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APPLICATIONS OF A DUALITY BETWEEN GENERALIZED TRIGONOMETRIC AND HYPERBOLIC FUNCTIONS II

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Dedicated to Professor Takeyuki Nagasawa on the occasion of his 60th birthday

(*Communicated by J. Matkowski*)

Abstract. Generalized trigonometric functions and generalized hyperbolic functions can be converted to each other by the duality formulas previously discovered by the authors. In this paper, we apply the duality formulas to prove dual pairs of Wilker-type inequalities, Huygens-type inequalities, and (relaxed) Cusa-Huygens-type inequalities for the generalized functions. In addition, multiple- and double-angle formulas not previously obtained are also given.

1. Introduction

Generalized trigonometric functions (GTFs) and generalized hyperbolic functions (GHFs) are natural mathematical generalizations of the trigonometric and hyperbolic functions, respectively. They have been applied not only to generalize π and the complete elliptic integrals, but also to analyze nonlinear differential equations involving p-Laplacian (see monographs [3, 6] and survey [15], and the references given there).

Although GTFs and GHFs have been actively studied, they have been treated separately (e.g., [3, 5, 7, 9, 14, 15]). In our previous work [8], the duality formulas that can transform GTFs and GHFs into each other are proved. As an application, we were able to construct generalized inequalities of the classical Mitrinović-Adamović inequalities to GTFs and GHFs such that they are dual pairs to each other in the sense explained in the next section.

In this paper, following [8], we will generalize the old and vigorously studied Wilker inequalities, Huygens inequalities and (relaxed) Cusa-Huygens inequalities to GTFs and GHFs. In fact, previous works, e.g., [5, 7, 9, 14, 15], have made various generalizations of these inequalities, but the trigonometric and hyperbolic versions are not in dual pairs. On the other hand, the pairs we create in the present paper are dual to each other.

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This paper is organized as follows. Section 2 summarizes the definitions of GTFs and GHFs and their properties, including the duality formulas obtained in [8]. Here, the conditions imposed on the parameters contained in these functions are more extended than usual. This extension reveals the duality between both generalized functions. In Section 3, we generalize the classical Wilker inequalities, Huygens inequalities, and (relaxed) Cusa-Huygens inequalities to GTFs and GHFs. It should be noted that the pairs of inequalities obtained there are dual to each other. In Section 4, as a further application of the duality formulas, we provide multiple- and double-angle formulas for GTFs and GHFs. Although some formulas have already been obtained in previous studies (cf. [12] and Table 1 in Section 4), we give formulas for parameters for which no formulas were previously known.

2. Preparation

In this section, we summarize the definitions and some properties of GTFs and GHFs (see [8] for more details). The relationship between GTFs and GHFs can be seen by making the range of parameters in the functions wider than the conventional definition.

Let us assume

$$\frac{q}{q+1} < p < \infty, \quad 0 < q < \infty, \tag{2.1}$$

and

$$F_{p,q}(y) := \int_0^y \frac{dt}{(1-t^q)^{1/p}}, \quad y \in [0,1).$$

We will denote by $\sin_{p,q}$ the inverse function of $F_{p,q}$, i.e.,

$$\sin_{p,q} x := F_{p,q}^{-1}(x).$$

Clearly, $\sin_{p,q} x$ is monotone increasing on $[0,\pi_{p,q}/2)$ onto $[0,1)$, where

$$\begin{aligned}\pi_{p,q} : &= 2F_{p,q}(1) = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}} \\ &= \begin{cases} (2/q)B(1-1/p,1/q), & 1 < p < \infty, \\ \infty, & q/(q+1) < p \leqslant 1, \end{cases}\end{aligned}$$

and B is the beta function. In almost all literature dealing with GTFs, the parameters $p,\ q$ are assumed to be $p,\ q > 1$, but we here allow them to be $p,\ q \leqslant 1$. Note that the condition $q/(q+1) < p \leqslant 1$ implies that $\sin_{p,q}$ is monotone increasing on the *infinite* interval $[0,\infty)$ and no longer similar to $\sin x$, but to $\tanh x$ (Figure 1).

Since $\sin_{p,q} \in C^1(0,\pi_{p,q}/2)$, we also define $\cos_{p,q}$ by

$$\cos_{p,q} x := \frac{d}{dx}(\sin_{p,q} x).$$

Then, it follows that

$$\cos_{p,q}^p x + \sin_{p,q}^q x = 1. \tag{2.2}$$

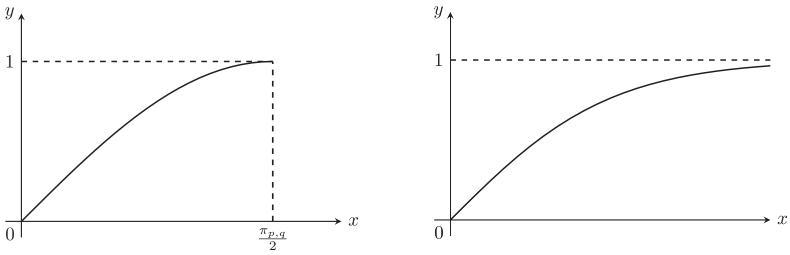

Figure 1: *The graphs of* $\sin_{p,q} x$*:* $p > 1$ *(left) and* $q/(q+1) < p \leqslant 1$ *(right).*

