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# $q$–LITTLEWOOD–PALEY $g$–FUNCTION

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*Dedicated to the Memory of Professor Ahmed Fitouhi, passed away on January 18, 2020.*

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*Abstract.* The purpose of this paper is to define and study, by the virtue of the $q$-Hardy-Littlewood maximal function $\mathscr{M}_q(f)$ the so called $L^p$-boundedness of the $q$-Littlewood-Paley $g$-function when $p \in (1,2]$.

## 1. Introduction

The theory of Littlewood-Paley was developed by Stein in his book [19], which remains the best reference in the study of this topic and has been an important impact in harmonic analysis. It plays an important role in the study of many functional spaces like the Hardy space, Lipshitz space, and BMO spaces. We point out that many authors have studied the Littlewood-Paley $g$-function, for instance one can cite [1, 2, 18, 20].

The usual Littlewood-Paley $g$-function is defined in the $n$-dimensional Euclidean space $\mathbb{R}^n$ according to [19] by:

$$g(f)(x) := \left( \int_0^\infty \left| \nabla P^t f(x) \right|^2 t dt \right)^{\frac{1}{2}},$$

where $(P^t)_{t>0}$ is the usual Poisson semigroup defined by:

$$P^t f(x) := \frac{\Gamma(\frac{n+1}{2})}{\pi^{\frac{n+1}{2}}} \int_{\mathbb{R}^n} \frac{t f(y)}{(t^2 + \mid x - y \mid^2)^{\frac{n+1}{2}}} dy$$

and $\nabla := \left( \frac{\partial}{\partial x_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial t} \right)$ is the gradient.

The well-known results is that the mapping $f \longmapsto g(f)$ is bounded from the Lebesgue space $L^p(\mathbb{R}^n, dx)$, $p \in (1, \infty)$ into itself.

The aim of this work is to define and study the $g$-function using many intermediary results in "Quantum calculus" or $q$-analogs, where the parameter $q$ is supposed to be a number from the interval $(0,1)$. Our interest in this paper is to prove one of well-known results:

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MAIN THEOREM. *For $p\in(1,2]$, there exit two constants $A_{p,q}>0$ and $B_{p,q}>0$ such that for $f\in L^p(\mathbb{R}_{q,+})$*

$$B_{p,q}\parallel f\parallel_{L^p(\mathbb{R}_{q,+})}\leqslant\parallel g(f)(x;q^2)\parallel_{L^p(\mathbb{R}_{q,+})}\leqslant A_{p,q}\parallel f\parallel_{L^p(\mathbb{R}_{q,+})}.$$

*This theorem will be proved in the last section of this paper. The techniques used are inspired in major part of the very interesting book of Stein* [19].

This work is organized as follows. In the second section, we recall some $q$-harmonic analysis results related to $q$-calculus. In the third section we will define and study the Poisson kernel and Poisson integral and we present some technical lemmas that will be useful for the proof of the main result of this manuscript. The last section will be devoted, by the virtue of the $q$-Hardy-Littlewood maximal function $\mathscr{M}_q(f)$ to study the so called $L^p$-boundedness of the $q$-Littlewood-Paley $g$-function when $p\in(1,2]$.

## 2. Preliminaries

The aim of this section is to introduce some notions of functions theory in the $q$-calculus. For $a\in\mathbb{C}$, the $q$-shifted factorial $(a;q)_k$ is defined as a product of $k$ factors

$$(a;q)_0=1,\ (a;q)_k=(1-a)(1-aq)\dots(1-aq^{k-1}),\quad k=1,2,\dots$$

This definition remains meaningful for $k=\infty$ as a convergent infinite product

$$(a;q)_\infty=\prod_{k=0}^{\infty}(1-aq^k).$$

We also write $(a_1,\dots,a_r;q)_k$ for the product of $r$ $q$-shifted factorials

$$(a_1,\cdots,a_r;q)_k=(a_1;q)_k\dots(a_r;q)_k,\quad k=1,2,\dots,\infty.$$

A $q$-hypergeometric series is a power series (for the moment still formal) in one complex variable $z$ with power series coefficients which depend, apart from $q$, on $r$ complex upper parameters $a_1,\dots,a_r$ and $s$ complex lower parameters $b_1,\dots,b_s$ as

$${}_r\phi_s(a_1,\cdots,a_r;b_1,\cdots,b_s;q,x):=\sum_{k=0}^{\infty}\frac{(a_1,\cdots,a_r;q)_k}{(b_1,\cdots,b_s;q)_k(q;q)_k}[(-1)^kq^{\frac{k(k-1)}{2}}]^{1+s-r}x^k,$$

where $r,s=1,2,\dots.$

The $q$-derivative of a function $f$ given on a subset of $\mathbb{R}$ or $\mathbb{C}$ is defined by

$$D_{q,x}f(x):=\frac{f(x)-f(qx)}{(1-q)x},\quad x,q\neq 0,$$

where $x$ and $qx$ should be in the domain of $f$. By continuity we set $(D_qf)(0)=f'(0)$ provided $f'(0)$ exists. For $k=0,1,2,\dots$

$$D^k_{q,x}f(x)=\frac{(-1)^k}{x^k(1-q)^k}\sum_{i=0}^{k}(-1)^i\frac{(q;q)_k}{(q;q)_i(q;q)_{k-i}}q^{-(k-i)(k-i-1)/2}f(q^{k-i}x).\qquad(1)$$

Moreover, for all $n \in \mathbb{N}$,

$$D_{q,x}(f^n(x)) = \frac{f^n(x) - f^n(qx)}{f(x) - f(qx)} D_{q,x}f(x) = \Big[\sum_{k=0}^{n-1} f^k(x) f^{-k}(qx)\Big] f^{n-1}(qx) D_{q,x}f(x). \quad (2)$$

$$\begin{aligned} D^2_{q,x}(f^n(x)) = q\Big[&\sum_{k=0}^{n-1}\sum_{i=0}^{k-1} f^i(x) f^{-i}(qx). \Big(D_{q,x}f(qx)\Big)\Big(D_{q,x}f(x)\Big) \\ &+ q\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-2} f^i(q^2x) f^{-i}(qx) \\ &\times \Big(D_{q,x}f(qx)\Big)^2\Big] f^{n-2}(qx) + \Big[\sum_{k=0}^{n-1} f^k(x) f^{-k}(qx)\Big] D^2_{q,x}f(x) f^{n-1}(qx). \quad (3)\end{aligned}$$

Note that when $q \uparrow 1^-$, the equation (2) tends to $nf^{n-1}(x)f'(x)$ and the equation (3) to $nf^{n-1}(x)f''(x) + n(n-1)f^{n-2}(x)f'(x)$.

