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THE LAI LAW FOR WEIGHTED SUMS

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Abstract. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with partial sums $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Lai law states that

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2n\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > \sqrt{r-1}, \\ = \infty, & \text{if } \varepsilon < \sqrt{r-1} \end{cases}$$

if and only if $EX = 0$, $EX^2 = 1$ and $E(X^2/\log|X|)^r < \infty$, where $r > 1$. The paper will extend the result to the weighted sums under some conditions both on the weights and the moment.

1. Introduction and the main result

The following theorem, related to the law of single logarithm, is well-known.

THEOREM A. *Let $r > 1$ and $\{X, X_n, n \geqslant 1\}$ be a sequence of independent and identically distributed random variables with partial sums $S_n = \sum_{k=1}^n X_k$, $n \geqslant 1$. Suppose that*

$$EX = 0, \quad EX^2 = 1 \text{ and } E(X^2/\log|X|)^r < \infty, \tag{1.1}$$

where, and in the following, $\log x = \log_e \max\{x, e\}, x > 0$. Then for all $\varepsilon > \sqrt{r-1}$,

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2n\log n}\} < \infty. \tag{1.2}$$

Conversely, if (1.2) holds for some $\varepsilon > 0$, then $EX = 0$ and $E(X^2/\log|X|)^r < \infty$.

One can label the result as the Lai law which was first established by Lai [10]. Chen and Wang [4] extended it to the moving processes partly, and furthermore showed that

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2n\log n}\} = \infty, \quad \text{for all } \varepsilon < \sqrt{r-1}.$$

Combining the results of Lai [10] and Chen and Wang [4],

$$\sum_{n=1}^{\infty} n^{r-2} P\{|S_n| > \varepsilon\sqrt{2n\log n}\} \begin{cases} < \infty, & \text{if } \varepsilon > \sqrt{r-1}, \\ = \infty, & \text{if } \varepsilon < \sqrt{r-1} \end{cases} \tag{1.3}$$

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if and only if (1.1) holds.

Recently, Liu and Meng [11] extended (1.3) to the case when $\{X_n, n \geqslant 1\}$ are no longer identically distributed, but rather their distributions come from a finite set of distributions. When $r = 1$, an analog of (1.3) is discussed by Chen and Qi [3].

Due to the estimation of least squares regression coefficients in linear regression and the non-parametric curve estimation, it is very interesting and meaningful to study the topic of the limiting behaviors for the weighted sums of random variables.

In this paper, we will focus on the array weights $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ of real numbers satisfying

$$\sum_{k=1}^{n} |a_{nk}|^{\alpha} = O(n) \tag{1.4}$$

for some $\alpha > 0$.

In fact, under condition (1.4), many authors have studied the strong laws of large numbers and the law of single logarithm for weighted sums of independent and identically distributed random variables. For example, Chow [5] proved the Kolmogorov strong law of large numbers for weighted sums, and Cuzick [7] generalized the result of Chow [5]. Bai and Cheng [1] proved the Marcinkiewicz-Zygmund strong law of large numbers and the law of single logarithm for weighted sums, and Chen and Gan [2] improved the result of Bai and Cheng [1] by giving the optimal moment condition for the law of single logarithm. The limiting behavior are also obtained for dependent random variables, one can refer to Wu and Wang [15] and their references for more detail.

For an array $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ of constants, set

$$\rho = \inf\left\{u : \sum_{n=1}^{\infty} n^{r-2} \exp\left(-\frac{un\log n}{\sum_{k=1}^{n} a_{nk}^2}\right) < \infty\right\}. \tag{1.5}$$

If (1.4) holds for $\alpha = 2$, it is easy to show that

$$\rho \in \left[(r-1)\liminf_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_{nk}^2, (r-1)\limsup_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_{nk}^2\right] \subset [0, +\infty),$$

and

$$\rho = (r-1)\lim_{n\to\infty} \frac{1}{n}\sum_{k=1}^{n} a_{nk}^2$$

if the limit exists. In particular, $\rho = r-1$ if $a_{n1} = \cdots = a_{nn} = 1$.

We now state the main results. Some preliminary lemmas and the proof of the main result will be detailed in the next section.

