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INTERPOLATION INEQUALITIES OF NUMERICAL RADIUS

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Abstract. In this paper, we give several generalization and refinement of numerical radius inequalities of bounded linear operators on a complex Hilbert space. It's shown that the bounds obtained here are stronger than the known bounds of numerical radius inequalities. Moreover, we use the properties of operator convex functions to obtain several interpolation inequalities of numerical radius.

1. Introduction

Let $\mathscr{H}$ denote a complex Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $\mathbb{B}(\mathscr{H})$ denote the collections of all bounded linear operator on $\mathscr{H}$. For $A\in\mathbb{B}(\mathscr{H})$, $\|A\|$ is the operator norm of A. An operator A on $\mathscr{H}$ is said to be positive if $\langle Ax,x\rangle\geqslant 0$ for all $x\in\mathscr{H}$. We write $A>0$ if A is positive and invertible. $A\in\mathbb{B}(\mathscr{H})$ is said to be hyponormal if $A^*A-AA^*\geqslant 0$. Recall that $\|A\|:=\sup_{\|x\|=1}\|Ax\|=\sup_{\|x\|=1,\|y\|=1}|\langle Ax,y\rangle|$. The numerical range of A is defined as $\omega(A)=\sup_{\|x\|=1}|\langle Ax,x\rangle|$. For $A\in\mathbb{B}(\mathscr{H})$, A^* denotes the adjoint of A and $|A|$, $|A^*|$ respectively denote the positive part of A,A^*, i.e., $|A|=(A^*A)^{\frac{1}{2}}$, $|A^*|=(AA^*)^{\frac{1}{2}}$. The numerical radius is a norm on $\mathbb{B}(\mathscr{H})$ satisfying the following inequalities

$$\frac{\|A\|}{2}\leqslant\omega(A)\leqslant\|A\|. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2=0$. The second inequality becomes an equality if A is normal ($A^*A=AA^*$).

Many authors discussed different proofs, generalizations, refinements and applications of inequality (1.1). For more information on this topic, the reader is referred to [4–7].

In [14], Kittaneh improved the second inequality in (1.1)

$$\omega(A)\leqslant\frac{1}{2}\||A|+|A^*|\|\leqslant\frac{1}{2}\|A\|+\frac{1}{2}\|A^2\|^{\frac{1}{2}}. \tag{1.2}$$

In [15], he also showed another refinement of inequalities (1.1)

$$\frac{1}{4}\||A|^2+|A^*|^2\|\leqslant\omega^2(A)\leqslant\frac{1}{2}\||A|^2+|A^*|^2\|. \tag{1.3}$$

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A generalization of the second inequality (1.3) has been given in [7]. That is for $A \in \mathbb{B}(\mathscr{H})$ and $r \geqslant 1$,

$$\omega^{2r}(A) \leqslant \frac{1}{2}\||A|^{2r} + |A^*|^{2r}\|. \tag{1.4}$$

In [6], Dragomir showed

$$\omega^{r}(B^*A) \leqslant \frac{1}{2}\||A|^{r} + |B|^{r}\|. \tag{1.5}$$

Moreover, he also applied a different approach to obtain

$$\omega^{2r}(B^*A) \leqslant \|\alpha|A|^{\frac{r}{\alpha}} + (1-\alpha)|B|^{\frac{r}{1-\alpha}}\|, \tag{1.6}$$

where $r \geqslant 1$, $0 < \alpha < 1$.

Recently, Bhunia and Paul obtained the following inequalities in [3]

$$\begin{aligned} &\frac{1}{4}\|A^*A + AA^*\| \\ \leqslant &\frac{1}{8}\left[\left(\|A+A^*\|^2 + \|A-A^*\|^2\right)^2 + \frac{1}{2}\left(\|A+A^*\|^2 - \|A-A^*\|^2\right)^2\right]^{\frac{1}{2}} \leqslant \omega^2(A). \end{aligned} \tag{1.7}$$

In this paper, we obtain the numerical radius inequality which improves the second inequality in (1.7). For $A \in \mathbb{B}(\mathscr{H})$, we prove

$$\begin{aligned} &\frac{1}{4}\|A^*A + AA^*\| \\ \leqslant &\frac{1}{8}\left[\left(\|A+A^*\|^2 + \|A-A^*\|^2\right)^2 + \frac{3}{4}\left(\|A+A^*\|^2 - \|A-A^*\|^2\right)^2\right]^{\frac{1}{2}} \\ \leqslant &\omega^2(A). \end{aligned}$$