We begin by putting

$$\mathbb{R}_q = \{\pm q^k, k \in \mathbb{Z}\}, \quad \mathbb{R}_{q,+} = \{q^k, k \in \mathbb{Z}\}, \quad \widetilde{\mathbb{R}}_{q,+} = \{q^k, k \in \mathbb{Z}\} \cup \{0\},$$

and $\Delta_{q,x} := \Lambda^{-1}_{q,x} D^2_{q,x}$, where the $q$-shift operators is $(\Lambda^{-1}_{q,x} f)(x) := f(q^{-1}x)$ and

$$\Delta_q := \Delta_{q,x} + \Delta_{q,t}, (x,t) \in \mathbb{R}_q \times \mathbb{R}_{q,+}. \quad (4)$$

For $a > 0$ and a function $f$ given on $(0,a]$ we define the $q$-integral by

$$\int_0^a f(x) d_q x := (1-q) a \sum_{n=0}^{\infty} f(aq^n) q^n.$$

The improper integral is defined in the following way

$$\int_0^{\infty} f(x) d_q x := (1-q) \sum_{k=-\infty}^{+\infty} f\left(q^k\right) q^k. \quad (5)$$

Note that for $n \in \mathbb{Z}$, we have

$$\int_0^{\infty} f(q^n x) d_q x = \frac{1}{q^n} \int_0^{\infty} f(x) d_q x, \qquad \int_0^{a} f(q^n x) d_q x = \frac{1}{q^n} \int_0^{aq^n} f(x) d_q x,$$

and if $f$ and $g$ are two suitable functions, the $q$-integration by parts is given by

$$\int_a^b D_{q,x}f(x) g(x) d_q x = \Big[f(x)g(x)\Big]_a^b - \int_a^b f(qx) D_{q,x}g(x) d_q x. \quad (6)$$

We denote by $\mu$ the measure on $\mathbb{R}_{q,+}$ given by

$$\mathrm{d}_q\mu(y) = \left(\frac{1+q}{1-q}\right)^{-1/2} \Gamma^{-1}_{q^2}(1/2) \mathrm{d}_q y = c_q \mathrm{d}_q y \quad (7)$$

where the $q$-gamma function $\Gamma_{q^2}$ (see [10, 11], Section 1.3.) is defined by

$$\Gamma_q(z) := \frac{(q;q)_\infty}{(q^z;q)_\infty}(1-q)^{1-z}, \quad q \in (0,1), \quad z \neq 0,-1,-2,\ldots$$

Let us now introduce some $q$-functional spaces which one will need in this work.

- ▷ $\mathscr{D}_{*,q}(\mathbb{R}_q)$ the space of even functions infinitely $q$-differentiable on $\mathbb{R}_q$ with compact support in $\mathbb{R}_q$. We equip this space with the topology of the uniform convergence of the functions and their $q$-derivatives.
- ▷ $\mathscr{C}_{*,q,0}(\mathbb{R}_q)$ the space of even functions $f$ defined on $\mathbb{R}_q$ continuous at 0, and satisfying
$$\lim_{x\to\infty} f(x) = 0 \text{ and } \| f \|_{\mathscr{C}_{*,q,0}} := \sup_{x\in\mathbb{R}_q} |f(x)| < +\infty.$$
- ▷ $L^p(\mathbb{R}_{q,+})$, $p \in [1,+\infty]$, the space of functions $f$ such that $\| f \|_{L^p(\mathbb{R}_{q,+})} < +\infty$, where
$$\| f \|_{L^p(\mathbb{R}_{q,+})} := \left[\int_0^\infty |f(x)|^p \mathrm{d}_q\mu(x)\right]^{\frac{1}{p}}, \text{ for } p < \infty,$$
where $\mathrm{d}_q\mu(x)$ is given by (7) and for $p = \infty$
$$\| f \|_{L^\infty(\mathbb{R}_{q,+})} := \sup_{x\in\mathbb{R}_{q,+}} | f(x) | .$$

Noting that

$$\| f \|_{L^p(\mathbb{R}_{q,+})} = \sup_{\{h\in\mathscr{D}_{*,q}(\mathbb{R}_q);\|h\|_{L^m(\mathbb{R}_{q,+})}=1\}} \left|\int_0^\infty f(x)h(x)\mathrm{d}_q\mu(x)\right|, \quad 1/p+1/m = 1. \tag{8}$$

### 2.1. One-parameter family of $q$-exponential functions

The one-parameter family of $q$-exponential functions with $\alpha \in \mathbb{R}$ has been considered in [9]

$$E_q^{(\alpha)}(x) := \sum_{n=0}^\infty q^{\alpha n^2/4} \frac{(1-q)^n}{(q;q)_n} x^n, \quad x \in \mathbb{R}. \tag{9}$$

Two particular cases of this family with $\alpha = 0$ and $\alpha = 1$ are well known: they are the $q$-exponential

$$e_q(x) = E_q^{(0)}(x) := \frac{1}{((1-q)x;q)_\infty} = \sum_{n=0}^\infty \frac{(1-q)^n}{(q;q)_n} x^n,$$

and its reciprocal

$$E_q(x) = e_q^{-1}(x) = E_q^{(1)}(-q^{-1/2}x) := (-(1-q)x;q)_\infty = \sum_{n=0}^\infty q^{n(n-1)/2} \frac{(1-q)^n}{(q;q)_n} x^n,$$

respectively. Another particular example of (9) corresponds to the value $\alpha = \frac{1}{2}$ and is

$$\mathscr{E}_q(x) = E_q^{(1/2)}(x) := \sum_{n=0}^{\infty} q^{n^2/2} \frac{(1-q)^n}{(q;q)_n} x^n. \tag{10}$$

Note that

$$D_{q,x} E_q^{(\alpha)}(x) = q^{\alpha/2} E^{\alpha}(q^{\alpha} x), \tag{11}$$

and

$$\lim_{n\to\infty} \mathscr{E}_q(q^{-n}) = \infty, \quad \lim_{n\to\infty} \mathscr{E}_q(-q^{-n}) = \lim_{n\to\infty} \mathscr{E}_q(q^n) = 0. \tag{12}$$

### 2.2. $q$-even translation and $q$-cosine Fourier transform

Let $f$ be a function in $L^1(\mathbb{R}_{q,+})$, the $q$-even translation operators $T_{q,x}$ [7] are defined by

$$T_{q,x} f(y) := \int_0^{\infty} f(z) D_q(x,y,z) \mathrm{d}_q \mu(z),$$

where $D_q(x,y,z)$ is defined for $x$ and $y$ in $\mathbb{R}_{q,+}$ by

$$D_q(x,y,z) := \int_0^{\infty} \cos(xt;q^2)\cos(yt;q^2)\cos(zt;q^2)\mathrm{d}_q\mu(t),$$

and the $q$-cosine function is given in [12] as a series of functions

$$\cos(x;q^2) := {}_1\phi_1(0;q;q^2,q^2x^2) = \sum_{n=0}^{\infty} (-1)^n \frac{q^{n(n+1)}}{(q;q)_{2n}} x^{2n} = \sum_{n=0}^{\infty} (-1)^n b_n(x;q^2),$$

and we have $\lim_{x\to\infty} \cos(x;q^2) = 0$, for $x = q^{1-n}$, $n \to \infty$.

