefinite. If* $X \geqslant W \geqslant Y, X \geqslant Z \geqslant Y$ *and* $X+Y \geqslant W+Z$*, then*

$$\Phi_1(X)+\Phi_1(Y) \geqslant \Phi_1(W)+\Phi_1(Z),$$

and

$$\Phi_2(X)+\Phi_2(Y) \geqslant \Phi_2(W)+\Phi_2(Z).$$

In particular, we have

$$\Phi(X)+\Phi(Y) \geqslant \Phi(W)+\Phi(Z).$$

Proof. We only prove the first inequality, the other two inequalities can be proved in a similar way. Let $N = X+Y-(W+Z)$. Then N is positive semidefinite. Note that $X \geqslant X-N$ and $X-N \geqslant W,Z$. It suffices to show

$$\Phi_1(X-N)+\Phi_1(Y) \geqslant \Phi_1(W)+\Phi_1(Z). \tag{14}$$

Let $C=Y, A=W-Y$ and $B=Z-Y$. (14) is then equivalent to

$$\Phi_1(A+B+C)+\Phi_1(C) \geqslant \Phi_1(A+C)+\Phi_1(B+C),$$

which is a direct consequence in Corollary 3.8. □

As promised, we shall give an extension of (12) and (13).

THEOREM 4.2. *Let $A,B,C \in \mathbb{M}_m(\mathbb{M}_n)$ be positive semidefinite. If Φ and Ψ are two functionals selected from* $\mathrm{tr}, \det, \mathrm{per}, \mathrm{d}_\chi^G, p_r, e_r$ *and* s_r, *then*

$$\Psi\big(\Phi_1(A+B+C)\big)+\Psi\big(\Phi_1(C)\big) \geqslant \Psi\big(\Phi_1(A+C)\big)+\Psi\big(\Phi_1(B+C)\big),$$

and

$$\Psi\big(\Phi_2(A+B+C)\big)+\Psi\big(\Phi_2(C)\big) \geqslant \Psi\big(\Phi_2(A+C)\big)+\Psi\big(\Phi_2(B+C)\big).$$

Proof. We only prove the first inequality, the proof of the second one is similar. It is easy to see that

$$\begin{aligned}\Phi_1(A+B+C) &\geqslant \Phi_1(A+C) \geqslant \Phi_1(C),\\ \Phi_1(A+B+C) &\geqslant \Phi_1(B+C) \geqslant \Phi_1(C),\end{aligned}$$

and by Corollary 3.8, we have

$$\Phi_1(A+B+C)+\Phi_1(C) \geqslant \Phi_1(A+C)+\Phi_1(B+C).$$

Applying Proposition 4.1, we can get

$$\Psi\big(\Phi_1(A+B+C)\big)+\Psi\big(\Phi_1(C)\big) \geqslant \Psi\big(\Phi_1(A+C)\big)+\Psi\big(\Phi_1(B+C)\big).$$

This completes the proof. □

Clearly, Theorem 4.2 is a versatile generalization of Paksoy-Turkmen-Zhang's result (5). Moreover, when $m=1$ or $n=1$, Theorem 4.2 reduces to Corollary 3.8. In particular, by setting $\Phi=\Psi=\det$ in Theorem 4.2, we get the following corollary, which is an extension of (12) as well as (2).

COROLLARY 4.3. *Let $A,B,C \in \mathbb{M}_m(\mathbb{M}_n)$ be positive semidefinite. Then*

$$\det\big(\det{}_1(A+B+C)\big)+\det\big(\det{}_1(C)\big) \geqslant \det\big(\det{}_1(A+C)\big)+\det\big(\det{}_1(B+C)\big),$$

and

$$\det\big(\det{}_2(A+B+C)\big)+\det\big(\det{}_2(C)\big) \geqslant \det\big(\det{}_2(A+C)\big)+\det\big(\det{}_2(B+C)\big).$$

Corollary 3.8 yields the following analogous result of Theorem 4.2.

COROLLARY 4.4. *Let* $A,B,C \in \mathbb{M}_m(\mathbb{M}_n)$ *be positive semidefinite. If* Φ *and* Ψ *are two functionals selected from* $\mathrm{tr}, \det, \mathrm{per}, \mathrm{d}_\chi^G, p_r, e_r$ *and* s_r*, then*

$$\Psi\big(\Phi_1(A+B+C)+\Phi_1(C)\big) \geqslant \Psi\big(\Phi_1(A+C)+\Phi_1(B+C)\big),$$

and

$$\Psi\big(\Phi_2(A+B+C)+\Phi_2(C)\big) \geqslant \Psi\big(\Phi_2(A+C)+\Phi_2(B+C)\big).$$

5. Concluding remarks

In this paper, we give a unified extension of (6) and (8) in Theorem 3.7. The key step of the proof of Theorem 3.7 is attributed to the inequality involving Kronecker product in Lemma 3.4. There are some similar inequalities on symmetric tensor in [3], so one could also present some analogous results for partial matrix function, we omit the details.

Moreover, some new inequalities related to both determinant and partial determinants are also presented in Theorem 4.2 and Corollary 4.3. Comparing Corollary 4.3 with inequality (3), we conclude at the end of the paper with the following conjecture.

