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SUPPORTING VECTORS FOR THE ℓ_p–NORM

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Abstract. Given a continuous linear operator $T : X \to Y$ between normed spaces X,Y, the set of supporting vectors of T is defined as $\operatorname{suppv}(T) := \{x \in X : \|T(x)\| = \|T\| \text{ and } \|x\| = 1\}$. The supporting vectors of nontrivial projections and operators on ℓ_p, for $p = 1,2,\infty$, have already been calculated in previous works. In this manuscript, we go on one step further and compute the supporting vectors of operators $T : \ell_p \to \ell_q$, where $1 < p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

1. Introduction

The origin of this work partially relies on the Frobenius norm and the existence of matrix norms which are not operator norms, that is, induced by vector norms [4, 7, 19]. For this kind of matrix norms, such as the Frobenius norm, it does not make sense to compute the set of supporting vectors. However, in the first section of this manuscript, we will establish some relationships between matrix norms and certain operator norms whose supporting vectors "behave" in a similar way as if the matrix norm were an operator norm.

As we have mentioned above, an operator norm is a norm induced by vector norms. In other words, given two normed spaces X,Y, the operator norm of a continuous linear operator $T : X \to Y$ is defined as

$$\|T\| := \sup\{\|T(x)\| : \|x\| \leqslant 1\},$$

that is, the sup of $\|T(x)\|$ when x ranges the closed unit ball of X, $\mathsf{B}_X := \{x \in X : \|x\| \leqslant 1\}$. It is an elementary exercise to check that, if the above sup is attained at some $x \in \mathsf{B}_X$, then $\|x\| = 1$.

The concept of supporting vector was formerly introduced for the first time in [6], although it appeared implicitly and scattered throughout the literature of Banach Space Theory (see, for instance, [1, 2, 3, 10, 13, 14]). A vector x in the unit sphere

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$\mathsf{S}_X := \{x \in X : \|x\| = 1\}$ of X is said to be a supporting vector of T provided that $\|T(x)\| = \|T\|$. The set of supporting vectors of T is denoted by $\operatorname{suppv}(T)$, thus,

$$\operatorname{suppv}(T) := \{x \in \mathsf{S}_X : \|T(x)\| = \|T\|\}.$$

We refer the reader to [8, 11] for a topological and geometrical study of the set of supporting vectors of a continuous linear operator. Supporting vectors were successfully applied to solve multiobjective optimization problems that typically arise in Bioengineering, Physics and Statistics (see [5, 9, 17, 18, 20]), improving considerably the results obtained by means of other techniques, such as Heuristic methods [15, 16, 21].

In [9, 22], the supporting vectors of continuous linear operators $T : \ell_p \to \ell_p$, for $p = 1, 2, \infty$, were fully characterized in an infinite dimensional setting and fully computed in a final dimensional setting. This manuscript is a continuation of the previous two works, where we consider a continuous linear operator $T : \ell_p \to \ell_q$ with $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$.

2. Operator norms

Throughout this section, X, Y will stand for normed spaces over the real or complex field, and $\mathscr{L}(X,Y)$ will denote the vector space of (non-necessarily continuous) linear operators from X to Y.

DEFINITION 1. Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y. Let $\|\cdot\|_A$ be a norm on A. We define the norm of a vector $x \in X$ induced by $\|\cdot\|_A$ as

$$\|x\|_a := \sup\{\|T(x)\| : T \in A,\ \|T\|_A \leqslant 1\}. \tag{1}$$

Observe that, in the previous definition, $\|\cdot\|_A$ plays the role of a matrix norm which is not necessarily an operator norm, such as the Frobenius norm.

Recall that a subset of linear operators $A \subseteq \mathscr{L}(X,Y)$ is said to separate points of X provided that for all $x, y \in X$ with $x \neq y$, there exists $T \in A$ such that $T(x) \neq T(y)$. Notice that A separates points of X if and only if A is not simultaneously zero on X, that is, $\bigcap_{T \in A} \ker(T) = \{0\}$.

PROPOSITION 1. *Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y. Let $\|\cdot\|_A$ be a norm on A. Equation (1) defines a seminorm on $X_A := \{x \in X : \|x\|_a < \infty\}$. This seminorm is a norm if and only if A is not simultaneously zero on X_A.*

Proof. Note that $\|0\|_a = \sup\{\|T(0)\| : T \in A,\ \|T\|_A \leqslant 1\} = 0$. On the other hand, for each $x, y \in X_A$ and each $\lambda \in \mathbb{K}$, we have

$$\begin{aligned}
\|\lambda x\|_a &= \sup\{\|T(\lambda x)\| : T \in A,\ \|T\|_A \leqslant 1\} \\
&= \sup\{|\lambda|\|T(x)\| : T \in A,\ \|T\|_A \leqslant 1\} \\
&= |\lambda| \sup\{\|T(x)\| : T \in A,\ \|T\|_A \leqslant 1\} \\
&= |\lambda|\|x\|_a.
\end{aligned}$$

and

$$\begin{aligned}\|x+y\|_a &= \sup\{\|T(x+y)\| : T\in A,\ \|T\|_A \leqslant 1\}\\ &\leqslant \sup\{\|T(x)\|+\|T(y)\| : T\in A,\ \|T\|_A \leqslant 1\}\\ &\leqslant \sup\{\|T(x)\| : T\in A,\ \|T\|_A \leqslant 1\}+\sup\{\|T(y)\| : T\in A,\ \|T\|_A \leqslant 1\}\\ &= \|x\|_a+\|y\|_a.\end{aligned}$$