In case $(p,q) = (2,2)$, it is obvious that $\sin_{p,q}$, $\cos_{p,q}$ and $\pi_{p,q}$ are reduced to the ordinary sin, cos and π, respectively. Therefore these functions and the constant are called *generalized trigonometric functions* (GTFs) and the *generalized* π, respectively. It is easy to check that $u = \sin_{p,q} x$ is a solution of the initial value problem of p-Laplacian

$$(|u'|^{p-2}u')' + \frac{(p-1)q}{p}|u|^{q-2}u = 0, \quad u(0) = 0,\ u'(0) = 1, \tag{2.3}$$

which is closely related to the eigenvalue problem of p-Laplacian.

In a similar way, we assume (2.1) and

$$G_{p,q}(y) := \int_0^y \frac{dt}{(1+t^q)^{1/p}}, \quad y \in [0,\infty).$$

We will denote by $\sinh_{p,q}$ the inverse function of $G_{p,q}$, i.e.,

$$\sinh_{p,q} x := G_{p,q}^{-1}(x).$$

Clearly, $\sinh_{p,q}$ is monotone increasing on $[0, \pi_{r,q}/2)$ onto $[0,\infty)$, where r is the positive constant determined by

$$\frac{1}{p} + \frac{1}{r} = 1 + \frac{1}{q}, \quad \text{i.e.,} \quad r = \frac{pq}{pq+p-q}. \tag{2.4}$$

Indeed, by $1 + t^q = 1/(1-s^q)$,

$$\lim_{y\to\infty} G_{p,q}(y) = \int_0^\infty \frac{dt}{(1+t^q)^{1/p}} = \int_0^1 \frac{ds}{(1-s^q)^{1/r}} = \frac{\pi_{r,q}}{2}.$$

The important point to note here is that for a fixed $q \in (0,\infty)$, if $r = r_q(p)$ is regarded as a function of p, then

$$r_q \text{ is bijective from } (q/(q+1),\infty) \text{ to itself, and} \tag{2.5}$$

$$r_q(r_q(p)) = p. \tag{2.6}$$

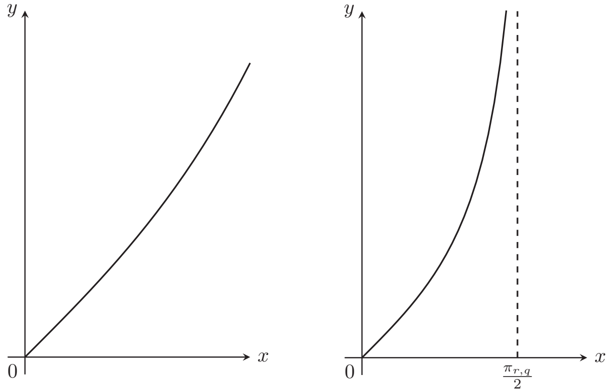

Figure 2: *The graphs of* $\sinh_{p,q} x$*:* $r \leqslant 1$ *(left) and* $r > 1$ *(right).*

In particular, $\pi_{r,q}$ has been defined under (2.1). If $r > 1$, i.e., $p < q$, then $\sinh_{p,q}$ is defined in the *bounded* interval $[0, \pi_{r,q}/2)$ with $\lim_{x \to \pi_{r,q}/2} \sinh_{p,q} x = \infty$ and no longer similar to $\sinh x$, but to $\tan x$ (Figure 2).

Since $\sinh_{p,q} \in C^1(0, \pi_{r,q}/2)$, we also define $\cosh_{p,q}$ by

$$\cosh_{p,q} x := \frac{d}{dx}(\sinh_{p,q} x).$$

Then, it follows that

$$\cosh_{p,q}^p x - \sinh_{p,q}^q x = 1. \tag{2.7}$$

In case $(p,q) = (2,2)$, it is obvious that $\sinh_{p,q}$, $\cosh_{p,q}$ and the interval $[0, \pi_{r,q}/2)$ are reduced to sinh, cosh and $[0, \infty)$, respectively. Therefore these functions are called *generalized hyperbolic functions* (GHFs). Just as $\sin_{p,q} x$ satisfies (2.3), $u = \sinh_{p,q} x$ is a solution of the initial value problem of p-Laplacian

$$(|u'|^{p-2}u')' - \frac{(p-1)q}{p}|u|^{q-2}u = 0, \quad u(0) = 0,\ u'(0) = 1.$$

We generalize the tangent and hyperbolic tangent functions in two ways. These functions are often generalized as

$$\tan_{p,q} x := \frac{\sin_{p,q} x}{\cos_{p,q} x}, \quad \tanh_{p,q} x := \frac{\sinh_{p,q} x}{\cosh_{p,q} x}$$

(e.g., [3, 4, 5, 6, 7, 9, 14, 15]). However, for practical purposes, the following modified functions are more convenient than the functions above:

$$\operatorname{tam}_{p,q} x := \frac{\sin_{p,q} x}{\cos_{p,q}^{p/q} x}, \quad \operatorname{tamh}_{p,q} x := \frac{\sinh_{p,q} x}{\cosh_{p,q}^{p/q} x}.$$

These modified functions were first introduced in [8, 12] with the symbols $\tau_{p,q}$, $\tilde{\tau}_{p,q}$, respectively. Note that if $p=q$, then $\mathrm{tam}_{p,q}x=\tan_{p,q}x$ and $\mathrm{tamh}_{p,q}x=\tanh_{p,q}x$.