On $\mathbb{R}_q$, these functions are bounded and there they satisfy $\left|\cos(x;q^2)\right| \leqslant \frac{1}{(q;q^2)_\infty^2}$.

In [4], for $f$ be an even function defined on $\mathbb{R}_q$ such that $x \mapsto D_{q,x}^k f(x)$ is continuous at 0 for all $k = 1, 2, \ldots$, the authors proved that the $q$-even translation $T_{q,x}$ can be written in the following form

$$T_{q,y} f(x) = \sum_{n=0}^{\infty} b_n(y;q^2)(1-q)^{2n} q^{-2n} \Delta_{q,x}^n f(x). \tag{13}$$

where

$$\Delta_{q,x}^n f(x) = \frac{q^{(2-n)n}(q;q)_{2n}}{(1-q)^{2n}} \sum_{k=-n}^{n} (-1)^{n-k} \frac{q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}} f(q^k x). \tag{14}$$

The $q$-cosine Fourier transform $\mathscr{F}_q$ and the $q$-convolution product are defined for suitable functions $f, g$ as follows

$$\mathscr{F}_q(f)(\lambda) = \int_0^{\infty} f(t)\cos(\lambda t;q^2)\mathrm{d}_q\mu(t), \tag{15}$$

$$f *_q g(x) = \int_0^{\infty} T_{q,x} f(y) g(y) d_q\mu(y).$$

Note that, the $q$-translation operators and $q$-cosine Fourier transform satisfy the following properties

$(P_1)$ *$L^1 - L^\infty$-boundedness* ([5], Proposition 5.1). For all $f \in L^1(\mathbb{R}_{q,+})$, $\mathscr{F}_q(f) \in L^\infty(\mathbb{R}_{q,+})$ and

$$\|\mathscr{F}_q(f)\|_{L^\infty(\mathbb{R}_{q,+})} \leqslant \frac{1}{(q;q^2)^2_\infty} \|f\|_{L^1(\mathbb{R}_{q,+})}.$$

$(P_2)$ *Inversion theorem* ([12]). Let $f \in L^1(\mathbb{R}_{q,+})$, such that $\mathscr{F}_q(f) \in L^1(\mathbb{R}_{q,+})$. Then

$$f(x) = \mathscr{F}_q(\mathscr{F}_q(f))(x), \quad x \in \mathbb{R}_{q,+}.$$

$(P_3)$ *Plancherel theorem* ([8], Theorem 7.7.). The $q$-cosine Fourier transform $\mathscr{F}_q$ extends uniquely to an isometric isomorphism of $L^2(\mathbb{R}_{q,+})$ onto itself. In particular,

$$\|\mathscr{F}_q(f)\|_{L^2(\mathbb{R}_{q,+})} = \|f\|_{L^2(\mathbb{R}_{q,+})}.$$

$(P_4)$ *$q$-Gaussian function* ([6], Proposition 6.2). The function $G(x,t;q^2)$ given by

$$G(x,t;q^2) := \frac{1}{A(t,q^2)} e_{q^2}^{-\frac{x^2}{q(1+q)t}}$$

where $A(t,q^2) = q^{-\frac{1}{2}}(1-q)^{\frac{1}{2}} \frac{(-\frac{1-q}{1+q}\frac{1}{t}, -\frac{1+q}{1-q}q^2t;q^2)_\infty}{(-\frac{1-q}{1+q}\frac{1}{qt}, -\frac{1+q}{1-q}q^3t;q^2)_\infty}$, $t > 0$, satisfies

$$\mathscr{F}_q(G(.,t;q^2))(\lambda) = e_{q^2}(-t\lambda^2).$$

$(P_5)$ *$q$-Young condition* [4]. Let $p,q,r \in [1,\infty]$ satisfy $1/p + 1/q - 1/r = 1$. Then the map $(f,g) \mapsto f *_q g$ extends to a continuous map from $L^p(\mathbb{R}_{q,+}) \times L^q(\mathbb{R}_{q,+})$ to $L^r(\mathbb{R}_{q,+})$ and we have

$$\|f *_q g\|_{L^r(\mathbb{R}_{q,+})} \leqslant \|f\|_{L^p(\mathbb{R}_{q,+})} \|g\|_{L^q(\mathbb{R}_{q,+})}$$

Specially, we need the positivity of the $q$-even translation operator [7] for proving the following inequality for $f \in L^1(\mathbb{R}_{q,+})$,

$$\|T_{q,x} f\|_{L^1(\mathbb{R}_{q,+})} \leqslant \|f\|_{L^1(\mathbb{R}_{q,+})}.$$

This positivity property holds if $q \in (0, q_0]$, where $q_0$ is the first zero of the function $q \mapsto {}_1\varphi_1(0;q;q,q)$.

## 3. $q$-Poisson kernel and $q$-Poisson integral

DEFINITION 1. [16] Consider the functions $P_t(x;q^2)$ called Poisson kernel given for $t \in \mathbb{R}_{q,+}$ by

$$P_t(x;q^2) := P(t,x;q^2) = d_q \frac{t}{t^2+x^2}, \quad \text{where } d_q = \frac{1}{\Gamma_{q^2}(\frac{1}{2})A(\frac{1}{q(1+q)^2};q^2)}. \tag{16}$$

PROPOSITION 1. [16] *For $x,t \in \mathbb{R}_{q,+}$, we have*

*(i)* $P_t(x;q^2) = \int_0^{+\infty} \frac{E_{q^2}^{-q^2u}}{\sqrt{u}} G(x, \frac{t^2}{q(1+q)^2u}; q^2) d_{q^2}\mu(u).$

*(ii)* $\| G(.,t;q^2) \|_{L^1(\mathbb{R}_{q,+})} = 1.$

Furthermore, from (1), the Poisson kernel satisfies the following.

LEMMA 1.

*(i)* $D_{q,t}P_t(x;q^2) = d_q \frac{x^2-qt^2}{(t^2+x^2)(q^2t^2+x^2)}$ *and* $D_{q,x}P_t(x;q^2) = -d_q \frac{(1+q)tx}{(t^2+x^2)(t^2+q^2x^2)}.$

*(ii)*

$$\Delta_{q,t}P_t(x;q^2) = d_q \frac{q^2t}{1-q} \frac{(q^3+q^2-q^{-1}-1)x^2+(1-q^2)t^2}{(t^2+x^2)(t^2+q^2x^2)(q^2t^2+x^2)},$$

*and*

$$\Delta_{q,x}P_t(x;q^2) = -d_q \frac{(1+q)q^2t}{1-q} \frac{(1-q)t^2+(q^2-q^{-1})x^2}{(t^2+x^2)(t^2+q^2x^2)(q^2t^2+x^2)}.$$

*(iii) For all* $k \in \mathbb{N}$, $\|D_{q,t}^k P_t(.;q^2)\|_{L^\infty(\mathbb{R}_{q,+})} \leqslant C_q t^{-(k+1)}$