All of these show that X_A is a vector subspace of X and (1) defines a seminorm on X. Suppose now that this seminorm is, in fact, a norm. Fix an arbitrary $x\in X_A\setminus\{0\}$. By assumption, $\|x\|_a>0$, so there exists $T\in A$ with $\|T\|_A\leqslant 1$ such that $0<\|T(x)\|\leqslant \|x\|_a$. This implies that $T(x)\neq 0$ because Y is normed. As a consequence, A is not simultaneously zero on X_A, or equivalently, A separates points of X_A. Conversely, assume that A separates points of X_A. Fix an arbitrary $x\in X_A\setminus\{0\}$. There exists $T\in A$ such that $T(x)\neq 0$. Since $\|\cdot\|_A$ is a norm on A, $\|T\|_A>0$. Also, $\|T(x)\|>0$ because Y is normed. Finally, $\widehat{T}:=\frac{T}{\|T\|_A}\in A$, $\left\|\widehat{T}\right\|_A=1$ and

$$\|x\|_a=\sup\{\|S(x)\| : S\in A,\ \|S\|_A\leqslant 1\}\geqslant \left\|\widehat{T}(x)\right\|_A=\frac{\|T(x)\|}{\|T\|_A}>0.$$

This shows that Equation (1) defines a norm on X_A. □

REMARK 1. Under the settings of Proposition 1, it is trivial that A is not simultaneously zero on $X\setminus X_A$ even if $\|\cdot\|_a$ is not a norm on X_A. Indeed, for every $x\in X\setminus X_A$, $\|x\|_a=\infty$, so for every $K>0$ there exists $T\in A$ with $\|T\|_A\leqslant 1$ such that $\|T(x)\|\geqslant K$, so in particular, $T(x)\neq 0$.

According to Proposition 1, if $A\subseteq \mathscr{L}(X,Y)$ is a vector space of linear operators that separates points of X_A, then Equation (1) defines a norm on X_A. The following result shows examples where $X_A\neq X$. However, we will first recall an elementary remark.

REMARK 2. Let X be a vector space. If $C\subseteq X$ is a generator system of X, that is, $\operatorname{span}(C)=X$, and $L\subseteq C$ is a linearly independent set, then there exists a Hamel basis B of X such that $L\subseteq B\subseteq C$. This is a direct consequence of Zorn's Lemma. On the other hand, every Hamel basis $B=\{b_i : i\in I\}$ of X defines a norm on X. For instance,

$$\|x\|:=|\lambda_1|+\cdots+\left|\lambda_p\right|,$$

where

$$x=\lambda_1 b_{i_1}+\cdots+\lambda_p b_{i_p}$$

is the unique linear decomposition of T with respect to B. Note that $\|b\|=1$ for all $b\in B$.

PROPOSITION 2. *Let X be an infinite dimensional vector space and Y an infinite dimensional normed space. There exists a vector subspace $A\subseteq\mathscr{L}(X,Y)$ that separates points of X and a norm $\|\cdot\|_A$ on A such that $X_A\subsetneq X$.*

Proof. For every $x^* \in X^*$ (the algebraic dual of X) and every $y \in Y$, define

$$\begin{aligned} T_{x^*,y} : X &\to Y \\ x &\mapsto T_{x^*,y}(x) := x^*(x)y. \end{aligned}$$

Define

$$A := \operatorname{span}\left\{T_{x^*,y} : x^* \in X^*, y \in Y\right\}.$$

It is trivial that A separates points of X. Next, observe that, by relying on Zorn's Lemma, we can extract a Hamel basis of A from $\left\{T_{x^*,y} : x^* \in X^*, y \in Y\right\}$. In fact, we will extract a Hamel basis in a different way. Notice that if $W_Y \subseteq Y$ is a Hamel basis for Y and $x_0^* \in X^* \setminus \{0\}$, then $\left\{T_{x_0^*,y} : y \in W_Y\right\}$ is a linearly independent set, which can be enlarged to a Hamel basis of A inside $\left\{T_{x^*,y} : x^* \in X^*, y \in Y\right\}$ in view of Remark 2. So, let us take a Hamel basis $W_Y \subseteq Y$ of Y in such a way that there exists a sequence $(y_n)_{n\in\mathbb{N}} \subseteq W_Y$ with $\|y_n\| \to \infty$ as $n \to \infty$. Let us denote the enlarged Hamel basis of $\left\{T_{x_0^*,y} : y \in W_Y\right\}$ by

$$B := \left\{T_{x_i^*,y_i} : i \in I\right\}.$$

For every $T \in A$, we can define

$$\|T\|_A := |\lambda_1| + \cdots + \left|\lambda_p\right|,$$

where

$$T = \lambda_1 T_{x_{i_1}^*,y_{i_1}} + \cdots + \lambda_p T_{x_{i_p}^*,y_{i_p}}$$

is the unique linear decomposition of T with respect to B. Notice that

$$\left\|T_{x_i^*,y_i}\right\| = 1$$

for all $i \in I$. In particular,

$$\left\|T_{x_0^*,y_n}\right\| = 1$$

for all $n \in \mathbb{N}$ since $(y_n)_{n\in\mathbb{N}} \subseteq W_Y$ and $\left\{T_{x_0^*,y} : y \in W_Y\right\} \subseteq B$. It only remains to show that $X_A \neq X$. Since $x_0^* \neq 0$, we can find $x_0 \in X$ such that $x_0^*(x_0) \neq 0$. Then

$$\left\|T_{x_0^*,y_n}(x_0)\right\| = \|x_0^*(x_0)y_n\| = |x_0^*(x_0)|\,\|y_n\| \overset{n\to\infty}{\longrightarrow} \infty.$$

As a consequence,

$$\|x_0\|_a := \sup\left\{\|T(x_0)\| : T \in A, \|T\|_A \leqslant 1\right\} = \infty. \quad \square$$

DEFINITION 2. Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y that separates points of X. A norm $\|\cdot\|_A$ on A is said to be faithful if $X_A = X$.