CONJECTURE 5.1. Let $A,B,C \in \mathbb{M}_m(\mathbb{M}_n)$ be positive semidefinite. Then

$$\begin{aligned}&\det\big(\det_2(A+B+C)\big)+\det\big(\det_2(A)\big)+\det\big(\det_2(B)\big)+\det\big(\det_2(C)\big)\\ &\quad\geqslant \det\big(\det_2(A+B)\big)+\det\big(\det_2(A+C)\big)+\det\big(\det_2(B+C)\big).\end{aligned}$$

In [17], it is shown that if $A \in \mathbb{M}_m(\mathbb{M}_n)$ is positive semidefinite, then

$$\det(\det_2 A) \geqslant \det A. \tag{15}$$

Choi [5] provided a complement for the first partial determinant

$$\det(\det_1 A) \geqslant \det A.$$

We can easily extend (15) to two matrices. By (8) and (1), we get

$$\det\big(\det_2(A+B)\big) \geqslant \det\big(\det_2 A+\det_2 B\big) \geqslant \det(\det_2 A)+\det(\det_2 B) \geqslant \det A+\det B.$$

It is natural to consider similar inequalities to (15). To our best knowledge, conclusions related to partial determinants are rare in the literature. This of course deserves further investigation.

Acknowledgements. The authors wish to thank Minghua Lin for his insightful comments which greatly improved the presentation. The corresponding authors L. Feng, X. Lin were supported by NSFC (Grant Nos. 12071484, 11871479), Hunan Provincial Natural Science Foundation (2020JJ4675, 2018JJ2479) and Mathematics and Inter- disciplinary Sciences Project of CSU.

REFERENCES

[1] R. BHATIA, *Matrix Analysis*, GTM 169, Springer-Verlag, New York, 1997.

[2] R. BHATIA, *Positive Definite Matrices*, Princeton University Press, Princeton, 2007.

[3] W. BERNDT, S. SRA, *Hlawka-Popoviciu inequalities on positive definite tensors*, Linear Algebra Appl. 486 (2015) 317–327.

[4] H. CHANG, V. E. PAKSOY, F. ZHANG, *An inequality for tensor product of positive operators and its applications*, Linear Algebra Appl. 498 (2016) 99–105.

[5] D. CHOI, *Inequalities related to trace and determinant of positive semidefinite block matrices*, Linear Algebra Appl. 532 (2017) 1–7.

[6] D. J. HARTFIEL, *An extension of Haynsworth's determinant inequality*, Proc. Amer. Math. Soc. 41 (1973) 463–465.

[7] E. V. HAYNSWORTH, *Applications of an inequality for the Schur complemment*, Proc. Amer. Math. Soc. 24 (1970) 512–516.

[8] R. A. HORN, C. R. JOHNSON, *Matrix Analysis, 2nd ed.*, Cambridge University Press, Cambridge, 2013.

[9] Y. LI, L. FENG, *Extensions of Brunn-Minkowski's inequality to multiple matrices*, Linear Algebra Appl. 603 (2020) 91–100.

[10] M. LIN, *A determinantal inequality for positive definite matrices*, Electron. J. Linear Algebra 27 (2014) 821–826.

[11] M. LIN, *An Oppenheim type inequality for a block Hadamard product*, Linear Algebra Appl. 452 (2014) 1–6.

[12] M. LIN, S. SRA, *A proof of Thompson's determinantal inequality*, Math. Notes 99 (2016) 164–165.

[13] Y. MAO, *Extensions of Hartfiel's inequality to multiple matrices*, Linear Algebra Appl. 589 (2020) 96–102.

[14] R. MERRIS, *Multilinear Algebra*, Gordon & Breach, Amsterdam, 1997.

[15] V. PAKSOY, R. TURKMEN, F. ZHANG, *Inequalities of generalized matrix functions via tensor products*, Electron. J. Linear Algebra 27 (2014) 332–341.

[16] D. PETZ, *Quantum Information Theory and Quantum Statistics. Theoretical and Mathematical Physics*, Springer, Berlin, 2008.

[17] R. C. THOMPSON, *A determinantal inequality for positive definite matrices*, Canad. Math. Bull. 4 (1961) 57–62.

[18] X. ZHAN, *Matrix Inequalities*, Springer, New York, 2002.

[19] X. ZHAN, *Matrix Theory*, Graduate Studies in Mathematics, vol. 147, Amer. Math. Soc., Providence, RI, 2013.

[20] F. ZHANG, *Matrix Theory: Basic Results and Techniques, 2nd edition*, Springer, New York, 2011.

[21] F. ZHANG, *Positivity of matrices with generalized matrix functions*, Acta Math. Sinica 28 (9) (2012) 1779–1786.

(Received July 4, 2020)

Yongtao Li
School of Mathematics
Hunan University
Changsha, Hunan, 410082, P. R. China

Xiqin Lin
School of Mathematics and Information Science
Shandong Institute of Business and Technology
Yantai, Shandong, 264005, P. R. China

Lihua Feng
School of Mathematics and Statistics
Central South University
New Campus, Changsha, Hunan, 410083, P. R. China
e-mail: fenglh@163.com