*Proof.* (i) and (ii) are obvious. (iii) follows from (1). □

DEFINITION 2. Let $f \in L^p(\mathbb{R}_{q,+})$, $p \in [1,+\infty]$, the $q$-Poisson integral of $f$ denoted $u(f)(x,t;q^2)$ is defined by

$$u(f)(x,t;q^2) = (P_t(.;q^2) *_q f)(x) = \int_0^\infty f(y) T_{q,x} P_t(y;q^2) \mathrm{d}_q\mu(y). \tag{17}$$

Note that from Proposition 1(i), $u(f)(x,t;q^2)$ can be written as

$$u(f)(x,t;q^2) = \int_0^{+\infty} \frac{E_{q^2}^{-q^2u}}{\sqrt{u}} T^{\frac{t^2}{u}} f(x) \mathrm{d}_{q^2}\mu(u),$$

where

$$T^t f(x) = (G(.,t/q(1+q)^2;q^2) *_q f)(x) \tag{18}$$

satisfying from property $(P_5)$ and Proposition 1(ii)

$$\| T^t f(x) \|_{L^p(\mathbb{R}_{q,+})} \leqslant \| f \|_{L^p(\mathbb{R}_{q,+})}. \tag{19}$$

LEMMA 2. *The $q$-cosine Fourier transform of the $q$-Poisson kernel $P_t(x;q^2)$ is given by*

$$\mathscr{F}_q(P_t(.;q^2))(\lambda) = d_q\theta_0\mathscr{E}_{q^2}(-q^{-1}\lambda t),$$

*where $\mathscr{E}_{q^2}(x)$ is given by* (10) *and $\theta_0 = Q(1,q)$ defined in* [14] *by*

$$Q(x,q) = \sum_{m=-\infty}^{\infty} \frac{1}{xq^m + x^{-1}q^{-m}}.$$

*Proof.* Note that from (5), $\theta_0 = Q(1,q) = \int_0^\infty \frac{1}{1+x^2} d_q x$. To prove the lemma, we need to prove first that the function $x \mapsto \theta_0\mathscr{E}_{q^2}(-q^{-1}x)$ is the unique solution of the problem $(P)$ given by

$$(P)\begin{cases} \Delta_{q,x}u = u \\ \lim\limits_{n\to\infty} u(q^n x) = \theta_0 \\ \lim\limits_{n\to-\infty} u(q^n x) = 0. \end{cases}$$

We proceed in the same way as in [13, 15]. For $f_1(x) = \mathscr{E}_{q^2}(-q^{-1}x)$ and $f_2(x) = \mathscr{E}_{q^2}(q^{-1}x)$, we can easily verified from (11) that they are solutions of the problem $(P)$. Thus, any solution can be written in the form $u(x) = p_1(x)f_1(x) + p_2(x)f_2(x)$, where $p_1$ and $p_2$ are two periodic functions. Moreover, by (12) and the second initial condition, we obtain that $u(x) = p_1(x)\mathscr{E}_{q^2}(-q^{-1}x)$. So, by the first initial condition, we deduce that $p_1(x) = \theta_0$. Now, applying (15) and replacing $P_t(x;q^2)$ by its expansion (16), we have by the substitution $x = yt$

$$\mathscr{F}_q(P_t(.;q^2))(\lambda) = d_q \int_0^\infty \frac{t}{t^2+x^2}\cos(\lambda x;q^2)\mathrm{d}_q\mu(x) = d_q \int_0^\infty \frac{1}{1+x^2}\cos(\lambda ty;q^2)\mathrm{d}_q\mu(y).$$

The result follows from the fact that $\lambda t \mapsto \int_0^\infty \frac{1}{1+x^2}\cos(\lambda ty;q^2)\mathrm{d}_q\mu(y)$ verifies the problem $(P)$. □

PROPOSITION 2. *For all $f \in L^p(\mathbb{R}_{q,+})$ and $p \in [1,\infty]$, we have*

$$\begin{aligned} u(f)(x,t;q^2) &= \theta_0 \int_0^\infty \mathscr{E}_{q^2}(-q\lambda t)\cos(\lambda x;q^2)\mathscr{F}_q(f)(\lambda)d_q\mu(\lambda) \\ &= d_q\theta_0\mathscr{F}_q\Big(\mathscr{E}_{q^2}(-q.t)\mathscr{F}_q(f)(.)\Big)(x) \end{aligned}$$

*Proof.* Applying the $q$-cosine Fourier transform to both sides of the formula (17), we get

$$\mathscr{F}_q(u(f)(.,t;q^2))(\lambda) = \mathscr{F}_q(P_t(.;q^2))(\lambda)\mathscr{F}_q(f)(\lambda).$$

So the result follows from property $(P_2)$ and Lemma 2. □

In the following, we give some technical lemmas concerning some properties of the Poisson integral $u(f)(x,t;q^2)$ and its $q$-derivatives.

LEMMA 3. *Let* $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$ *be a positive function and* $p \in (1,\infty)$. *Then*

*(i)* $u(f)(x,t;q^2) \geqslant 0$.

*(ii)* $\Delta_q u(f)(x,t;q^2) = \Delta_{q,x} u(f)(x,t;q^2) + \Delta_{q,t} u(f)(x,t;q^2) = 0$

*(iii) For all* $k \in \mathbb{N}$, *there exists* $C_q > 0$ *such that* $\left| D^k_{q,t} u(f)(x,t;q^2) \right| \leqslant C_q t^{-(k+1)}$.

*Proof.* The lemma follows directly from Lemma 1. □

LEMMA 4. *Let* $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$ *be a positive function and* $p \in (1,\infty)$. *Then, for* $x$ *large, there exist respectively* $C_{1,q}, C_{2,q} > 0$ *such that*

*(i)* $u(f)(x,t;q^2) \leqslant C_{1,q}(t^2+x^2)^{-1/2}$.

*(ii)* $\left| D_{q,x} u(f)(x,t;q^2) \right| \leqslant C_{2,q}(t^2+x^2)^{-1}$.