LEMMA 1. *Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y that separates points of X. If $\|\cdot\|_A$ is a faithful norm on A and $|||\cdot|||_A$ is an equivalent norm on A to $\|\cdot\|_A$, then $|||\cdot|||_A$ is faithful as well.*

Proof. Indeed, let $M,N>0$ such that $M\|\cdot\|_A \leqslant |||\cdot|||_A \leqslant N\|\cdot\|_A$. Then for every $x\in X$ and every $T\in A$, if $|||T|||_A \leqslant 1$, then $\|MT\|_A \leqslant 1$, thus $\|MT(x)\| \leqslant \|x\|_a$, so $\|T(x)\| \leqslant \frac{1}{M}\|x\|_a$. As a consequence,

$$|||x|||_a := \sup\{\|T(x)\| : T\in A, |||T|||_A \leqslant 1\} \leqslant \frac{1}{M}\|x\|_a < \infty,$$

for every $x\in X$. □

Under the settings of the previous definition, we have that X is a normed space endowed with $\|\cdot\|_a$. Thus, we can consider the vector space of continuous linear operators from X to Y, $\mathscr{CL}(X,Y)$, and the corresponding operator norm on $\mathscr{CL}(X,Y)$, which is

$$\|T\|_{(A)} := \sup\{\|T(x)\| : \|x\|_a \leqslant 1\} \text{ for all } T\in\mathscr{CL}(X,Y). \tag{2}$$

The following theorem relates $\|T\|_{(A)}$ with $\|T\|_A$.

THEOREM 1. *Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y that separates points of X. Let $\|\cdot\|_A$ be a faithful norm on A. Then:*

1. *$A\subseteq\mathscr{CL}(X,Y)$ and $\|T\|_{(A)} \leqslant \|T\|_A$ for all $T\in A$. Thus, $\mathsf{B}_A \subseteq A\cap \mathsf{B}_{\mathscr{CL}(X,Y)}$, where $\mathsf{B}_A := \{T\in A : \|T\|_A \leqslant 1\}$ and $\mathsf{B}_{\mathscr{CL}(X,Y)} := \{T\in\mathscr{CL}(X,Y) : \|T\|_{(A)} \leqslant 1\}$.*

2. *If there exists $x_0\in X\setminus\{0\}$ such that the sup in Equation (1) is attained at some $T_0\in\mathsf{B}_A$, then $\|T_0\|_A = \|T_0\|_{(A)} = 1$ and $\frac{x_0}{\|x_0\|_a}\in\operatorname{suppv}(T_0)$.*

Proof.

1. Fix an arbitrary $T\in A\setminus\{0\}$. Since $\widehat{T} := \frac{T}{\|T\|_A}\in A$ and $\left\|\widehat{T}\right\|_A = 1$, for every $\|x\|_a \leqslant 1$,

   $$\frac{\|T(x)\|}{\|T\|_A} = \left\|\widehat{T}(x)\right\|_A \leqslant \sup\{\|S(x)\| : S\in A,\ \|S\|_A\leqslant 1\} = \|x\|_a \leqslant 1.$$

   Therefore, $\|T(x)\| \leqslant \|T\|_A$. This proves that $T\in\mathscr{CL}(X,Y)$ and $\|T\|_{(A)} \leqslant \|T\|_A$.

2. Suppose that there exists $x_0\in X\setminus\{0\}$ such that the sup in Equation (1) is attained at some $T_0\in\mathsf{B}_A$. Notice that, according to Theorem 1(1), $T_0\in\mathscr{CL}(X,Y)$. Then $\|T_0(x_0)\| = \|x_0\|_a > 0$, thus

   $$1 = \left\|T_0\left(\frac{x_0}{\|x_0\|_a}\right)\right\| \leqslant \|T_0\|_{(A)} \leqslant \|T_0\|_A \leqslant 1,$$

so $\|T_0\|_{(A)} = \|T_0\|_A = 1$. This also shows that $\frac{x_0}{\|x_0\|_a} \in \operatorname{suppv}(T_0)$. □

The following corollary is a direct consequence of Theorem 1(2).

COROLLARY 1. *Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y that separates points of X. Let $\|\cdot\|_A$ be a faithful norm on A. Let $x \in X$ be thought of as a linear operator from A to Y,*

$$\begin{aligned} x : A &\to Y \\ T &\mapsto x(T) := T(x). \end{aligned}$$

Then

$$\operatorname{suppv}(x) \subseteq \left\{T \in A : \|T\|_{(A)} = \|T\|_A = 1\right\}.$$

Proof. It simply suffices to observe that $\|x\|_a$ is precisely the operator norm of x when A is endowed with $\|\cdot\|_A$. □

Theorem 1 will allow to define the concept of supporting vector for matrix norms which are not necessarily operator norms.

DEFINITION 3. Let X be a vector space, Y a normed space and $A \subseteq \mathscr{L}(X,Y)$ a vector space of linear operators from X to Y that separates points of X. Let $\|\cdot\|_A$ be a faithful norm on A. For every $T \in A$, the set of supporting vectors of T is defined as the usual set of supporting vectors of T when X is endowed with the induced norm by $\|\cdot\|_A$ as in Equation (1). In other words,

$$\operatorname{suppv}(T) := \left\{x \in X : \|x\|_a = 1 \text{ and } \|T(x)\| = \|T\|_{(A)}\right\}.$$

In the upcoming results, we will assume that X is already endowed with a norm.