THEOREM 1.1. *Let* $r > 1$, $\alpha > 0, \beta > 0$ *with* $1/\alpha + 1/\beta = 1/2$. *Let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying* (1.4), *and let* $\{X, X_n, n \geqslant 1\}$ *be a* *sequence of independent and identically distributed random variables. Suppose that*

$$EX=0,\ \ EX^2=1 \text{ and } \begin{cases} E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{(r-1)\beta}<\infty, & \text{if } \alpha<2r,\\ E|X|^{(r-1)\beta}<\infty, & \text{if } \alpha=2r,\\ E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{2r}<\infty, & \text{if } \alpha>2r. \end{cases} \tag{1.6}$$

Then

$$\sum_{n=1}^{\infty} n^{r-2}P\{|\sum_{k=1}^{n} a_{nk}X_k|>\varepsilon\sqrt{2n\log n}\}\begin{cases}<\infty, & \text{if } \varepsilon>\sqrt{\rho},\\ =\infty, & \text{if } \varepsilon<\sqrt{\rho},\end{cases} \tag{1.7}$$

where ρ *is defined by* (1.5).

Conversely, if

$$\sum_{n=1}^{\infty} n^{r-2}P\{|\sum_{k=1}^{n} a_{nk}X_k|>\varepsilon\sqrt{2n\log n}\}<\infty \tag{1.8}$$

holds for some $\varepsilon>0$ *and for any array* $\{a_{nk}, 1\leqslant k\leqslant n, n\geqslant 1\}$ *satisfying* (1.4), *then* $EX=0$, $E(X^2/\log|X|)^r<\infty$ *and* $E(X^2/\log|X|)^{(r-1)\beta/2}<\infty$.

REMARK 1.1. By the Hölder inequality, (1.4) implies

$$\sum_{k=1}^{n}|a_{nk}|^s=O(n)$$

holds for all $0<s\leqslant\alpha$.

REMARK 1.2. Suppose that $r>1$, $\alpha>0$, $\beta>0$ with $1/\alpha+1/\beta=1/2$. Then the case $\alpha<2r$ is equivalent to the case $2r<(r-1)\beta$, and in this case, $\alpha<2r<(r-1)\beta$. The case $\alpha=2r$ is equivalent to the case $2r=(r-1)\beta$, and in this case, $\alpha=2r=(r-1)\beta$. The case $\alpha>2r$ is equivalent to the case $2r>(r-1)\beta$, and in this case, $\alpha>2r>(r-1)\beta$.

REMARK 1.3. In two cases $\alpha>2r$ and $\alpha<2r$, the moment conditions

$$E(X^2/\log|X|)^r<\infty \quad \text{and} \quad E(X^2/\log|X|)^{(r-1)\beta/2}<\infty$$

are necessary and sufficient, respectively. But in the case $\alpha=2r$, the moment condition $E|X|^{(r-1)\beta}<\infty$ is only sufficient for (1.7). It may be difficult to prove (1.7) under the necessary moment condition $E(X^2/\log|X|)^{(r-1)\beta/2}<\infty$.

By Theorem 1.1 and the Borel-Cantelli lemma, we have the following single law of logarithm at once.

THEOREM 1.2. *Let* $\alpha>0$, $\beta>0$ *with* $1/\alpha+1/\beta=1/2$. *Let* $\{a_{nk}, 1\leqslant k\leqslant n, n\geqslant 1\}$ *be an array of constants satisfying* (1.4), *and let* $\{X, X_{nk}, 1\leqslant k\leqslant n, n\geqslant 1\}$

be an array of independent and identically distributed random variables. Suppose that

$$EX = 0, \ \ EX^2 = 1 \text{ and } \begin{cases} E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{\beta} < \infty, & \text{if } \alpha < 4, \\ E|X|^{\beta} < \infty, & \text{if } \alpha = 4, \\ E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{4} < \infty, & \text{if } \alpha > 4. \end{cases}$$

Then

$$\limsup_{n\to\infty} \frac{|\sum_{k=1}^{n} a_{nk} X_{nk}|}{\sqrt{2n\log n}} = \sqrt{\rho} \ \text{ a.s.,} \tag{1.9}$$

where ρ *is defined by* (1.5) *when* $r = 2$.