In the second section, we get some improvements and interpolations of numerical radius inequalities via operator convex function. These results present several general forms and refinements of some known inequalities in the literature, such as

$$\begin{aligned} f\left(\omega^{2r}(B^*A)\right) &\leqslant \left\|\int_0^1 f\left((1-t)\left(\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right) + t\omega^{2r}(B^*A)I\right)dt\right\| \\ &\leqslant \left\|f\left(\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right)\right\|, \quad \forall \alpha \in [0,1], \quad r \geqslant 1. \end{aligned}$$

If $f = t^2$, then we can get

$$\begin{aligned} \omega^{2r}(B^*A) &\leqslant \left\|\int_0^1 \left((1-t)\left(\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right) + t\omega^{2r}(B^*A)I\right)^2 dt\right\|^{\frac{1}{2}} \\ &\leqslant \left\|\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right\|, \quad \forall \alpha \in [0,1], \quad r \geqslant 1. \end{aligned}$$

This is an improvement of inequality (1.6).

The purpose of this work is to eatablish some new numerical radius inequalities for Hilbert space operators that improve the existing results in the literature.

2. Some refinements of numerical radius inequalities

In this section, we mainly establish several refinement of numerical radius inequalities (1.3). Furthermore, we get a refinement of inequality (1.1). To prove our results we need the following basic lemmas.

LEMMA 2.1. ([1]) *Let $x,y,e \in \mathscr{H}$ with $\|e\| = 1$. Then*

$$|\langle x,e\rangle\langle e,y\rangle| \leqslant \frac{1}{2}(\|x\|\|y\| + |\langle x,y\rangle|).$$

LEMMA 2.2. ([2]) *Let $A,D \in \mathbb{B}(\mathscr{H})$ be positive, then*

$$\|AD\| \leqslant \frac{1}{4}\|A+D\|^2.$$

LEMMA 2.3. ([4]) *Let $A,B \in \mathbb{B}(\mathscr{H})$ be selfadjoint. Then*

$$\|A+B\| \leqslant \sqrt{\omega^2(A+\mathrm{i}B) + \|A\|\|B\| + \omega(BA)}.$$

LEMMA 2.4. ([9]) *Let $A \in \mathbb{B}(\mathscr{H})$ be a positive operator. Then*

$$\langle Ax,x\rangle^{\lambda}\left(1+2(\lambda-1)\left(1-\frac{\left\langle A^{\frac{1}{2}}x,x\right\rangle}{\langle Ax,x\rangle^{\frac{1}{2}}}\right)\right) \leqslant \left\langle A^{\lambda}x,x\right\rangle$$

for any $\lambda \geqslant 1$ and $x \in \mathscr{H}$ with $\|x\| = 1$.

LEMMA 2.5. ([13]) *Let $A \in \mathbb{B}(\mathscr{H})$. Let f and g be nonnegative functions on $[0,\infty]$ which are continuous and satisfy the relation $f(t)g(t) = t$ for all $t \in [0,\infty]$. Then*

$$|\langle Ax,y\rangle| \leqslant \|f(|A|)x\|\,\|g(|A^*|)y\|$$

for all $x,y \in \mathscr{H}$.

The following theorem shows that inequality (2.1) is a refinement of inequality (1.7).

THEOREM 2.6. *Let $A \in \mathbb{B}(\mathscr{H})$. Then*

$$\begin{aligned}
&\frac{1}{4}\|A^*A+AA^*\| \\
\leqslant&\frac{1}{8}\left[\left(\|A+A^*\|^2+\|A-A^*\|^2\right)^2+\frac{3}{4}\left(\|A+A^*\|^2-\|A-A^*\|^2\right)^2\right]^{\frac{1}{2}} \qquad (2.1)\\
\leqslant&\omega^2(A).
\end{aligned}$$