LEMMA 2. *Let X, Y be normed spaces. Then $A := \mathscr{C}\mathscr{L}(X,Y)$ separates points of X and its operator norm is faithful. Hence, Equation (1) defines a norm $\|\cdot\|_a$ on X such that $\|x\|_a \leqslant \|x\|$ for every $x \in X$.*

Proof. We will prove first that A is not simultaneously zero on X. Let $x_0 \in X \setminus \{0\}$. In virtue of the Hahn-Banach Theorem, we can find $x_0^* \in \mathsf{S}_{X^*}$ satisfying that $x_0^*(x_0) = \|x_0\| > 0$. Consider the continuous linear operator given by

$$\begin{aligned} T_0 : X &\to Y \\ x &\mapsto T_0(x) := x_0^*(x)y_0, \end{aligned}$$

where y_0 is a fixed element of S_Y. Observe that $T_0(x_0) = x^*(x_0)y_0 \neq 0$. It only remains to show that the operator norm of $A := \mathscr{C}\mathscr{L}(X,Y)$ is faithful. Indeed, for every $x \in X$ and every $T \in A := \mathscr{C}\mathscr{L}(X,Y)$ with $\|T\| \leqslant 1$, $\|T(x)\| \leqslant \|T\|\|x\| \leqslant \|x\|$, therefore

$$\|x\|_a := \sup\{\|T(x)\| : T \in \mathscr{C}\mathscr{L}(X,Y), \|T\| \leqslant 1\} \leqslant \|x\| < \infty. \quad \square$$

In the last result of this section, we will prove that, by bearing in mind Lemma 2, an equivalent norm on $A := \mathscr{CL}(X,Y)$ induces an equivalent norm on X. Notice that this latter fact is precisely the converse of a well known result, compiled in the following remark.

REMARK 3. Let X,Y be normed spaces. If $\|\cdot\|_0$ is an equivalent norm on X, then $\|\cdot\|_0$ trivially induces an equivalent norm on $\mathscr{CL}(X,Y)$.

THEOREM 2. *Let X,Y be normed spaces. Let $\|\cdot\|_A$ be an equivalent norm on $A := \mathscr{CL}(X,Y)$. Then:*

1. *$\|\cdot\|_A$ is faithful and its induced norm $\|\cdot\|_a$ as in Equation (1) defines an equivalent norm on X.*

2. *The induced norm $\|\cdot\|_{(A)}$ on $A := \mathscr{CL}(X,Y)$ by $\|\cdot\|_a$ is equivalent to the original operator norm of $A := \mathscr{CL}(X,Y)$.*

3. *$\|\cdot\|_{(A)}$ is also equivalent to $\|\cdot\|_A$.*

Proof. Consider constants $c,d > 0$ such that $c\|T\| \leqslant \|T\|_A \leqslant d\|T\|$ for all $T \in \mathscr{CL}(X,Y)$.

1. In the first place, in view of Lemma 2, the original operator norm on $\mathscr{CL}(X,Y)$ is faithful. By applying Lemma 1, $\|\cdot\|_A$ is faithful as well. Let us prove now that induced norm $\|\cdot\|_a$ by $\|\cdot\|_A$ as in Equation (1) defines an equivalent norm on X. Fix an arbitrary $x \in X$. For every $T \in A$ with $\|T\|_A \leqslant 1$,
$$\|T(x)\| \leqslant \|T\|\|x\| \leqslant \frac{1}{c}\|T\|_A\|x\| \leqslant \frac{1}{c}\|x\|.$$
Thus, $\|x\|_a \leqslant \frac{1}{c}\|x\|$, that is, $c\|x\|_a \leqslant \|x\|$. Next, if we look at the proof of Lemma 2, then we can construct a continuous linear operator $T_0 \in A$ such that $\|T_0\| = 1$ and $\|T_0(x)\| = \|x\|$. Notice that
$$\left\|\frac{1}{d}T_0\right\|_A \leqslant \|T_0\| = 1,$$
therefore,
$$\|x\|_a \geqslant \left\|\frac{1}{d}T_0(x)\right\| = \frac{1}{d}\|x\|.$$
In other words,
$$\frac{1}{d}\|x\| \leqslant \|x\|_a \leqslant \frac{1}{c}\|x\|.$$

2. In accordance with Remark 3, since $\|\cdot\|_a$ is equivalent to the original norm of X, the induced operator norm $\|\cdot\|_{(A)}$ on $A := \mathscr{CL}(X,Y)$ by $\|\cdot\|_a$ is equivalent to the original operator norm of $\mathscr{CL}(X,Y)$.

3. In view of Theorem 2(2), $\|\cdot\|_{(A)}$ is equivalent to the original operator norm of $\mathscr{CL}(X,Y)$, which is equivalent, by hypothesis, to $\|\cdot\|_A$. As a consequence, $\|\cdot\|_A$ and $\|\cdot\|_{(A)}$ are equivalent. $\square$

3. Entrywise sup norm is an operator norm, that is, induced by vector norms

The purpose of this section is to show that the entrywise sup norm of a matrix $A \in \mathbb{R}^{m\times n}$,

$$\|A\|_{(\infty)} := \max\left\{\left|a_{ij}\right| : 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant m\right\},$$

is an operator norm, in other words, it is induced by vector norms. In the first place, let us observe that

$$\|A\|_{(\infty)} = \max\left\{\|Ae_i\|_\infty : 1 \leqslant i \leqslant m\right\},$$

where e_i is the i^{th}-canonical column vector. This observation allows to generalize this situation as follows: Let X be a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$. For every continuous linear operator $T : X \to \ell_\infty$, we can define

$$\|T\|_{(\infty)} := \sup\left\{\|T(e_n)\|_\infty : n \in \mathbb{N}\right\}.$$

Since the Schauder basis is normalized, $\|T\|_{(\infty)} \leqslant \|T\|$.