Throughout this paper, C always stands for a positive constant which may differ from one place to another. For events A and B, we denote $I(A,B) = I(A \cap B)$, where $I(A)$ is the indicator function of an event A.

2. Lemmas and proof of main result

The main tool in the proof of the main result is from the invariance principle' way to estimate the rate of convergence (see Sakhanenko [12, 13, 14]), which is powerful in the field of limit theory (for example, see Csörgo, Szyszkowicz and Wu [6], Jiang and Zhang [9], Chen and Wang [4], Liu and Meng [11], etc.) and is listed as the following lemma.

LEMMA 2.1. *For any* $q > 2$, *there exists* $B = B(q) > 0$ *satisfying that for any sequence of independent random variables* $\{\xi_k, 1 \leqslant k \leqslant n\}$ *with mean zero and* $E|\xi_k|^q < \infty, 1 \leqslant k \leqslant n$, *there is a sequence* $\{\eta_k, 1 \leqslant k \leqslant n\}$ *of independent normal random variables with* $E\eta_k = 0$, $E\eta_k^2 = E\xi_k^2$ *and for all* $y > 0$,

$$P\left\{\max_{1\leqslant m\leqslant n} \Big|\sum_{k=1}^{m} \xi_k - \sum_{k=1}^{m} \eta_k\Big| > y\right\} \leqslant By^{-q} \sum_{k=1}^{n} E|\xi_k|^q. \tag{2.1}$$

In the following, we always set $a_n = \sqrt{n\log n}$, $b_n = n^{1/\beta}\sqrt{\log n}$, $n \geqslant 1$, $a_0 = b_0 = 0$.

LEMMA 2.2. *Let* $r > 1$, $\alpha > 0$, $\beta > 0$ *with* $1/\alpha + 1/\beta = 1/2$, *and* X *be a random variable. Let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying (1.4). Then*

$$\sum_{n=1}^{\infty} n^{r-2} \sum_{k=1}^{n} P\{|a_{nk}X| > a_n\} \leqslant \begin{cases} CE\left(\frac{|X|}{\log^{1/2}|X|}\right)^{(r-1)\beta}, & \text{if } \alpha < 2r, \\ CE|X|^{(r-1)\beta}, & \text{if } \alpha = 2r, \\ CE\left(\frac{|X|}{\log^{1/2}|X|}\right)^{2r}, & \text{if } \alpha > 2r. \end{cases} \tag{2.2}$$

Proof. *Case* 1: $\alpha \leqslant 2r$. We observe by the Markov inequality that, for any $s>0$,

$$\begin{aligned} P\{|a_{nk}X|>a_n\} &= P\{|a_{nk}X|>a_n, |X|>b_n\}+P\{|a_{nk}X_k|>a_n, |X|\leqslant b_n\} \\ &\leqslant a_n^{-\alpha}|a_{nk}|^{\alpha}E|X|^{\alpha}I(|X|>b_n)+a_n^{-s}|a_{nk}|^{s}E|X|^{s}I(|X|\leqslant b_n). \end{aligned} \tag{2.3}$$

It is easy to show that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-\alpha}\left(\sum_{k=1}^{n}|a_{nk}|^{\alpha}\right)E|X|^{\alpha}I(|X|>b_n) \\ &\leqslant C\sum_{n=1}^{\infty} n^{r-1-\alpha/2}(\log n)^{-\alpha/2}E|X|^{\alpha}I(|X|>b_n) \\ &= C\sum_{n=1}^{\infty} n^{r-1-\alpha/2}(\log n)^{-\alpha/2}\sum_{k=n}^{\infty}E|X|^{\alpha}I(b_k<|X|\leqslant b_{k+1}) \\ &= C\sum_{k=1}^{\infty}E|X|^{\alpha}I(b_k<|X|\leqslant b_{k+1})\sum_{n=1}^{k} n^{r-1-\alpha/2}(\log n)^{-\alpha/2} \\ &\leqslant \begin{cases} C\sum_{k=1}^{\infty} k^{r-\alpha/2}(\log k)^{-\alpha/2}E|X|^{\alpha}I(b_k<|X|\leqslant b_{k+1}), & \text{if } \alpha<2r, \\ C\sum_{k=1}^{\infty} E|X|^{\alpha}I(b_k<|X|\leqslant b_{k+1}), & \text{if } \alpha=2r \end{cases} \\ &\leqslant \begin{cases} E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{(r-1)\beta}, & \text{if } \alpha<2r, \\ E|X|^{(r-1)\beta}, & \text{if } \alpha=2r. \end{cases} \end{aligned} \tag{2.4}$$