In [8], we proved the following duality properties between GTFs and GHFs. This property remains important in the present paper.

THEOREM 2.1. [8] *Let p and q satisfy (2.1) and r be the positive number defined as (2.4). Then, for $x\in[0,\pi_{p,q}/2)$,*

$$\sin_{p,q}x=\frac{\sinh_{r,q}x}{\cosh_{r,q}^{r/q}x}=\mathrm{tamh}_{r,q}x,$$
$$\cos_{p,q}x=\frac{1}{\cosh_{r,q}^{r/p}x},$$
$$\mathrm{tam}_{p,q}x=\sinh_{r,q}x.$$

THEOREM 2.2. [8] *Let p and q satisfy (2.1) and r be the positive number defined as (2.4). Then, for $x\in[0,\pi_{r,q}/2)$,*

$$\sinh_{p,q}x=\frac{\sin_{r,q}x}{\cos_{r,q}^{r/q}x}=\mathrm{tam}_{r,q}x,$$
$$\cosh_{p,q}x=\frac{1}{\cos_{r,q}^{r/p}x},$$
$$\mathrm{tamh}_{p,q}x=\sin_{r,q}x.$$

REMARK 2.3. In [8], we have supposed q to be $1<q<\infty$. However, the proofs therein are perfectly valid in the case $0<q<\infty$ as well. The same is true for Theorem 2.4 below.

Theorems 2.1 and 2.2 tell us the counterparts to GHFs of the properties already known for GTFs, and vice versa. For example, Theorem 2.1 immediately converts (2.2) into (2.7) (with p replaced by r); that is,

$$\cos_{p,q}^{p}x+\sin_{p,q}^{q}x=1 \tag{2.8}$$

into

$$\cosh_{r,q}^{r}x-\sinh_{r,q}^{q}x=1; \tag{2.9}$$

and Theorem 2.2 (with (2.6)) does vice versa. Hence, it follows from (2.5) that (2.8) and (2.9) correspond one-to-one. In this sense, we say that inequalities (2.8) and (2.9) (i.e., (2.7) with p replaced by r) are *dual* to each other.

Moreover, using our theorems, the authors [8, Theorem 1.3] have shown the generalization of Mitrinović-Adamović inequalities. The generalized inequalities will be applied in the next section, and are discussed now. The classical Mitrinović-Adamović inequalities are as follows:

$$\cos^{1/3} x < \frac{\sin x}{x} < 1, \quad x \in \left(0, \frac{\pi}{2}\right),$$
$$\cosh^{1/3} x < \frac{\sinh x}{x} < \cosh x, \quad x \in (0, \infty).$$

The latter is also called the Lazarević inequality. Klén et al. [5, Theorems 3.6 and 3.8] extend these inequalities to the one-parameter case: for $p \in (1, \infty)$,

$$\cos_p^{1/(p+1)} x < \frac{\sin_p x}{x} < 1, \quad x \in \left(0, \frac{\pi_p}{2}\right),$$
$$\cosh_p^{1/(p+1)} x < \frac{\sinh_p x}{x} < \cosh_p x, \quad x \in (0, \infty),$$

where $\sin_p x := \sin_{p,p} x$ and the other functions are defined in the same way. Moreover, Ma et al. [7, Lemma 2] obtain the inequalities for the two-parameter case: for $p,\ q \in (1, \infty)$,

$$\cos_{p,q}^{1/(q+1)} x < \frac{\sin_{p,q} x}{x} < 1, \quad x \in \left(0, \frac{\pi_{p,q}}{2}\right), \tag{2.10}$$
$$\cosh_{p,q}^{1/(q+1)} x < \frac{\sinh_{p,q} x}{x} < \cosh_{p,q} x \quad \text{for appropriate } x. \tag{2.11}$$

The proofs of Klén et al. [5] and Ma et al. [7] are similar, and both prove the inequalities for the trigonometric case and the hyperbolic case separately in the same way.