The next remark is crucial towards the development of the proof of the following theorem.

REMARK 4. Let X a Banach space and $A \subseteq X$. The absolutely convex hull of A is defined as the intersection of all absolutely convex subsets of X containing A. The absolutely convex hull of A is usually denoted as $\mathrm{aco}(A)$. It is well-known folklore that

$$\mathrm{aco}(A) = \left\{\sum_{i=1}^{k} t_i a_i : k \in \mathbb{N},\ \forall i = 1, \cdots, k\ \ t_i \in \mathbb{K},\ a_i \in A, \sum_{i=1}^{k} |t_i| \leqslant 1\right\}.$$

The closed absolutely convex hull of A is defined as the intersection of all closed absolutely convex subsets of X containing A. The closed absolutely convex hull of A is usually denoted as $\overline{\mathrm{aco}}(A)$. It is well-known folklore that

$$\overline{\mathrm{aco}}(A) = \mathrm{cl}\left(\mathrm{aco}(A)\right).$$

THEOREM 3. *Let X be a Banach space with a normalized Schauder basis $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$. For every continuous linear operator $T : X \to \ell_\infty$,*

$$\|T\|_{(\infty)} = \sup\left\{\|T(x)\|_\infty : \|x\|_E \leqslant 1\right\},$$

where $\|\cdot\|_E$ is the norm on X induced, through the Minkowski functional, by the closed absolutely convex hull of $\{e_n : n \in \mathbb{N}\}$.

Proof. Let $E := \overline{\mathrm{aco}}\left(\{e_n : n \in \mathbb{N}\}\right)$ denote the closed absolutely convex hull of $\{e_n : n \in \mathbb{N}\}$. Since $e_n \in E$ for all $n \in \mathbb{N}$, we have that

$$\|T(e_n)\|_\infty \leqslant \sup\left\{\|T(x)\|_\infty : \|x\|_E \leqslant 1\right\}$$

for all $n \in \mathbb{N}$. As a consequence,

$$\|T\|_{(\infty)} := \sup\left\{\|T(e_n)\|_\infty : n \in \mathbb{N}\right\} \leqslant \sup\left\{\|T(x)\|_\infty : \|x\|_E \leqslant 1\right\}.$$

On the other hand, if $x \in \operatorname{aco}(\{e_n : n \in \mathbb{N}\})$, there exist $\lambda_1, \cdots, \lambda_k \in \mathbb{K}$ and $n_1, \cdots, n_k \in \mathbb{N}$ such that $|\lambda_1| + \cdots + |\lambda_k| \leqslant 1$ and $x = \lambda_1 e_{n_1} + \cdots + \lambda_k e_{n_k}$. Then

$$\|T(x)\|_\infty \leqslant \sum_{i=1}^{k} |\lambda_i| \, \|T(e_{n_i})\| \leqslant \sup\{\|T(e_n)\|_\infty : n \in \mathbb{N}\} = \|T\|_{(\infty)}.$$

Finally, notice that

$$\begin{aligned} \sup\{\|T(x)\|_\infty : \|x\|_E \leqslant 1\} &= \sup\{\|T(x)\|_\infty : x \in E\} \\ &= \sup\{\|T(x)\|_\infty : x \in \overline{\operatorname{aco}}(\{e_n : n \in \mathbb{N}\})\} \\ &= \sup\{\|T(x)\|_\infty : x \in \operatorname{aco}(\{e_n : n \in \mathbb{N}\})\} \\ &\leqslant \|T\|_{(\infty)}. \quad \square \end{aligned}$$

4. Supporting vectors of operators on ℓ_p

Recall that a Schauder basis $(e_n)_{n\in\mathbb{N}}$ on a Banach space is called binormalized provided that $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ and $(e_n^*)_{n\in\mathbb{N}} \subseteq \mathsf{S}_{X^*}$. Any Banach space admitting a Schauder basis can be equivalently renormed to turn the basis into a binormalized basis (see [12]).

LEMMA 3. *Let X be a Banach space with a binormalized Schauder basis $(e_n)_{n\in\mathbb{N}}$. For every $x \in X$, $(e_n^*(x))_{n\in\mathbb{N}}$ is bounded and the linear operator*

$$\begin{aligned} X &\longrightarrow \ell_\infty \\ x &\mapsto (e_n^*(x))_{n\in\mathbb{N}} \end{aligned} \tag{3}$$

is continuous. As a consequence, if $(x_k)_{k\in\mathbb{N}} \subseteq X$ is a sequence converging to $x \in X$, then $((e_n^*(x_k))_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ converges uniformly on $n \in \mathbb{N}$ to $(e_n^*(x))_{n\in\mathbb{N}}$.