Taking $s>\max\{\alpha,(r-1)\beta\}$, we have that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2}\cdot a_n^{-s}\left(\sum_{k=1}^{n}|a_{nk}|^{s}\right)E|X|^{s}I(|X|\leqslant b_n) \\ &\leqslant C\sum_{n=1}^{\infty} n^{r-2-s/2+s/\alpha}(\log n)^{-s/2}E|X|^{s}I(|X|\leqslant b_n) \\ &= C\sum_{n=1}^{\infty} n^{r-2-s/2+s/\alpha}(\log n)^{-s/2}\sum_{k=1}^{n}E|X|^{s}I(b_{k-1}<|X|\leqslant b_k) \\ &= C\sum_{k=1}^{\infty}E|X|^{s}I(b_{k-1}<|X|\leqslant b_k)\sum_{n=k}^{\infty} n^{r-2-s/2+s/\alpha}(\log n)^{-s/2} \\ &\leqslant C\sum_{k=1}^{\infty} k^{r-1-s/2+s/\alpha}(\log k)^{-s/2}E|X|^{s}I(b_{k-1}<|X|\leqslant b_k) \\ &\leqslant CE\left(\frac{|X|}{\log^{1/2}|X|}\right)^{(r-1)\beta}, \end{aligned} \tag{2.5}$$

since $s>(r-1)\beta$. Then (2.2) holds by (2.3)–(2.5).

Case 2: $\alpha > 2r$. The proof is similar to that of Case 1. However, we use a different truncation for X. We observe by the Markov inequality that, for any $t > 0$,

$$\begin{aligned} P\{|a_{nk}X| > a_n\} &= P\{|a_{nk}X| > a_n, |X| > a_n\} + P\{|a_{nk}X_k| > a_n, |X| \leqslant a_n\} \\ &\leqslant a_n^{-t}|a_{nk}|^t E|X|^t I(|X| > a_n) + a_n^{-\alpha}|a_{nk}|^{\alpha} E|X|^{\alpha} I(|X| \leqslant a_n). \end{aligned} \tag{2.6}$$

Taking $0 < t < 2r$, we have that

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-t} \left(\sum_{k=1}^{n} |a_{nk}|^t\right) E|X|^t I(|X| > a_n) \\ &\leqslant C \sum_{n=1}^{\infty} n^{r-1-t/2} (\log n)^{-t/2} E|X|^t I(|X| > a_n) \\ &\leqslant C \sum_{n=1}^{\infty} n^{r-t/2} (\log n)^{-t/2} E|X|^t I(a_n < |X| \leqslant a_{n+1}) \\ &\leqslant CE \left(\frac{|X|}{\log^{1/2}|X|}\right)^{2r}. \end{aligned} \tag{2.7}$$

By the same argument as (2.4),

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-\alpha} \left(\sum_{k=1}^{n} |a_{nk}|^{\alpha}\right) E|X|^{\alpha} I(|X| \leqslant a_n) \\ &\leqslant C \sum_{n=1}^{\infty} n^{r-1-\alpha/2} (\log n)^{-\alpha/2} E|X|^{\alpha} I(|X| \leqslant a_n) \\ &\leqslant CE \left(\frac{|X|}{\log^{1/2}|X|}\right)^{2r}, \end{aligned} \tag{2.8}$$

since $\alpha > 2r$. Then (2.2) holds by (2.6)–(2.8). □

LEMMA 2.3. *Let* $r > 1$, $\alpha > 0$, $\beta > 0$ *with* $1/\alpha + 1/\beta = 1/2$, *and let* X *be a random variable. Let* $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ *be an array of constants satisfying* (1.4). *Then for any* $s > \max\{\alpha, (r-1)\beta\}$,

$$\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n) \leqslant \begin{cases} E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{(r-1)\beta}, & \text{if } \alpha < 2r, \\ E|X|^{(r-1)\beta}, & \text{if } \alpha = 2r, \\ E\left(\frac{|X|}{\log^{1/2}|X|}\right)^{2r}, & \text{if } \alpha > 2r. \end{cases} \tag{2.9}$$