However, (2.10) and (2.11) (with p replaced by r) are not dual to each other. A dual pair of Mitrinović-Adamović-type inequalities is as follows:

THEOREM 2.4. (Mitrinović-Adamović-type inequalities with duality, [8]) *Let p and q satisfy* (2.1) *and r be the positive number defined as* (2.4). *Then,*

$$\cos_{p,q}^{1/(q+1)} x < \frac{\sin_{p,q} x}{x} < 1, \quad x \in \left(0, \frac{\pi_{p,q}}{2}\right), \tag{2.12}$$
$$\cosh_{p,q}^{1/(q+1)} x < \frac{\sinh_{p,q} x}{x} < \cosh_{p,q}^{p/q} x, \quad x \in \left(0, \frac{\pi_{r,q}}{2}\right). \tag{2.13}$$

Moreover, (2.12) *and* (2.13) *(with p replaced by r) are dual to each other.*

REMARK 2.5. If $p = q$, then (2.12) and (2.13) are equal to (2.10) and (2.11); hence, to the one-parameter ones above.

In our approach in [8], Theorem 2.4 allows us to obtain the inequalities over the wider range (2.1) of parameters, and (2.13) immediately follows from (2.12) by Theorem 2.1.

3. Dual pairs of inequalities

In this section, we generalize the Wilker, Huygens, and (relaxed) Cusa-Huygens inequalities for GTFs and GHFs to a form with duality using our duality formulas (Theorems 2.1 and 2.2), just as we generalized the Mitrinović-Adamović inequalities as Theorem 2.4.

3.1. Wilker-type inequalities

The classical Wilker inequalities are as follows:

$$\left(\frac{\sin x}{x}\right)^2+\frac{\tan x}{x}>2, \quad x\in\left(0,\frac{\pi}{2}\right),$$
$$\left(\frac{\sinh x}{x}\right)^2+\frac{\tanh x}{x}>2, \quad x\in(0,\infty).$$

Klén et al. [5, Corollary 3.19] and Yin et al. [14, Theorem 3.1] extend these inequalities to the one-parameter case: for $p\in(1,\infty)$,

$$\left(\frac{\sin_p x}{x}\right)^p+\frac{\tan_p x}{x}>2, \quad x\in\left(0,\frac{\pi_p}{2}\right),$$
$$\left(\frac{\sinh_p x}{x}\right)^p+\frac{\tanh_p x}{x}>2, \quad x\in(0,\infty).$$

Moreover, Neuman [9, Corollary 6.3 (6.13)] obtains the inequalities for the two-parameter case: for $p,\ q\in(1,\infty)$,

$$\left(\frac{\sin_{p,q} x}{x}\right)^q+\frac{\tan_{p,q} x}{x}>2, \quad x\in\left(0,\frac{\pi_{p,q}}{2}\right), \tag{3.1}$$
$$\left(\frac{\sinh_{p,q} x}{x}\right)^q+\frac{\tanh_{p,q} x}{x}>2 \quad \text{for appropriate } x. \tag{3.2}$$

However, (3.1) and (3.2) (with p replaced by r) are not dual to each other. A dual pair of Wilker-type inequalities is as follows:

THEOREM 3.1. (Wilker-type inequalities with duality) *Let p and q satisfy (2.1) and r be the positive number defined as (2.4). Then,*

$$\left(\frac{\sin_{p,q} x}{x}\right)^p+\left(\frac{\operatorname{tam}_{p,q} x}{x}\right)^r>2, \quad x\in\left(0,\frac{\pi_{p,q}}{2}\right), \tag{3.3}$$
$$\left(\frac{\sinh_{p,q} x}{x}\right)^p+\left(\frac{\operatorname{tamh}_{p,q} x}{x}\right)^r>2, \quad x\in\left(0,\frac{\pi_{r,q}}{2}\right). \tag{3.4}$$

Moreover, (3.3) and (3.4) (with p replaced by r) are dual to each other.

REMARK 3.2. If $p = q$, then (3.3) and (3.4) are equal to (3.1) and (3.2); hence, to the one-parameter ones above, respectively.

Proof. We prove (3.3). From the inequality of arithmetic and geometric means and (2.12) in Theorem 2.4, it follows that

$$\begin{aligned}\left(\frac{\sin_{p,q}x}{x}\right)^p+\left(\frac{\tan_{p,q}x}{x}\right)^r &\geqslant 2\left(\frac{\sin_{p,q}x}{x}\right)^{p/2}\left(\frac{\tan_{p,q}x}{x}\right)^{r/2}\\ &=2\left(\frac{\sin_{p,q}x}{x}\right)^{p/2+r/2}\left(\frac{1}{\cos_{p,q}^{p/q}x}\right)^{r/2}\\ &>2\left(\frac{\sin_{p,q}x}{x}\right)^{p/2+r/2}\left(\frac{\sin_{p,q}x}{x}\right)^{-rp(q+1)/(2q)}\\ &=2.\end{aligned}$$

Next we show (3.4). For any $x \in (0, \pi_{p,q}/2)$, we have proved (3.3). Then, Theorem 2.1 gives the dual inequality to (3.3):

$$\left(\frac{\tanh_{r,q}x}{x}\right)^p+\left(\frac{\sinh_{r,q}x}{x}\right)^r>2.$$

Owing to (2.5) and (2.6), this means (3.4). □

3.2. Huygens-type inequalities

The classical Huygens inequalities are as follows:

$$\frac{2\sin x}{x}+\frac{\tan x}{x}>3, \quad x\in\left(0,\frac{\pi}{2}\right),$$
$$\frac{2\sinh x}{x}+\frac{\tanh x}{x}>3, \quad x\in(0,\infty).$$