Proof. First off, keep in mind that, since $(e_n)_{n\in\mathbb{N}}$ is binormalized, $(e_n)_{n\in\mathbb{N}} \subseteq \mathsf{S}_X$ and $(e_n^*)_{n\in\mathbb{N}} \subseteq \mathsf{S}_{X^*}$. Notice that $|e_n^*(x)| \leqslant \|e_n^*\| \, \|x\| = \|x\|$ for all $n \in \mathbb{N}$ and all $x \in X$. Therefore,

$$\|(e_n^*(x))_{n\in\mathbb{N}}\|_\infty \leqslant \|x\|$$

for all $x \in X$, hence the linear operator (3) is continuous and has norm less than or equal to 1. In fact, (3) maps $(e_n)_{n\in\mathbb{N}}$ to the canonical basis of ℓ_∞, therefore, (3) has norm 1. Finally, if $(x_k)_{k\in\mathbb{N}} \subseteq X$ is a sequence converging to $x \in X$, then $((e_n^*(x_k))_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ converges to $(e_n^*(x))_{n\in\mathbb{N}}$ in the sup norm of ℓ_∞, therefore, $((e_n^*(x_k))_{n\in\mathbb{N}})_{k\in\mathbb{N}}$ converges uniformly on $n \in \mathbb{N}$ to $(e_n^*(x))_{n\in\mathbb{N}}$. $\square$

By bearing in mind Lemma 3, from now on, whenever X is endowed with a binormalized Schauder basis $(e_n)_{n\in\mathbb{N}}$, if $x \in X$, then we will define

$$\|x\|_\infty := \|(e_n^*(x))_{n\in\mathbb{N}}\|_\infty = \sup_{n\in\mathbb{N}} |e_n^*(x)|.$$

In virtue of Lemma 3, $\|x\|_\infty \leqslant \|x\|$ for each $x \in X$.

LEMMA 4. *Let X, Y be Banach spaces admitting binormalized Schauder bases $(e_n)_{n\in\mathbb{N}} \subseteq X, (u_i)_{i\in\mathbb{N}} \subseteq Y$, respectively. $T : X \to Y$ be a nonzero continuous linear operator. For every $x \in X$,*

$$T(x) = \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} e_n^*(x) u_i^* (T(e_n)) \right) u_i.$$

Proof. Fix an arbitrary $x \in X$ and observe that $x = \sum_{n=1}^{\infty} e_n^*(x) e_n$. In view of Lemma 3, $\|x\|_\infty \leqslant \|x\|$. For every $k \in \mathbb{N}$, let $x_k := \sum_{n=1}^{k} e_n^*(x) e_n$. Observe that $(x_k)_{k\in\mathbb{N}}$ converges to x by definition of Schauder basis. For each $k \in \mathbb{N}$,

$$\begin{aligned} T(x_k) &= T\left(\sum_{n=1}^{k} e_n^*(x) e_n \right) \\ &= \sum_{n=1}^{k} e_n^*(x) T(e_n) \\ &= \sum_{n=1}^{k} e_n^*(x) \sum_{i=1}^{\infty} u_i^* (T(e_n)) u_i \\ &= \sum_{i=1}^{\infty} \left(\sum_{n=1}^{k} e_n^*(x) u_i^* (T(e_n)) \right) u_i. \end{aligned}$$

Thus,

$$u_i^* (T(x_k)) = \sum_{n=1}^{k} e_n^*(x) u_i^* (T(e_n))$$

for all $k, i \in \mathbb{N}$. Since $(x_k)_{k\in\mathbb{N}}$ converges to x, we have that $(T(x_k))_{k\in\mathbb{N}}$ converges to $T(x)$, hence $(T(x_k))_{k\in\mathbb{N}}$ converges to $T(x)$ in the sup norm of c_0, so in particular $(u_i^* (T(x_k)))_{k\in\mathbb{N}}$ converges to $u_i^* (T(x))$ uniformly for all $i \in \mathbb{N}$. Then

$$u_i^* (T(x)) = \sum_{n=1}^{\infty} e_n^*(x) u_i^* (T(x_k))$$

for all $i \in \mathbb{N}$. Finally,

$$T(x) = \sum_{i=1}^{\infty} u_i^* (T(x)) u_i = \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} e_n^*(x) u_i^* (T(e_n)) \right) u_i. \quad \square$$

We will rely on Lemma 4 to characterize the supporting vectors of continuous linear operators from ℓ_p to ℓ_q. In the upcoming theorem, if $x \in \ell_p$, then

$$x_k := (x(1), x(2), \cdots, x(k), 0, \cdots, 0, \cdots).$$

It is well known that $(x_k)_{k\in\mathbb{N}}$ converges to x in the ℓ_p-norm, and since $\|\cdot\|_\infty \leqslant \|\cdot\|_p$, $(x_k(n))_{k\in\mathbb{N}}$ converges uniformly on $n \in \mathbb{N}$ to $x(n)$.

THEOREM 4. *Let* $T : \ell_p \to \ell_q$ *be a nonzero continuous linear operator, where* $\frac{1}{p} + \frac{1}{q} = 1$. *Then:*

1. *For every* $x \in \ell_p$,

$$T(x) = \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n) T(e_n)(i) \right) e_i, \qquad \|T(x)\|_q = \left(\sum_{i=1}^{\infty} \left| \sum_{n=1}^{\infty} x(n) T(e_n)(i) \right|^q \right)^{\frac{1}{q}}$$

and

$$\|T(x)\|_q^q \leqslant \|x\|_p^q \sum_{i=1}^{\infty} \sum_{n=1}^{\infty} |T(e_n)(i)|^q.$$

2. *If* $(\|T(e_n)\|_q)_{n\in\mathbb{N}} \in \ell_q$, *then*

$$\|T\| \leqslant \left\| (\|T(e_n)\|_q)_{n\in\mathbb{N}} \right\|_q = \left(\sum_{n=1}^{\infty} \|T(e_n)\|_q^q \right)^{\frac{1}{q}}.$$