Proof. *Case* 1: $\alpha \leqslant 2r$. By (2.4) and (2.5) we get that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n) \\
&= \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n, |X| > b_n) \\
&\quad + \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n, |X| \leqslant b_n) \\
&\leqslant \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-\alpha} \left(\sum_{k=1}^{n} |a_{nk}|^\alpha \right) E|X|^\alpha I(|X| > b_n) \\
&\quad + \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \left(\sum_{k=1}^{n} |a_{nk}|^s \right) E|X|^s I(|X| \leqslant b_n) \\
&\leqslant \begin{cases} CE\left(\frac{|X|}{\log^{1/2}|X|}\right)^{(r-1)\beta}, & \text{if } \alpha < 2r, \\ CE|X|^{(r-1)\beta}, & \text{if } \alpha = 2r. \end{cases}
\end{aligned}
$$

Case 2: $\alpha > 2r$. Taking $0 < t < 2r$, we have by (2.7) and (2.8) that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n) \\
&= \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n, |X| > a_n) \\
&\quad + \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-s} \sum_{k=1}^{n} E|a_{nk}X|^s I(|a_{nk}X| \leqslant a_n, |X| \leqslant a_n) \\
&\leqslant \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-t} \left(\sum_{k=1}^{n} |a_{nk}|^t \right) E|X|^t I(|X| > a_n) \\
&\quad + \sum_{n=1}^{\infty} n^{r-2} \cdot a_n^{-\alpha} \left(\sum_{k=1}^{n} |a_{nk}|^\alpha \right) E|X|^\alpha I(|X| \leqslant b_n) \\
&\leqslant CE\left(\frac{|X|}{\log^{1/2}|X|} \right)^{2r}.
\end{aligned}
$$

Therefore (2.9) holds. □

LEMMA 2.4. *Let X be a random variable, let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of constants satisfying* (1.4) *for some* $\alpha \geqslant 2$. *If* $E|X|^2 < \infty$, *then*

$$
a_n^{-1} \sum_{k=1}^{n} E|a_{nk}X| I(|a_{nk}X| > a_n) \to 0 \tag{2.10}
$$

as $n \to \infty$, and hence, in addition, if $EX = 0$, then

$$a_n^{-1}\left|\sum_{k=1}^{n} a_{nk}EXI(|a_{nk}X| \leqslant a_n)\right| \to 0 \tag{2.11}$$

as $n \to \infty$.

Proof. Note that

$$\begin{aligned} a_n^{-1}\sum_{k=1}^{n} E|a_{nk}X|I(|a_{nk}X| > a_n) &\leqslant a_n^{-2}\sum_{k=1}^{n} E|a_{nk}X|^2 I(|a_{nk}X| > a_n) \\ &\leqslant a_n^{-2}\sum_{k=1}^{n} E|a_{nk}X|^2 \\ &\leqslant (\log n)^{-1}\cdot\left(n^{-1}\sum_{k=1}^{n}|a_{nk}|^2\right)\cdot EX^2 \\ &\to 0 \end{aligned}$$

as $n \to \infty$.Hence (2.10) holds. If, in addition, $EX = 0$, then we get by (2.10) that

$$\begin{aligned} a_n^{-1}\left|\sum_{k=1}^{n} a_{nk}EXI(|a_{nk}X| \leqslant a_n)\right| &= a_n^{-1}\left|\sum_{k=1}^{n} a_{nk}EXI(|a_{nk}X| > a_n)\right| \\ &\leqslant a_n^{-1}\sum_{k=1}^{n} E|a_{nk}X|I(|a_{nk}X| > a_n) \to 0 \end{aligned}$$

as $n \to \infty$. Hence (2.11) holds. □

Proof of Theorem 1.1. Sufficiency. Set

$$S_n = \sum_{k=1}^{n} a_{nk}X_k, T_n = \sum_{k=1}^{n} a_{nk}X_k I(|a_{nk}X_k| \leqslant a_n).$$