Klén et al. [5, Theorem 3.16] extend these inequalities to the one-parameter case: for $p \in (1,\infty)$,

$$\frac{p\sin_p x}{x}+\frac{\tan_p x}{x}>p+1, \quad x\in\left(0,\frac{\pi_p}{2}\right),$$
$$\frac{p\sinh_p x}{x}+\frac{\tanh_p x}{x}>p+1, \quad x\in(0,\infty).$$

Moreover, Neuman [9, Corollary 6.3 (6.14)] obtains the inequalities for the two-parameter case: for $p,\ q \in (1,\infty)$,

$$\frac{q\sin_{p,q}x}{x}+\frac{\tan_{p,q}x}{x}>q+1, \quad x\in\left(0,\frac{\pi_{p,q}}{2}\right), \tag{3.5}$$
$$\frac{q\sinh_{p,q}x}{x}+\frac{\tanh_{p,q}x}{x}>q+1 \quad \text{for appropriate } x. \tag{3.6}$$

However, (3.5) and (3.6) (with p replaced by r) are not dual to each other. A dual pair of Huygens-type inequalities is as follows:

THEOREM 3.3. (Huygens-type inequalities with duality) *Let p and q satisfy* (2.1) *and r be the positive number defined as* (2.4). *Then,*

$$\frac{p\sin_{p,q}x}{x}+\frac{r\operatorname{tam}_{p,q}x}{x}>p+r, \quad x\in\left(0,\frac{\pi_{p,q}}{2}\right), \tag{3.7}$$

$$\frac{p\sinh_{p,q}x}{x}+\frac{r\operatorname{tamh}_{p,q}x}{x}>p+r, \quad x\in\left(0,\frac{\pi_{r,q}}{2}\right). \tag{3.8}$$

Moreover, (3.7) *and* (3.8) *(with p replaced by r) are dual to each other.*

REMARK 3.4. If $p=q$, then (3.7) and (3.8) are equal to (3.5) and (3.6); hence, to the one-parameter ones above, respectively.

Proof. Let α, β be

$$\alpha=\frac{p(q+1)}{pq+p-q}=1+\frac{r}{p}, \quad \beta=\frac{p(q+1)}{q}=1+\frac{p}{r}.$$

Since α, $\beta>1$ and $1/\alpha+1/\beta=1$, the following Young inequality holds true for positive numbers A, B:

$$A+B\geqslant(\alpha A)^{1/\alpha}(\beta B)^{1/\beta}.$$

Therefore, by this inequality and (2.12) in Theorem 2.4,

$$\begin{aligned}\frac{p\sin_{p,q}x}{x}+\frac{r\operatorname{tam}_{p,q}x}{x}&\geqslant\left(\alpha\frac{p\sin_{p,q}x}{x}\right)^{1/\alpha}\left(\beta\frac{r\operatorname{tam}_{p,q}x}{x}\right)^{1/\beta}\\&=(\alpha p)^{1/\alpha}(\beta r)^{1/\beta}\frac{\sin_{p,q}x}{x}\left(\frac{1}{\cos_{p,q}^{p/q}x}\right)^{1/\beta}\\&>(p+r)^{1/\alpha}(r+p)^{1/\beta}\frac{\sin_{p,q}x}{x}\left(\frac{\sin_{p,q}x}{x}\right)^{-p(q+1)/(\beta q)}\\&=p+r.\end{aligned}$$

Next we show (3.8). For any $x\in(0,\pi_{p,q}/2)$, we have proved (3.7). Then, Theorem 2.1 gives the dual inequality to (3.7):

$$\frac{p\operatorname{tamh}_{r,q}x}{x}+\frac{r\sinh_{r,q}x}{x}>p+r.$$

Owing to (2.5) and (2.6), this means (3.8). □

3.3. Relaxed Cusa-Huygens-type inequalities

The classical Cusa-Huygens inequalities are as follows:

$$\frac{\sin x}{x}<\frac{2+\cos x}{3}, \quad x\in\left(0,\frac{\pi}{2}\right),$$

$$\frac{\sinh x}{x}<\frac{2+\cosh x}{3}, \quad x\in(0,\infty).$$

Ma et al. [7, Theorems 2 and 3] obtain the inequalities for the two-parameter case: for $p,\ q \in (1,2]$,

$$\frac{\sin_{p,q} x}{x} < \frac{q + \cos_{p,q} x}{q+1}, \quad x \in \left(0, \frac{\pi_{p,q}}{2}\right), \tag{3.9}$$

$$\frac{\sinh_{p,q} x}{x} < \frac{q + \cosh_{p,q} x}{q+1} \quad \text{for appropriate } x. \tag{3.10}$$

The inequalities for the one parameter $p = q \in (1,2]$ are given by Klén et al. [5, Theorems 3.22 and 3.24]. Unfortunately, these generalized inequalities are shown only for $p,\ q \in (1,2]$, and (3.9) and (3.10) (with p replaced by r) are not dual to each other.