3. *If there exists a sequence* $(y_k)_{k\in\mathbb{N}} \subseteq \mathsf{B}_{\ell_p}$ *such that* $y_k(n) = 0$ *for all* $n > k$ *and*

$$\left| \sum_{n=1}^{k} y_k(n) T(e_n)(i) \right|^q \geqslant \sum_{n=1}^{k} |T(e_n)(i)|^q \quad \forall i \in \mathbb{N}\, \forall k \in \mathbb{N},$$

then $(\|T(e_n)\|_q)_{n\in\mathbb{N}} \in \ell_q$,

$$\|T\| = \left\| (\|T(e_n)\|_q)_{n\in\mathbb{N}} \right\|_q = \left(\sum_{n=1}^{\infty} \|T(e_n)\|_q^q \right)^{\frac{1}{q}}$$

and $\operatorname{suppv}(T) \supseteq W$, *where*

$$W := \left\{ x \in \mathsf{B}_{\ell_p} : \left| \sum_{n=1}^{k} x(n) T(e_n)(i) \right|^q \geqslant \sum_{n=1}^{k} |T(e_n)(i)|^q \quad \forall i \in \mathbb{N}\, \forall k \in \mathbb{N} \right\}.$$

Proof.

1. Fix an arbitrary $x \in \ell_p$. According to Lemma 4,

$$T(x) = \sum_{i=1}^{\infty} \left(\sum_{n=1}^{\infty} x(n) T(e_n)(i) \right) e_i.$$

By definition of $\|\cdot\|_q$ in ℓ_q,

$$\|T(x)\|_q = \left(\sum_{i=1}^{\infty} \left| \sum_{n=1}^{\infty} x(n) T(e_n)(i) \right|^q \right)^{\frac{1}{q}}.$$

Next, attending to Hölder's inequality,

$$\begin{aligned}\|T(x)\|_q^q &= \sum_{i=1}^{\infty}\left|\sum_{n=1}^{\infty} x(n)T(e_n)(i)\right|^q \\ &\leqslant \sum_{i=1}^{\infty}\left(\sum_{n=1}^{\infty}|x(n)T(e_n)(i)|\right)^q \\ &\leqslant \sum_{i=1}^{\infty}\left(\left(\sum_{n=1}^{\infty}|x(n)|^p\right)^{\frac{1}{p}}\left(\sum_{n=1}^{\infty}|T(e_n)(i)|^q\right)^{\frac{1}{q}}\right)^q \\ &= \|x\|_p^q \sum_{i=1}^{\infty}\sum_{n=1}^{\infty}|T(e_n)(i)|^q.\end{aligned}$$

2. For every $x \in \mathsf{B}_{\ell_p}$ and every $k \in \mathbb{N}$, following the previous chain of inequalities, we have that

$$\begin{aligned}\|T(x_k)\|_q^q &\leqslant \|x_k\|_p^q \sum_{i=1}^{\infty}\sum_{n=1}^{k}|T(e_n)(i)|^q \leqslant \sum_{i=1}^{\infty}\sum_{n=1}^{k}|T(e_n)(i)|^q \\ &= \sum_{n=1}^{k}\sum_{i=1}^{\infty}|T(e_n)(i)|^q = \sum_{n=1}^{k}\|T(e_n)\|_q^q \\ &\leqslant \sum_{n=1}^{\infty}\|T(e_n)\|_q^q = \left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q^q,\end{aligned}$$

thus

$$\|T(x_k)\|_q \leqslant \left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q.$$

Since $(x_k)_{k\in\mathbb{N}}$ converges to x in the ℓ_p-norm and T is continuous, we have that $(T(x_k))_{k\in\mathbb{N}}$ converges to $T(x)$ in the ℓ_q-norm. In particular, $(\|T(x_k)\|_q)_{k\in\mathbb{N}}$ converges to $\|T(x)\|_q$, hence

$$\|T(x)\|_q \leqslant \left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q.$$

The arbitrariness of $x \in \mathsf{B}_{\ell_p}$ shows that

$$\|T\| \leqslant \left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q = \left(\sum_{n=1}^{\infty}\|T(e_n)\|_q^q\right)^{\frac{1}{q}}.$$

3. Notice that, for all $k \in \mathbb{N}$,

$$\begin{aligned}\|T\|^q \geqslant \|T(y_k)\|_q^q &= \sum_{i=1}^{\infty}\left|\sum_{n=1}^{k} y_k(n)T(e_n)(i)\right|^q \\ &\geqslant \sum_{i=1}^{\infty}\sum_{n=1}^{k}|T(e_n)(i)|^q = \sum_{n=1}^{k}\sum_{i=1}^{\infty}|T(e_n)(i)|^q \\ &= \sum_{n=1}^{k}\|T(e_n)\|_q^q,\end{aligned}$$

thus $\sum_{n=1}^{k} \|T(e_n)\|_q^q$ is convergent and

$$\left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q^q = \sum_{n=1}^{\infty} \|T(e_n)\|_q^q \leqslant \|T\|^q,$$

so

$$\left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q = \left(\sum_{n=1}^{\infty} \|T(e_n)\|_q^q\right)^{\frac{1}{q}} \leqslant \|T\|.$$

As a consequence, in view of the previous item,

$$\|T\| = \left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q = \left(\sum_{n=1}^{\infty} |T(e_n)|^q\right)^{\frac{1}{q}}.$$