*Proof.* (i) Since $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$, there exists $a \in \mathbb{R}_{q,+}$, such that $\mathrm{supp}(f) \subset [0,a]$. Then from (13) and (14), we can write

$$\begin{aligned}
T_{q,y}P_t(x;q^2) &= \sum_{n=0}^{\infty} b_n(y;q^2) \sum_{k=-n}^{n} q^{(2-n)n} \frac{(q;q)_{2n}}{(1-q)^{2n}} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}} P_t(q^k x;q^2) \\
&= \sum_{n=0}^{\infty} q^n y^{2n} \sum_{k=-n}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \frac{t}{t^2+q^{2k}x^2} \\
&= \sum_{n=0}^{\infty} q^n y^{2n} \Big[ \sum_{k=1}^{n} \frac{(-1)^{n+k} q^{(n+k)(n+k-1)/2}}{(q;q)_{n+k}(q;q)_{n-k}} \frac{t}{t^2+q^{-2k}x^2} \\
&\quad + \sum_{k=0}^{n} \frac{(-1)^{n-k} q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}} \frac{t}{t^2+q^{2k}x^2} \Big].
\end{aligned}$$

Thus

$$\begin{aligned}
T_{q,y}P_t(x;q^2) &\leqslant \sum_{n=0}^{\infty} q^n y^{2n} \sum_{k=1}^{n} \frac{t}{(q;q)_{n-k}(q;q)_{n+k}} \left[ \frac{1}{t^2+q^{-2k}x^2} + \frac{1}{t^2+q^{2k}x^2} \right] \\
&\leqslant t \sum_{n=0}^{\infty} q^n y^{2n} \sum_{k=1}^{n} \frac{1}{(q;q)_{n-k}(q;q)_{n+k}} \frac{2q^{-4k}}{t^2+q^{-2k}x^2} \\
&\leqslant \frac{2t}{t^2+x^2} \sum_{n=0}^{\infty} q^n y^{2n} \sum_{k=1}^{n} q^{-4k} \\
&\leqslant \frac{2}{\mid q^4-1 \mid\mid 1-qy^2 \mid} \frac{t}{t^2+x^2}.
\end{aligned}$$

Therefore by using the fact that $t \leqslant (t^2+x^2)^{1/2}$, we obtain

$$\begin{aligned} u(f)(x,t;q^2) = (P_t(.;q^2)*_q f)(x) &= \int_0^a f(y)T_{q,x}P_t(y;q^2)\mathrm{d}_q\mu(y) \\ &\leqslant \frac{2a}{|q^4-1|}\sup_{y\in[0,a]}\frac{|f(y)|}{|1-qy^2|}(t^2+x^2)^{-1/2} \\ &\leqslant C_{1,q}(t^2+x^2)^{-1/2}. \end{aligned}$$

Thus the first inequality is proven.

Now, we will prove the second inequality. By derivation under the $q$-integral sign

$$D_{q,x}u(f)(x,t;q^2) = D_{q,x}(P_t(.;q^2)*_q f)(x) = \int_0^a f(y)D_{q,x}T_{q,x}P_t(y;q^2)\mathrm{d}_q\mu(y).$$

But from Lemma 1(i)

$$\begin{aligned} &D_{q,x}T_{q,x}P_t(y;q^2) \\ &= \sum_{n=0}^{\infty} b_n(y;q^2)D_{q,x}\Delta_{q,x}^n(P_t(y;q^2)) \\ &= \sum_{n=0}^{\infty} b_n(y;q^2)\sum_{k=-n}^{n} q^{(2-n)n}\frac{(q;q)_{2n}}{(1-q)^{2n}}\frac{(-1)^{n-k}q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}}D_{q,x}P_t(q^kx;q^2) \\ &= \sum_{n=0}^{\infty} q^n y^{2n}\sum_{k=-n}^{n}\frac{(-1)^{n-k}q^{(n-k)(n-k-1)/2}}{(q;q)_{n-k}(q;q)_{n+k}}\frac{tx}{(t^2+q^{2k+2}x^2)(t^2+q^{2k}x^2)}, \end{aligned}$$

then,

$$\begin{aligned} &\left|D_{q,x}T_{q,x}P_t(y;q^2)\right| \\ &\leqslant t\,|\,x\,|\sum_{n=0}^{\infty} q^n y^{2n}\sum_{k=1}^{n}\frac{q^{2k}}{(q;q)_{n-k}(q;q)_{n+k}}\left[\frac{1}{t^2+q^{-2k-2}x^2}+\frac{1}{t^2+q^{2k+2}x^2}\right] \\ &\leqslant 2t\,|\,x\,|\sum_{n=0}^{\infty} q^n y^{2n}\sum_{k=1}^{n}\frac{q^{2k}q^{-8k-8}}{(t^2+q^{-2k-2}x^2)^2} \\ &\leqslant \frac{2t\,|\,x\,|}{(t^2+x^2)^2}\sum_{n=0}^{\infty} q^n y^{2n}\sum_{k=1}^{n} q^{-6k-8} \\ &\leqslant \frac{2q^{-1}}{|1-q^{-6}||1-qy^2|}\frac{t\,|\,x\,|}{(t^2+x^2)^2}. \end{aligned}$$

In the same manner, by the fact that: $t\,|\,x\,| \leqslant t^2+x^2$

$$\begin{aligned} D_{q,x}u(f)(x,t;q^2) &\leqslant \frac{2a}{|q^4-1|}\sup_{y\in[0,a]}\frac{|f(y)|}{|1-qy^2|}(t^2+x^2)^{-1} \\ &\leqslant C_{2,q}(t^2+x^2)^{-1}, \end{aligned}$$

which gives (ii). $\square$

LEMMA 5. *Let $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$ be a positive function and $p \in (1,\infty)$. Then*

*(i)* $\lim_{R\to\infty} \int_0^R \int_0^R \Delta_{q,t}(u^p(f)(x,t)) t d_q t d_q \mu(x) = \int_0^\infty f^p(x) d_q \mu(x).$

*(ii)* $\lim_{R\to\infty} \int_0^R \int_0^R \Delta_{q,x}(u^p(f)(x,t)) t d_q \mu(x) d_q t = 0.$

*(iii)* $\int_0^\infty \int_0^\infty \Delta_q(u^p(f)(x,t)) t d_q t d_q \mu(x) = \| f \|_{L^p(\mathbb{R}_{q,+})}.$

*Proof.* Let $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$ be an even positive function and $p \in (1,\infty)$. To prove (i), from (2) and $q$-integration by part formula (6), we get

$$\begin{aligned}
\int_0^R \Delta_{q,t}(u^p(f)(x,t)) t d_q t &= \int_0^R D^2_{q,t}(u^p(f)(x,q^{-1}t)) t d_q t \\
&= \left[ t D_{q,t}(u^p(f)(x,q^{-1}t)) \right]_0^R - \int_0^R D_{q,t}(u^p(f)(x,q^{-1}t)) d_q t \\
&= R \Big( \sum_{k=0}^{p-1} u^k(f)(x,q^{-1}R) u^{p-k-1}(f)(x,R) \Big) D_{q,t} u(f)(x,q^{-1}R) \\
&\quad - u^p(f)(x,q^{-1}R) + f^p(x).
\end{aligned}$$

Thus

$$\begin{aligned}
&\int_0^R \int_0^R \Delta_{q,t}(u^p(f)(x,t)) t d_q t \mathrm{d}_q \mu(x) \\
&= R \int_0^R \Big( \sum_{k=0}^{p-1} u^k(f)(x,q^{-1}R) u^{p-k-1}(f)(x,R) \Big) D_{q,t} u(f)(x,q^{-1}R) \mathrm{d}_q \mu(x) \\
&\quad - \int_0^R u^p(f)(x,q^{-1}R) \mathrm{d}_q \mu(x) + \int_0^R f^p(x) \mathrm{d}_q \mu(x).
\end{aligned}$$

From Lemma 3(iii), we get easily respectively

$$\Big| \int_0^R u^p(f)(x,q^{-1}R) \mathrm{d}_q \mu(x) \Big| \leqslant C_{1,q} R^{-p} R = C_{1,q} R^{-(p-1)} \underset{R\to\infty}{\longrightarrow} 0,$$

and

$$\begin{aligned}
&\Big| R \int_0^R \Big( \sum_{k=0}^{p-1} u^k(f)(x,q^{-1}R) u^{p-k-1}(f)(x,R) \Big) D_{q,t} u(f)(x,q^{-1}R) \mathrm{d}_q \mu(x) \Big| \\
&\leqslant C_{2,q} R^2 \Big( \sum_{k=0}^{p-1} R^{-k} R^{-(p-k-1)} \Big) R^{-2} \\
&= p C_{2,q} R^{-(p-1)} \underset{R\to\infty}{\longrightarrow} 0,
\end{aligned}$$

which leads to the result.