We first prove that

$$\sum_{n=1}^{\infty} n^{r-2}P\{|S_n| > \varepsilon\sqrt{2n\log n}\} < \infty, \ \ \forall\, \varepsilon > \sqrt{\rho}. \tag{2.12}$$

Note that

$$\{|S_n| > \varepsilon\sqrt{2n\log n}\} \subset \{\max_{1\leqslant k\leqslant n}|a_{nk}X_k| > a_n\} \cup \left\{|T_n| > \varepsilon\sqrt{2n\log n}\right\},$$

and by Lemma 2.2,

$$\sum_{n=1}^{\infty} n^{r-2}P\{\max_{1\leqslant k\leqslant n}|a_{nk}X_k| > a_n\} \leqslant \sum_{n=1}^{\infty}\sum_{k=1}^{n} P\{|a_{nk}X| > a_n\} < \infty, \tag{2.13}$$

and by Lemma 2.4,

$$\frac{1}{a_n}|ET_n| \to 0 \tag{2.14}$$

as $n \to \infty$. Hence to prove (2.12), it is enough to prove that

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{|T_n - ET_n| > \varepsilon\sqrt{2n\log n}\right\} < \infty, \ \ \forall\, \varepsilon > \sqrt{\rho}. \tag{2.15}$$

We can get from Lemma 2.1 that for any $n \geqslant 1$, there exists independent normal random variables Z_{nk} with $EZ_{nk} = 0$ and $EZ_{nk}^2 = E(a_{nk}X_k I(|a_{nk}X_k| \leqslant a_n) - Ea_{nk}X_k I(|a_{nk}X_k| \leqslant a_n))^2$, $1 \leqslant k \leqslant n$, such that for any $q > 2$ and all $y > 0$

$$\begin{aligned} &P\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > y\right\} \\ &\leqslant Ay^{-q} \sum_{k=1}^{n} E|a_{nk}X_k I(|a_{nk}X_k| \leqslant a_n) - Ea_{nk}X_k I(|a_{nk}X_k| \leqslant a_n)|^q. \end{aligned} \tag{2.16}$$

Note that

$$\begin{aligned} \left\{|T_n - ET_n| > \varepsilon\sqrt{2n\log n}\right\} &\subset \left\{|(T_n - ET_n) - \sum_{k=1}^{k} Z_{nk}| > \varepsilon_1\sqrt{2n\log n}\right\} \\ &\quad \cup \left\{|\sum_{k=1}^{n} Z_{nk}| > \varepsilon_2\sqrt{2n\log n}\right\}, \end{aligned}$$

where $\varepsilon_1 > 0, \varepsilon_2 > \sqrt{\rho}$ with $\varepsilon = \varepsilon_1 + \varepsilon_2$. Hence

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{r-2} P\left\{|T_n - ET_n| > \varepsilon\sqrt{2n\log n}\right\} \\ &\leqslant \sum_{n=1}^{\infty} n^{r-2} P\left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > \varepsilon_1\sqrt{2n\log n}\right\} \\ &\quad + \sum_{n=1}^{\infty} n^{r-2} P\left\{|\sum_{k=1}^{n} Z_{nk}| > \varepsilon_2\sqrt{2n\log n}\right\} \\ &= I_1 + I_2. \end{aligned}$$

By (2.16) and Lemma 2.3 we can derive that for $q > \max\{\alpha, (r-1)\beta\}$

$$\begin{aligned} I_1 &\leqslant C\sum_{n=1}^{\infty} \frac{n^{r-2}}{a_n^q} \sum_{k=1}^{n} E|a_{nk}X_k I(|a_{nk}X_k| \leqslant a_n) - Ea_{nk}X_k I(|a_{nk}X_k| \leqslant a_n)|^q \\ &\leqslant C\sum_{n=1}^{\infty} \frac{n^{r-2}}{a_n^q} \sum_{k=1}^{n} E|a_{nk}X_k|^q I(|a_{nk}X_k| \leqslant a_n) \\ &< \infty. \end{aligned}$$