Finally, let us show that $\operatorname{suppv}(T) \supseteq W$. Note that, if $x \in W$, then $x \in \mathsf{B}_{\ell_p}$ and

$$\left|\sum_{n=1}^{k} x(n)T(e_n)(i)\right|^q \geqslant \sum_{n=1}^{k} |T(e_n)(i)|^q \quad \forall i \in \mathbb{N}\, \forall k \in \mathbb{N},$$

then, by reproducing the chain of inequalities right above, we obtain that

$$\begin{aligned}
\|T\|^q &\geqslant \|T(x_k)\|_q^q \\
&= \sum_{i=1}^{\infty} \left|\sum_{n=1}^{k} x_k(n)T(e_n)(i)\right|^q \\
&= \sum_{i=1}^{\infty} \left|\sum_{n=1}^{k} x(n)T(e_n)(i)\right|^q \\
&\geqslant \sum_{i=1}^{\infty} \sum_{n=1}^{k} |T(e_n)(i)|^q \\
&= \sum_{n=1}^{k} \sum_{i=1}^{\infty} |T(e_n)(i)|^q \\
&= \sum_{n=1}^{k} \|T(e_n)\|_q^q
\end{aligned}$$

for all $k \in \mathbb{N}$. Since $(x_k)_{k\in\mathbb{N}}$ converges to x in the ℓ_p-norm and T is continuous, we have that $(T(x_k))_{k\in\mathbb{N}}$ converges to $T(x)$ in the ℓ_q-norm. In particular, $(\|T(x_k)\|_q)_{k\in\mathbb{N}}$ converges to $\|T(x)\|_q$, hence

$$\|T\|^q \geqslant \|T(x)\|_q^q \geqslant \sum_{n=1}^{\infty} \|T(e_n)\|_q^q = \left\|(\|T(e_n)\|_q)_{n\in\mathbb{N}}\right\|_q^q = \|T\|^q.$$

In other words, $\|T(x)\|_q = \|T\|$, so $x \in \operatorname{suppv}(T)$. $\square$

Following the proof of Theorem 4, we immediately obtain the following corollary that covers the finite dimensional setting.

COROLLARY 2. *Let* $A = (a_{ij}) \in \mathscr{M}_{m\times n}(\mathbb{R})$ *such that* $a_{ij} \geqslant 0$ *for all* $i = 1,\dots,m$ *and all* $j = 1,\dots,n$. *If* $x \in \mathbb{R}^n$ *satisfies that* $x_j \geqslant 0$ *for all* $j = 1,\dots,n$, $\sum_{j=1}^n x_j^p = 1$ *and*

$$\left(\sum_{j=1}^n x_j a_{ij}\right)^q \geqslant \sum_{j=1}^n a_{ij}^q \ \ \forall i = 1,\dots,m,$$

then $x \in \operatorname{suppv}(A)$ *when* A *is seen as a linear operator between* $\ell_p^n := (\mathbb{R}^n, \|\cdot\|_p)$ *and* $\ell_q^m := (\mathbb{R}^m, \|\cdot\|_q)$.

We will conclude this section and the manuscript with an example in two dimensions that illustrates the results of Theorem 4 and Corollary 2.

EXAMPLE 1. Denote $\ell_p^2 := \left(\mathbb{R}^2, \|\cdot\|_p\right)$ and $\ell_q^2 := \left(\mathbb{R}^2, \|\cdot\|_q\right)$, with $1 < p,q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. Consider

$$\begin{aligned} T: \quad & \ell_p^2 \to \ell_q^2 \\ & (x,y) \mapsto T(x,y) := (x+y, 2x+2y). \end{aligned}$$

Notice that the matrix associated to T with respect to the canonical basis of $\mathbb{R}^2$ is

$$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}.$$

Also, $T(e_1) = T(1,0) = (1,2)$ and $T(e_2) = T(0,1) = (1,2)$. On the one hand,

$$\sum_{n=1}^2 |T(e_n)(i)|^q = \begin{cases} 2 & \text{if } i = 1, \\ 2^{q+1} & \text{if } i = 2. \end{cases}$$

Let us take now $x := \left(2^{\frac{-1}{p}}, 2^{\frac{-1}{p}}\right) \in \mathsf{S}_{\ell_p^2}$. On the other hand,

$$\left|\sum_{n=1}^2 x(n)T(e_n)(i)\right|^q = \begin{cases} \left(2^{\frac{-1}{p}} + 2^{\frac{-1}{p}}\right)^q & \text{if } i = 1, \\ \left(2\cdot 2^{\frac{-1}{p}} + 2\cdot 2^{\frac{-1}{p}}\right)^q & \text{if } i = 2. \end{cases}$$

Then

$$\begin{aligned} \left|\sum_{n=1}^2 x(n)T(e_n)(1)\right|^q &= \left(2^{\frac{-1}{p}} + 2^{\frac{-1}{p}}\right)^q = \left(2\cdot 2^{\frac{-1}{p}}\right)^q \\ &= \left(2^{1-\frac{1}{p}}\right)^q = \left(2^{\frac{1}{q}}\right)^q \\ &= 2 \geqslant \sum_{n=1}^2 |T(e_n)(1)|^q, \end{aligned}$$

and

$$\left|\sum_{n=1}^{2} x(n)T(e_n)(2)\right|^q = \left(2\cdot 2^{\frac{-1}{p}} + 2\cdot 2^{\frac{-1}{p}}\right)^q = \left(2^{2-\frac{1}{p}}\right)^q$$
$$= \left(2^{1+\frac{1}{q}}\right)^q = 2^{q+1}$$
$$\geqslant \sum_{n=1}^{2} |T(e_n)(2)|^q.$$

As a consequence, in view of Corollary 2,

$$x := \left(2^{\frac{-1}{p}}, 2^{\frac{-1}{p}}\right) \in \operatorname{suppv}(T).$$

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