(ii) From the fact that

$$\begin{aligned}
&\int_0^R \Delta_{q,x}(u^p(f)(x,t))\mathrm{d}_q\mu(x)\\
&= \int_0^R D^2_{q,x}(u^p(f)(q^{-1}x,t))d_q\mu(x) = \big[D_{q,x}(u^p(f)(q^{-1}x,t))\big]_0^R\\
&= \Big(\sum_{k=0}^{p-1} u^k(f)(q^{-1}R,t)u^{p-k-1}(f)(q^{-1}R,t)\Big)D_{q,x}u(f)(q^{-1}R,t)\\
&\quad -\Big(\sum_{k=0}^{p-1} u^k(f)(0,t)u^{p-k-1}(f)(q^{-1}R,t)\Big)D_{q,x}u(f)(0,t).
\end{aligned}$$

Thus, from Lemma 4 we get

$$\begin{aligned}
&\Big|\int_0^R\int_0^R \Delta_{q,x}(u^p(f)(x,t))t\mathrm{d}_q\mu(x)d_qt\Big|\\
&\leqslant \int_0^R \Big|\Big(\sum_{k=0}^{p-1} u^k(f)(q^{-1}R,t)u^{p-k-1}(f)(q^{-1}R,t)\Big)D_{q,x}u(f)(q^{-1}R,t)\\
&\quad -\Big(\sum_{k=0}^{p-1} u^k(f)(0,t)u^{p-k-1}(f)(q^{-1}R,t)\Big)D_{q,x}u(f)(0,t)\Big|td_qt\\
&\leqslant 2pC_{2,q}R^{-p-1}\int_0^R td_qt\\
&= C_qR^{-(p-1)} \underset{R\to\infty}{\longrightarrow} 0\,.
\end{aligned}$$

This completes the proof of (ii). □

## 4. The $q$-Littlewood-Paley $g$-function

In this section, we define and study the so called $L^p$-boundedness of the $q$-Littlewood-Paley $g$-function when $p \in (1,2]$. For this, we need first use the $q$-Hardy-Littlewood maximal $\mathscr{M}_q(f)$ function.

DEFINITION 3. Let $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$. The $q$-Hardy-Littlewood maximal $\mathscr{M}_q(f)$ function is defined by

$$\mathscr{M}_q(f)(x) := \sup_{t\in\mathbb{R}_{q,+}} \mid u(f)(x,t)\mid,\ \ x \in \mathbb{R}_q.$$

PROPOSITION 3. *Let $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$ and $p \in (1,\infty)$. Then there exists $C_{p,q} > 0$ such that*

$$\| \mathscr{M}_q(f)(x) \|_{L^p(\mathbb{R}_{q,+})} \leqslant C_{p,q} \| f \|_{L^p(\mathbb{R}_{q,+})}$$

*Proof.* By changing variable $y = t^2/u$ in Proposition 1(i), we obtain

$$P_t(x;q^2) = t^{-2}\int_0^{+\infty} \phi(y/t^2)T^y f(x)\mathrm{d}_{q^2}\mu(y). \tag{20}$$

where $\phi(y) = y^{-3/2}E_{q^2}^{-q^2y^{-1}}$ and $T^y f(x)$ is given by (18).

We verify easily that $\phi(y)$ and $yD_{q,y}\phi(y)$ belong to $L^1(\mathbb{R}_{q,+})$. Then by $q$-integration by part in (20) and relation (19) shows that

$$\begin{aligned}
| P_t(x;q^2) | &= \Big| -\int_0^{\infty}\Big(\int_0^y T^t f(x)d_q t\Big)D_{q,y}\phi(qy/t^2)\mathrm{d}_q\mu(y)\Big| \\
&= \Big| \int_0^{\infty}\Big(1/y\int_0^y T^t f(x)d_q t\Big).\Big(yD_{q,y}\phi(qy/t^2)\Big)\mathrm{d}_q\mu(y)\Big| \\
&\leqslant \sup_{y\in\mathbb{R}_{q,+}}\Big|1/y\int_0^y T^t f(x)d_q t\Big|\Big(t^{-2}\int_0^{\infty}\Big|yD_{q,y}\phi(qy/t^2)\Big|\Big)\mathrm{d}_q\mu(y) | \\
&\leqslant C_q \sup_{y\in\mathbb{R}_{q,+}}\Big|1/y\int_0^y T^t f(x)d_q t\Big| \\
&\leqslant C_q\mathscr{M}_q^T(f)(x),
\end{aligned}$$

where $\mathscr{M}_q^T(f)(x) = \sup_{y\in\mathbb{R}_{q,+}}\Big|1/y\int_0^y T^t f(x)d_q t\Big|.$

So that $\mathscr{M}_q(f)(x) \leqslant \mathscr{M}_q^T(f)(x)$.

The result follows by the Hopf-Dunford-Schwartz ergodic theorem [7, Theorem 7, p. 693]. □

DEFINITION 4. The $q$-Littlewood-Paley $g$-function for $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$ is given by

$$g(f)(x;q^2) := \Big(\int_0^{\infty}\Big|\nabla_q u(f)(x,t;q^2)\Big|^2 t d_q t\Big)^{\frac{1}{2}},$$

where $u(f)(x,t)$ is the $q$-Poisson integral and $\nabla_q$ is the $q$-gradient, defined by

$$\nabla_q^2 u(f)(x,t;q^2) := (D_{q,x}u(f)(x,t;q^2))^2 + (D_{q,t}u(f)(x,t;q^2))^2$$

THEOREM 1. *For $p \in (1,2]$, there exist two constants $A_{p,q} > 0$ and $B_{p,q} > 0$, such that for $f \in L^p(\mathbb{R}_{q,+})$,*

$$B_{p,q} \parallel f \parallel_{L^p(\mathbb{R}_{q,+})} \leqslant \parallel g(f)(x;q^2) \parallel_{L^p(\mathbb{R}_{q,+})} \leqslant A_{p,q} \parallel f \parallel_{L^p(\mathbb{R}_{q,+})}.$$