We hope to find inequalities that hold for $p,\ q$ satisfying (2.1) and are dual to each other. Therefore, consider the following relaxed inequalities instead of the classical Cusa-Huygens inequalities:

$$\frac{\sin x}{x} < \sqrt{\frac{2+\cos^2 x}{3}}, \quad x \in \left(0, \frac{\pi}{2}\right),$$

$$\frac{\sinh x}{x} < \sqrt{\frac{2+\cosh^2 x}{3}}, \quad x \in (0,\infty).$$

Neuman [9, Theorem 6 (6.7), (6.9)] generalizes the inequalities to the two-parameter case: for $p,\ q \in (1,\infty)$,

$$\frac{\sin_{p,q} x}{x} < \left(\frac{q + \cos_{p,q}^p x}{q+1}\right)^{1/p}, \quad x \in \left(0, \frac{\pi_{p,q}}{2}\right), \tag{3.11}$$

$$\frac{\sinh_{p,q} x}{x} < \left(\frac{q + \cosh_{p,q}^p x}{q+1}\right)^{1/p} \quad \text{for appropriate } x. \tag{3.12}$$

However, (3.11) and (3.12) (with p replaced by r) are not dual to each other. A dual pair of Cusa-Huygens-type inequalities is as follows:

THEOREM 3.5. (Relaxed Cusa-Huygens-type inequalities with duality) *Let p and q satisfy* (2.1) *and r be the positive number defined as* (2.4)*. Then,*

$$\frac{\sin_{p,q} x}{x} < \left(\frac{p + r\cos_{p,q}^p x}{p+r}\right)^{1/q}, \quad x \in \left(0, \frac{\pi_{p,q}}{2}\right), \tag{3.13}$$

$$\frac{\sinh_{p,q} x}{x} < \left(\frac{p + r\cosh_{p,q}^p x}{p+r}\right)^{1/q}, \quad x \in \left(0, \frac{\pi_{r,q}}{2}\right). \tag{3.14}$$

Moreover, (3.13) *and* (3.14) *(with p replaced by r) are dual to each other.*

REMARK 3.6. If $p = q$, then (3.13) and (3.14) are equal to (3.11) and (3.12), respectively.

Proof. We prove

$$\frac{(p+r+rx^q)\sin_{p,q}^q x}{x^q} < p+r,$$

which is equivalent to (3.13). Let $f(x) := p+r-(p+r+rx^q)\sin_{p,q}^q x/x^q$. Then,

$$f'(x) = \frac{q\sin_{p,q}^q x}{x^{q+1}}\left(p+r-\frac{(p+r)x\cos_{p,q}x}{\sin_{p,q}x}-\frac{rx^{q+1}\cos_{p,q}x}{\sin_{p,q}x}\right).$$

From (2.12) in Theorem 2.4, it follows that

$$\frac{x\cos_{p,q}x}{\sin_{p,q}x} < \cos_{p,q}^{q/(q+1)} x, \quad \frac{x^{q+1}\cos_{p,q}x}{\sin_{p,q}x} < 1-\cos_{p,q}^p x.$$

Therefore,

$$f'(x) > \frac{q\sin_{p,q}^q x}{x^{q+1}}\left(p-(p+r)\cos_{p,q}^{q/(q+1)} x + r\cos_{p,q}^p x\right).$$

Now let $g(t) = p-(p+r)t^{q/(q+1)}+rt^p$. Then,

$$g'(t) = prt^{-1/(q+1)}(t^{p-q/(q+1)}-1)$$

Since $q/(q+1) < p$, we see $g'(t) < 0$. Therefore, $g(t) > \lim_{t\to 1-0} g(t) = 0$ and

$$f'(x) > \frac{q\sin_{p,q}^q x}{x^{q+1}} g(\cos_{p,q}x) > 0.$$

Moreover, $f(x) > \lim_{x\to +0} f(x) = 0$, which means (3.13).

Next we show (3.14). For any $x \in (0, \pi_{p,q}/2)$, we have proved (3.13). Then, Theorem 2.1 gives the dual inequality to (3.13):

$$\frac{\sinh_{r,q}x}{x\cosh_{r,q}^{r/q}x} < \left(\frac{p+r\cosh_{r,q}^{-r}x}{p+r}\right)^{1/q};$$

hence,

$$\frac{\sinh_{r,q}x}{x} < \left(\frac{p\cosh_{r,q}^r x+r}{p+r}\right)^{1/q}.$$

Owing to (2.5) and (2.6), this means (3.14). □

4. Multiple- and double-angle formulas

Several multiple- and double-angle formulas for GTFs and GHFs are already known (see [8, Theorems 1.4 and 1.6] and [12, Theorem 1.1] for multiple-angle formulas; Table 1 for double-angle formulas). In this section, we apply the duality formulas (Theorems 2.1 and 2.2) to obtain multiple- and double-angle formulas which are not covered in [8] and [12], for these generalized functions.