Let N be a standard normal random variable. Note that

$$EZ_{nk}^2 = E(a_{nk}X_kI(|a_{nk}X_k| \leqslant a_n) - Ea_{nk}X_kI(|a_{nk}X_k| \leqslant a_n))^2 \leqslant a_{nk}^2$$

for all $1 \leqslant k \leqslant n$ and $n \geqslant 1$ since $EX^2 = 1$, and $P\{|N| > x\} \sim \sqrt{2/\pi}\, x^{-1}e^{-x^2/2}$. Hence for large enough n,

$$P\left\{|\sum_{k=1}^{n} Z_{nk}| > \varepsilon_2\sqrt{2n\log n}\right\} = P\left\{|N| > \frac{\varepsilon_2\sqrt{2n\log n}}{\sqrt{\sum_{k=1}^{n} EZ_{nk}^2}}\right\} \leqslant C\exp\left\{-\frac{\varepsilon_2^2 n\log n}{\sum_{k=1}^{n} a_{nk}^2}\right\}.$$

Then $I_2 < \infty$ since $\varepsilon_2^2 > \rho$.

Now we prove that

$$\sum_{n=1}^{\infty} n^{r-2}P\{|S_n| > \varepsilon\sqrt{2n\log n}\} = \infty, \tag{2.17}$$

holds for any $\varepsilon < \sqrt{\rho}$. If $\varepsilon \leqslant 0$, (2.17) holds trivially. So we can assume that $\varepsilon > 0$ and $\rho > 0$. Note that

$$\{|T_n| > \varepsilon\sqrt{2n\log n}\} \subset \{\max_{1\leqslant k\leqslant n} |a_{nk}X_k| > a_n\} \cup \left\{|S_n| > \varepsilon\sqrt{2n\log n}\right\}$$

Hence, by (2.13) and (2.14), to prove (2.17) it suffices to prove that

$$\sum_{n=1}^{\infty} n^{r-2}P\left\{|T_n - ET_n| > \varepsilon\sqrt{2n\log n}\right\} = \infty, \ \ \forall\, \varepsilon < \sqrt{\rho}. \tag{2.18}$$

Note that for any $0 < \varepsilon < \sqrt{\rho}$, if we take $\varepsilon_3 > 0$ and $0 < \varepsilon_4 < \sqrt{\rho}$ with $\varepsilon_4 = \varepsilon_3 + \varepsilon$, then

$$\begin{aligned}\left\{|\sum_{k=1}^{n} Z_{nk}| > \varepsilon_4\sqrt{2n\log n}\right\} &\subset \left\{|(T_n - ET_n) - \sum_{k=1}^{n} Z_{nk}| > \varepsilon_3\sqrt{2n\log n}\right\} \\ &\quad \cup \left\{|T_n - ET_n| > \varepsilon\sqrt{2n\log n}\right\},\end{aligned}$$

Then by $I_1 < \infty$, to prove (2.18), it is enough to show that

$$\sum_{n=1}^{\infty} n^{r-2}P\left\{|\sum_{k=1}^{n} Z_{nk}| > \varepsilon_4\sqrt{2n\log n}\right\} = \infty. \tag{2.19}$$

Set $A = \sup_{n\geqslant 1}(n^{-1}\sum_{k=1}^{n} a_{nk}^2)^{1/2}$, then $|a_{nk}| \leqslant A\sqrt{n}$. Note that by $EX = 0$ and $EX^2 = 1$,