We will go to prove the theorem. For this purpose, we will use the function $g_1(f)$ given in [17] by

$$g_1 f(x;q^2) := \Big(\int_0^{\infty} t\Big|D_{q,t}u(f)(x,t;q^2)\Big|^2 d_q t\Big)^{1/2}, \quad f \in L^p(\mathbb{R}_{q,+}).$$

Obviously, we have

$$g_1(f)(x;q^2) \leqslant g(f)(x;q^2). \tag{21}$$

To prove the theorem, we need the following lemma

LEMMA 6. *For $f_1, f_2 \in \mathscr{D}_{*,q}(\mathbb{R}_q)$, there exist $A_q > 0$, such that*

$$\int_0^\infty \int_0^\infty tD_{q,t}u(f_1)(x,t;q^2)D_{q,t}u(f_2)(x,t;q^2)d_qtd_q\mu(x) = A_q \int_0^\infty f_1(x)f_2(x)d_q\mu(x).$$

*Proof.* Let $f_1, f_2 \in \mathscr{D}_{*,q}(\mathbb{R}_q)$, we have by applying twice Hölder's inequality

$$\begin{aligned}\int_0^\infty \int_0^\infty t & \left|D_{q,t}u(f_1)(x,t;q^2)\right|\left|D_{q,t}u(f_2)(x,t;q^2)\right|d_qtd_q\mu(x)\\ &\leqslant \int_0^\infty g(f_1)(x;q^2)g(f_2)(x;q^2)d_q\mu(x)\\ &\leqslant \| g(f_1) \|_{L^2(\mathbb{R}_{q,+})}\| g(f_2) \|_{L^2(\mathbb{R}_{q,+})} .\end{aligned}$$

Then using Fubini, property $(P_3)$ and Proposition 2, we obtain

$$\begin{aligned}&\int_0^\infty \int_0^\infty tD_{q,t}u(f_1)(x,t;q^2)\overline{D_{q,t}u(f_2)(x,t;q^2)}d_qt\mathrm{d}_q\mu(x)\\ &= \int_0^\infty \int_0^\infty t\mathscr{F}_q\Big(D_{q,t}u(f_1)(.,t;q^2)\Big)(\lambda)\overline{\mathscr{F}_q\Big(D_{q,t}u(f_2)(.,t;q^2)\Big)}(\lambda)\mathrm{d}_q\mu(\lambda)d_qt\\ &= \int_0^\infty \int_0^\infty tq^4\lambda^2\mathscr{E}_q^2(-q^2\lambda t)\mathscr{F}_q(f_1)(\lambda)\overline{\mathscr{F}_q(f_2)(\lambda)}d_qt\mathrm{d}_q\mu(\lambda)\\ &= A_q\int_0^\infty \mathscr{F}_q(f_1)(\lambda)\overline{\mathscr{F}_q(f_2)(\lambda)}\mathrm{d}_q\mu(\lambda),\;\; A_q = \int_0^\infty u\,\mathscr{E}_q^2(-u)d_qu\\ &= A_q\int_0^\infty f_1(x)\overline{f_2(x)}\mathrm{d}_q\mu(x). \quad \square\end{aligned}$$

*Proof of the Theorem 1.* Let $p \in (1,2]$, it is clear that from the density of $\mathscr{D}_{*,q}(\mathbb{R}_q)$ in $L^p(\mathbb{R}_{q,+})$ ([8], Theorem 4.28), taken $f \in \mathscr{D}_{*,q}(\mathbb{R}_q)$. Note that from (3), we have

$$\begin{aligned}\Delta_{q,x}(f^n(x)) = q\Big[&\sum_{k=0}^{n-1}\sum_{i=0}^{k-1} f^i(q^{-1}x)f^{-i}(x).\Big(D_{q,x}f(x)\Big)\Big(D_{q,x}f(q^{-1}x)\Big)\Big)\\ &+q\sum_{k=0}^{n-1}\sum_{i=0}^{n-k-2} f^i(qx)f^{-i}(x)\\ &\times\Big(D_{q,x}f(x)\Big)^2\Big]f^{n-2}(x) + \Big[\sum_{k=0}^{n-1} f^k(q^{-1}x)f^{-k}(x)\Big]f^{n-1}(x)\Delta_{q,x}f(x).\end{aligned}$$

Thus, using Lemma 3(ii) we obtain

$$\begin{aligned}&\Delta_q(u(f)^p(x,t;q^2))\\ &= q\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{k-1} u(f)^i(q^{-1}x,t;q^2)f^{-i}(x,t;q^2).\Big(D_{q,x}u(f)(x,t;q^2).D_{q,x}u(f)(q^{-1}x,t;q^2)\\ &+D_{q,t}u(f)(x,t;q^2)\times D_{q,t}u(f)(x,q^{-1}t;q^2)\Big)\\ &+q\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u(f)^i(qx,t;q^2)u(f)^{-i}(x,t;q^2).\Big(\nabla_q u(f)(x,t;q^2)\Big)^2\Big]u(f)^{p-2}(x,t;q^2).\end{aligned}$$

So that from Lemma 3(i)

$$\Delta_q(u(f)^p(x,t;q^2)) \geqslant q^2\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u(f)^i(qx,t;q^2)u(f)^{-i}(x,t;q^2)\Big] \times u(f)^{p-2}(x,t;q^2)\Big(\nabla_{q,x}u(f)(x,t;q^2)\Big)^2,$$

we deduce that

$$\Big(\nabla_q u(f)(x,t;q^2)\Big)^2 \leqslant q^{-2}\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u(f)^i(qx,t;q^2)u(f)^{-i}(x,t;q^2)\Big]^{-1} \times u(f)^{2-p}(x,t;q^2)\Delta_q(u(f)^p(x,t;q^2)).$$

Now, from Lemma 4 (i) and the fact that $f$ is positive, and the function $x\mapsto u(f)(x,t;q^2)$ is decreasing we deduce that

$$\begin{aligned}&\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u(f)^i(qx,t;q^2)u(f)^{-i}(x,t;q^2)\Big]^{-1}\\ \leqslant&\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} u(f)^i(qx,t;q^2)u(f)^{-i}(x,t;q^2)\Big]^{-1}\\ \leqslant&\Big[\sum_{k=0}^{p-1}\sum_{i=0}^{p-k-2} 1\Big]^{-1}=\frac{2}{p(p-1)}.\end{aligned}$$

we deduce that

$$\Big|\nabla_{q,x}u(f)(x,t;q^2)\Big|^2 \leqslant \frac{2q^{-2}}{p(p-1)}u(f)^{2-p}(x,t;q^2)\Delta_q(u(f)^p(x,t;q^2)). \tag{22}$$

Using (22), we obtain

$$\begin{aligned}\Big[g(f)(x;q^2)\Big]^2 &\leqslant \frac{2q^{-2}}{p(p-1)}\int_0^\infty u(f)^{2-p}(x,t;q^2)\Delta_q(u(f)^p(x,t;q^2))td_qt\\ &\leqslant \frac{2q^{-2}}{p(p-1)}\mathscr{M}_q^{2-p}(f)(x)\int_0^\infty \Delta_q(u(f)^p(x,t;q^2))td_qt\end{aligned}$$