The multiple-angle formulas in the following theorem assure that GTFs for $(2q/(2+q), q/2)$ can be represented in terms of GTFs for $(2q/(2+q), q)$. Moreover, the counterparts to GHFs are obtained as their dual inequalities.

Table 1: *The parameters for which the double angle formulas of GTF have been obtained.*

q	$(q/(q-1),2)$	$(2,q)$	$(q/(q-1),q)$
2	$(2,2)$ Abu al-Wafa	$(2,2)$ Abu al-Wafa	$(2,2)$ Abu al-Wafa
3	$(3/2,2)$ Miyakawa-Takeuchi [8]	$(2,3)$ Cox-Shurman [1]	$(3/2,3)$ Dixon [2]
4	$(4/3,2)$ Sato-Takeuchi [10]	$(2,4)$ Fagnano	$(4/3,4)$ Edmunds et al. [4]
6	$(6/5,2)$ Takeuchi [13]	$(2,6)$ Shinohara [11]	$(6/5,6)$ Takeuchi [13]
q	$(2q/(2+q),q/2)$	$(2q/(2+q),q)$	$(q/2,q)$
2	$(1,1)$ Napier	$(1,2)$ V. Riccati	$(1,2)$ V. Riccati
3	$(6/5,3/2)$ **Theorem 4.4**	$(6/5,3)$ Miyakawa-Takeuchi [8]	$(3/2,3)$ Dixon [2]
4	$(4/3,2)$ Sato-Takeuchi [10]	$(4/3,4)$ Edmunds et al. [4]	$(2,4)$ Fagnano
6	$(3/2,3)$ Dixon [2]	$(3/2,6)$ Miyakawa-Takeuchi [8]	$(3,6)$ Miyakawa-Takeuchi [8]

THEOREM 4.1. *Let $0<q<\infty$. Then, for*

$$x \in [0, \pi_{2q/(2+q),q/2}/(2^{2/q+1})) = [0, \pi_{2q/(2+q),q}/2),$$

$$\sin_{2q/(2+q),q/2}(2^{2/q}x) = \frac{2^{2/q}\sin_{2q/(2+q),q}x}{(1+\sin_{2q/(2+q),q}^{q/2}x)^{2/q}},$$

$$\cos_{2q/(2+q),q/2}(2^{2/q}x) = \left(\frac{1-\sin_{2q/(2+q),q}^{q/2}x}{1+\sin_{2q/(2+q),q}^{q/2}x}\right)^{1/q+1/2}.$$

Moreover, for same x,

$$\sinh_{2q/(2+q),q/2}(2^{2/q}x) = 2^{2/q}\sinh_{2,q}x(\cosh_{2,q}x+\sinh_{2,q}^{q/2}x)^{2/q},$$
$$\cosh_{2q/(2+q),q/2}(2^{2/q}x) = (\cosh_{2,q}x+\sinh_{2,q}^{q/2}x)^{2/q+1}.$$

Proof. The former half is shown as follows. Let $y \in [0,\infty)$. Setting $t^q = u^q/(4(1-u^{q/2}))$ in

$$\sinh_{2,q}^{-1} y = \int_0^y \frac{dt}{\sqrt{1+t^q}},$$

we have

$$\begin{aligned}\sinh_{2,q}^{-1} y &= 2^{-2/q-1}\int_0^{2^{2/q}y/(y^{q/2}+\sqrt{y^q+1})^{2/q}} \frac{2(1-u^{q/2})^{1/2}}{2-u^{q/2}} \cdot \frac{2-u^{q/2}}{(1-u^{q/2})^{1/q+1}}\,du \\ &= 2^{-2/q}\int_0^{2^{2/q}y/(y^{q/2}+\sqrt{y^q+1})^{2/q}} \frac{du}{(1-u^{q/2})^{1/q+1/2}};\end{aligned}$$

that is,

$$\sinh_{2,q}^{-1} y = 2^{-2/q} \sin_{2q/(2+q),q/2}^{-1} \left(\frac{2^{2/q} y}{(y^{q/2} + \sqrt{y^q+1})^{2/q}} \right). \tag{4.1}$$

Letting $y \to \infty$ in (4.1) and using $r_q(2) = 2q/(2+q)$, we get

$$\frac{\pi_{2q/(2+q),q}}{2} = \frac{\pi_{2q/(2+q),q/2}}{2^{2/q+1}}.$$

From (4.1), we see that for $x \in [0, \pi_{2q/(2+q),q/2}/(2^{2/q+1})) = [0, \pi_{2q/(2+q),q}/2)$,

$$\sin_{2q/(2+q),q/2}(2^{2/q}x) = \frac{2^{2/q} \sinh_{2,q} x}{(\sinh_{2,q}^{q/2} x + \cosh_{2,q} x)^{2/q}} = \frac{2^{2/q} \operatorname{tamh}_{2,q} x}{(\operatorname{tamh}_{2,q}^{q/2} x + 1)^{2/q}}.$$

Theorem 2.2 with $r_q(2) = 2q/(2+q)$ shows that the right-hand side becomes

$$\frac{2^{2/q} \sin_{2q/(2+q),q} x}{(\sin_{2q/(2+q),q}^{q/2} x + 1)^{2/q}}.$$

The formula of $\cos_{2q/(q+2),q/2}$ immediately follows from (2.2).