$$\begin{aligned}1 &\geqslant \frac{\sum_{k=1}^{n} E|a_{nk}X_k|^2 I(|a_{nk}X_k| \leqslant a_n) - \sum_{k=1}^{n}(Ea_{nk}X_kI(|a_{nk}X_k| \leqslant a_n))^2}{\sum_{k=1}^{n} a_{nk}^2} \\ &= 1 - \frac{\sum_{k=1}^{n} E|a_{nk}X_k|^2 I(|a_{nk}X_k| > a_n) + \sum_{k=1}^{n}(Ea_{nk}X_kI(|a_{nk}X_k| > a_n))^2}{\sum_{k=1}^{n} a_{nk}^2} \\ &\geqslant 1 - \frac{2\sum_{k=1}^{n} E|a_{nk}X_k|^2 I(|a_{nk}X_k| > a_n)}{\sum_{k=1}^{n} a_{nk}^2} \geqslant 1 - \frac{2\sum_{k=1}^{n} E|a_{nk}X_k|^2 I(|A\sqrt{n}X_k| > a_n)}{\sum_{k=1}^{n} a_{nk}^2} \\ &= 1 - 2EX^2I(|AX| > \sqrt{\log n}) \to 1\end{aligned}$$

as $n\to\infty$. Then there exists $\delta<1$ close to 1 enough with $u_1=\varepsilon_4^2/\delta<\rho$ such that

$$\sum_{k=1}^{n} EZ_{nk}^2=\sum_{k=1}^{n} E|a_{nk}X_k|^2 I(|a_{nk}X_k|\leqslant a_n)-\sum_{k=1}^{n}(Ea_{nk}X_k I(|a_{nk}X_k|\leqslant a_n))^2\geqslant \delta\sum_{k=1}^{n} a_{nk}^2$$

for n large enough. Using $P\{|N|>x\}\sim\sqrt{2/\pi}\,x^{-1}e^{-x^2/2}$ again,

$$P\left\{|\sum_{k=1}^{n} Z_{nk}|>\varepsilon_4\sqrt{2n\log n}\right\}=P\left\{|N|>\frac{\varepsilon_4\sqrt{2n\log n}}{\sqrt{\sum_{k=1}^n EZ_{nk}^2}}\right\}\geqslant P\left\{|N|>\frac{\varepsilon_4\sqrt{2n\log n}}{\sqrt{\delta\sum_{k=1}^n a_{nk}^2}}\right\}$$
$$\sim\sqrt{\frac{\delta}{\pi\varepsilon_4^2}}\cdot\sqrt{\frac{\sum_{k=1}^n a_{nk}^2}{n\log n}}\exp\left\{-\frac{u_1 n\log n}{\sum_{k=1}^n a_{nk}^2}\right\}.$$

Taking $u_2\in(u_1,\rho)$, then

$$\sqrt{\frac{\sum_{k=1}^n a_{nk}^2}{n\log n}}\exp\left\{-\frac{u_1 n\log n}{\sum_{k=1}^n a_{nk}^2}\right\}\Big/\exp\left\{-\frac{u_2 n\log n}{\sum_{k=1}^n a_{nk}^2}\right\}$$
$$=\sqrt{\frac{\sum_{k=1}^n a_{nk}^2}{n\log n}}\exp\left\{\frac{(u_2-u_1)n\log n}{\sum_{k=1}^n a_{nk}^2}\right\}\to\infty$$

as $n\to\infty$. Therefore (2.19) holds by the fact that the series

$$\sum_{n=1}^{\infty} n^{r-2}\exp\left\{-\frac{u_2 n\log n}{\sum_{k=1}^n a_{nk}^2}\right\}=\infty.$$

Necessary. Set $a_{nk}=1$ for all $1\leqslant k\leqslant n$ and $n\geqslant 1$. Then (1.8) can be rewritten as

$$\sum_{n=1}^{\infty} n^{r-2}P\{|\sum_{k=1}^{n} a_{nk}X_{nk}|>\varepsilon\sqrt{2n\log n}\}<\infty$$

for some $\varepsilon>0$. Then by Lai [10], $EX=0$, and $E(X^2/\log|X|)^r<\infty$. Set $a_{nk}=0$ if $1\leqslant k\leqslant n-1$ and $a_{nn}=n^{1/\alpha}$, Then (1.8) can be rewritten as

$$\sum_{n=1}^{\infty} n^{r-2}P\{|n^{1/\alpha}X_n|>\varepsilon\sqrt{2n\log n}\}<\infty$$

for some $\varepsilon>0$, which is equivalent to $E(X^2/\log|X|)^{(r-1)\beta/2}<\infty$. The proof is completed. □

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