Thus observe that $(2-p)/2+p/2=1$ and by Hölder's inequality, we get

$$\begin{aligned}&\parallel g(f)(x;q^2)\parallel_{L^p(\mathbb{R}_{q,+})}^p\\ \leqslant&\Big[\frac{2q^{-2}}{p(p-1)}\Big]^{p/2}\int_0^\infty \mathscr{M}_q^{(2-p)(p/2)}(f)(x)\Big[\int_0^\infty \Delta_q(u(f)^p(x,t;q^2))td_qt\Big]^{p/2}\mathrm{d}_q\mu(x)\\ \leqslant&\Big[\frac{2q^{-2}}{p(p-1)}\Big]^{p/2}\parallel \mathscr{M}_q(f)\parallel_{L^p(\mathbb{R}_{q,+})}^{(2-p)(p/2)}\Big[\int_0^\infty \Delta_q(u(f)^p(x,t;q^2))td_qt\mathrm{d}_q\mu(x)\Big]^{p/2}\end{aligned}$$

Due to Lemma 4(iii), we obtain

$$\begin{aligned}\| g(f)(x;q^2) \|^p_{L^p(\mathbb{R}_{q,+})} &\leqslant \left[\frac{2q^{-2}}{p(p-1)}\right]^{p/2} \| \mathscr{M}_q(f) \|^{(2-p)(p/2)}_{L^p(\mathbb{R}_{q,+})} \| f \|^{p^2/2}_{L^p(\mathbb{R}_{q,+})}\\ &\leqslant C_{1,q}\left[\frac{2q^{-2}}{p(p-1)}\right]^{p/2} \| f \|^{(2-p)(p/2)}_{L^p(\mathbb{R}_{q,+})} \| f \|^{p^2/2}_{L^p(\mathbb{R}_{q,+})}\\ &\leqslant A_{p,q} \| f \|^p_{L^p(\mathbb{R}_{q,+})} .\end{aligned}$$

Proving now the left inequality. Computing relations (8), (21), Lemma 6 and Hölder inequality, we have

$$\begin{aligned}\frac{1}{A_q}\left| \int_0^\infty f(x)h(x)\mathrm{d}_q\mu(x)\right| &\leqslant \int_0^\infty g_1(f)(x;q^2)g_1(h)(x;q^2)\mathrm{d}_q\mu(x)\\ &\leqslant \| g_1(f) \|_{L^p(\mathbb{R}_{q,+})} \| g_1(h) \|_{L^m(\mathbb{R}_{q,+})}, \quad 1/p+1/m=1\\ &\leqslant C_{q,m} \| g_1(f) \|_{L^p(\mathbb{R}_{q,+})}\\ &\leqslant C_{q,m} \| g(f) \|_{L^p(\mathbb{R}_{q,+})} .\end{aligned}$$

Which completes the proof of the theorem by taking the supremum. □

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## REFERENCES

[1] A. ACHOUR AND K. TRIMECHE, *La g-fonction de Littlewood-Paley associeé à un opérateur differentiel singulier sur* $(0,\infty)$, Ann. Inst. Fourier (Grenoble) **33** (1983), 203–226.

[2] H. ANNABI AND A. FITOUHI, *La g-fonction de Littlewood-Paley associee a une classe d'operateurs differentiels sur* $(0,\infty)$ *contenant l'opérateur de Bessel*, C. R. Acad. Sci. Paris Ser. I Math. **9** (1986), 411–413.

[3] J. M. DAVIS, I. A. GRAVAGNE, R. J. MARKS II, J. E. MILLER, A. A. RAMOS, *Stability of switched linear systems on nonuniform time domains*, in: IEEE Proc. of the 42nd Meeting of the Southeastern Symposium on System Theory, Texas, 2010.

[4] L. DHAOUADI, A. FITOUHI AND J. EL KAMEL, *Inequalities in q-Fourier analysis*, J. Inequal. Pure Appl. Math. **171** (2006) 1–14.

[5] A. FITOUHI AND F. BOUZEFFOUR, *The q-cosine Fourier transform and the q-heat equation*, Ramanujan J. **28** (2012) 443–461.

[6] A. FITOUHI, M. HAMZA AND F. BOUZEFFOUR, *The q-* $j_\alpha$ *Bessel function*, J. Approx. Theory **115** (2002) 114–116.

[7] A. FITOUHI AND L. DHAOUADI, *Positivity of the generalized translation associated with the q-Hankel transform*, Constr. Approx. **34** (2011) 453–472.

[8] A. FITOUHI AND A. NEMRI, *Distribution And Convolution Product in Quantum Calculus*, Afr. Diaspora. J. Math., **7** (2008), 39–57.

[9] R. FLOREANINI, J. LETOURNEUX AND L. VINET, *More on the q-oscillator algebra and q-orthogonal polynomials*, J. Phys. A **28** (1995), 287–293.

[10] G. GASPER AND M. RAHMAN, *Basic hypergeometric series*, 2nd edn, Cambridge University Press, 2004.

[11] T. H. KOORNWINDER, *q-Special functions*, a tutorial arXiv:math/9403216v1.

[12] T. H. KOORNWINDER AND R. F. SWARTTOUW, *On $q$-Analogues of the Fourier and Hankel transforms*, Trans. Amer Math. Soc. **333** (1992) 445–461.

[13] A. B. OLDE DAALHUIS, *Asymptotic expansions for $q$-gamma, $q$- exponential and $q$-Bessel functions*, J. Math. Anal. Appl. **186** (1994) 896–913.

[14] M. A. OLSHANETSKY AND V. B. K. ROGOV, *The $q$-Fourier transform of $q$-generalized functions*, (Russian) Mat. Sb. **190** (1999) 717–736.

[15] H. MABROUK, *$q$-heat operator and $q$-Poisson's operator*, Fract. Calc. Appl. Anal. **9** (2006), 265–286.

[16] A. NEMRI, *On the connection between heat and wave problems in quantum calculus and applications*, Math. Mech. Solids **18** (2013) 849–860.

[17] A. NEMRI AND B. SELMI, *Lipschitz and Besov spaces in quantum calculus*, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **19** (2016), no. 3, 1650021, 19 pp.

[18] F. SOLTANI, *Littlewood-Paley $g$-function in the Dunkl analysis on $\mathbb{R}^d$*, J. Ineq. Pure Appl. Math. **6** (2005), Article 84.

[19] E. M. STEIN, *Topics in harmonic analysis related to the Littlewood-Paley theory*, Annals of Mathematics Studies, Vol. 63. Princeton, NJ/Tokyo: Princeton University Press/University of Tokyo Press, 1970.

[20] K. STEMPAK, *La theorie de Littlewood-Paley pour la transformation de Fourier Bessel*, C. R. Acad. Sci. Paris Ser. I Math. **303** (1986), 15–18.

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