The latter half is proved as follows. By Theorem 2.2 with $r_{q/2}(2q/(2+q)) = 2q/(2+q)$ and the former half,

$$\sinh_{2q/(2+q),q/2}(2^{2/q}x) = \frac{\sin_{2q/(2+q),q/2}(2^{2/q}x)}{\cos_{2q/(2+q),q/2}^{4/(2+q)}(2^{2/q}x)} = \frac{2^{2/q} \sin_{2q/(2+q),q} x}{(1 - \sin_{2q/(2+q),q}^{q/2} x)^{2/q}}.$$

Theorem 2.1 with $r_q(2q/(2+q)) = 2$ shows that the right-hand side becomes

$$2^{2/q} \sinh_{2,q} x (\cosh_{2,q} x + \sinh_{2,q}^{q/2} x)^{2/q}.$$

The formula of $\cosh_{2q/(q+2),q/2}$ immediately follows from (2.2). □

REMARK 4.2. If $q = 2$, then the formulas of $\sin_{2q/(2+q),q/2}$ and $\sinh_{2q/(2+q),q/2}$ are

$$1 - e^{-2x} = \frac{2 \tanh x}{1 + \tanh x},$$
$$e^{2x} - 1 = 2 \sinh x (\cosh x + \sinh x).$$

The following double-angle formula is proved by [8, Theorem 3.8].

LEMMA 4.3. [8] *For* $x \in [0, \pi_{6/5,3}/4)$,

$$\sin_{6/5,3}(2x) = \frac{4 \cos_{6/5,3}^{1/5} x (3 \cos_{6/5,3}^{3/5} x + 1)(1 - \cos_{6/5,3}^{3/5} x)^{1/3}}{\left(16 \cos_{6/5,3}^{3/5} x + (3 \cos_{6/5,3}^{3/5} x + 1)^3 (1 - \cos_{6/5,3}^{3/5} x)\right)^{2/3}}.$$

Now, we are in a position to show the double-angle formula of $\sin_{6/5,3/2}$.

THEOREM 4.4. *For* $x \in [0, \pi_{6/5,3/2}/4)$,

$$\sin_{6/5,3/2}(2x) = (\Theta \circ \Phi \circ \Psi)(\cos_{6/5,3/2} x),$$

where

$$\Theta(x) = \left(\frac{2x}{1+x}\right)^{2/3},$$
$$\Phi(x) = \frac{8\sqrt{x(3x+1)^3(1-x)}}{16x+(3x+1)^3(1-x)},$$
$$\Psi(x) = \frac{2x^{3/5}}{1+x^{6/5}}.$$

Proof. From Theorem 4.1 with $q = 3$, for $x \in [0, \pi_{6/5,3/2}/(2^{5/3})) = [0, \pi_{6/5,3}/2)$,

$$\sin_{6/5,3/2}(2^{2/3}x) = \frac{2^{2/3}\sin_{6/5,3} x}{(1+\sin_{6/5,3}^{3/2} x)^{2/3}} = \Theta(\sin_{6/5,3}^{3/2} x);$$

hence,

$$\sin_{6/5,3}^{3/2} x = \Theta^{-1}(\sin_{6/5,3/2}(2^{2/3}x)) = \frac{\sin_{6/5,3/2}^{3/2}(2^{2/3}x)}{2-\sin_{6/5,3/2}^{3/2}(2^{2/3}x)}.$$

Thus, from (2.2),

$$\cos_{6/5,3}^{3/5} x = \frac{2\cos_{6/5,3/2}^{3/5}(2^{2/3}x)}{1+\cos_{6/5,3/2}^{6/5}(2^{2/3}x)} = \Psi(\cos_{6/5,3/2}(2^{2/3}x)). \tag{4.2}$$

Now, let $x \in [0, \pi_{6/5,3/2}/4)$ and $y := x/(2^{2/3})$. It follows from Theorem 4.1 with $q = 3$ that since $2y \in [0, \pi_{6/5,3/2}/(2^{5/3})) = [0, \pi_{6/5,3}/2)$, we get

$$\sin_{6/5,3/2}(2x) = \sin_{6/5,3/2}(2^{2/3}\cdot 2y) = \Theta(\sin_{6/5,3}^{3/2}(2y)). \tag{4.3}$$

Here, Lemma 4.3 and (4.2) yield

$$\sin_{6/5,3}^{3/2}(2y) = \Phi(\cos_{6/5,3}^{3/5} y) = \Phi(\Psi(\cos_{6/5,3/2}(2^{2/3}y))) = \Phi(\Psi(\cos_{6/5,3/2} x)).$$

Therefore, from (4.3), we have

$$\sin_{6/5,3/2}(2x) = \Theta(\Phi(\Psi(\cos_{6/5,3/2} x))) = (\Theta \circ \Phi \circ \Psi)(\cos_{6/5,3/2} x).$$

The proof is completed. □

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