Proof. Let $A = B + \mathrm{i}C$ be the Cartesian decomposition of A, where $B = \mathrm{Re}(A) := \frac{A+A^*}{2}$ and $C = \mathrm{Im}(A) := \frac{A-A^*}{2\mathrm{i}}$. We observe that

$$\frac{1}{4}\|A^*A + AA^*\| = \frac{1}{2}\left\|B^2 + C^2\right\|.$$

Then,

$$\begin{aligned}
\frac{1}{4}\|A^*A + AA^*\| &= \frac{1}{2}\left\|B^2 + C^2\right\| \\
&\leqslant \frac{1}{2}\left[\omega^2\left(B^2 + iC^2\right) + \left\|B^2\right\|\left\|C^2\right\| + \omega\left(B^2C^2\right)\right]^{\frac{1}{2}} \quad \text{(by Lemma 2.3)} \\
&\leqslant \frac{1}{2}\left[\left\|B^4 + C^4\right\| + \|B\|^2\|C\|^2 + \left\|B^2C^2\right\|\right]^{\frac{1}{2}} \\
&\qquad\qquad \text{(since } \omega^2(B^2 + iC^2) \leqslant \|B^4 + C^4\|\text{)} \\
&\leqslant \frac{1}{2}\left[\|B\|^4 + \|C\|^4 + \|B\|^2\|C\|^2 + \frac{1}{4}\left\|B^2 + C^2\right\|^2\right]^{\frac{1}{2}} \quad \text{(by Lemma 2.2)} \\
&\leqslant \frac{1}{2}\left[\|B\|^4 + \|C\|^4 + \|B\|^2\|C\|^2 + \frac{1}{4}\left(\|B\|^2 + \|C\|^2\right)^2\right]^{\frac{1}{2}} \\
&\leqslant \frac{1}{2}\left[\|B\|^4 + \|C\|^4 + \frac{1}{2}\left(\|B\|^4 + \|C\|^4\right) + \frac{1}{4}\left(\|B\|^2 + \|C\|^2\right)^2\right]^{\frac{1}{2}}.
\end{aligned}$$

This implies the first inequality in (2.1). Since $\|B\| \leqslant \omega(A)$ and $\|C\| \leqslant \omega(A)$, the second inequality in (2.1) holds. □

THEOREM 2.7. *Let $A, B \in \mathbb{B}(\mathscr{H})$, then*

$$\begin{aligned}
\|A + B\| &\leqslant \sqrt{\|A + iB\|^2 + \|A\|\|B\| + \omega(B^*A)} \\
&\leqslant \|A\| + \|B\|.
\end{aligned} \tag{2.2}$$

Proof. Let $x, y \in \mathscr{H}$ and $\|x\| = \|y\| = 1$, then we have

$$\begin{aligned}
|\langle (A+B)x, y\rangle|^2 &\leqslant (|\langle Ax, y\rangle| + |\langle Bx, y\rangle|)^2 \\
&= |\langle Ax, y\rangle|^2 + |\langle Bx, y\rangle|^2 + 2|\langle Ax, y\rangle\langle Bx, y\rangle| \\
&\qquad\qquad \text{(by } (|a| + |b|)^2 = |a|^2 + |b|^2 + 2|ab|\text{)} \\
&= |\langle Ax, y\rangle|^2 + |\langle Bx, y\rangle|^2 + 2|\langle Ax, y\rangle\langle y, Bx\rangle| \\
&\leqslant |\langle Ax, y\rangle|^2 + |\langle Bx, y\rangle|^2 + \|Ax\|\|Bx\| + |\langle Ax, Bx\rangle| \quad \text{(by Lemma 2.1)} \\
&= |\langle Ax, y\rangle + i\langle Bx, y\rangle|^2 + \|Ax\|\|Bx\| + |\langle Ax, Bx\rangle| \\
&\qquad\qquad \text{(by } |a|^2 + |b|^2 = |a + ib|^2\text{)} \\
&= |\langle (A + iB)x, y\rangle|^2 + \|Ax\|\|Bx\| + |\langle Ax, Bx\rangle|.
\end{aligned}$$

Now, taking the supremum over $x,y \in \mathscr{H}$ with $\|x\| = \|y\| = 1$ in the above inequalities we get

$$\|A+B\|^2 \leqslant \|A+iB\|^2 + \|A\|\|B\| + \omega(B^*A).$$

Hence

$$\|A+B\| \leqslant \sqrt{\|A+iB\|^2 + \|A\|\|B\| + \omega(B^*A)}.$$

Then we prove the second inequality of (2.2),

$$\begin{aligned}(\|A\|+\|B\|)^2 &= \|A\|^2 + \|B\|^2 + \|A\|\|B\| + \|A\|\|B\| \\ &\geqslant \|A\|^2 + \|B\|^2 + \|A\|\|B\| + \|B^*A\| \\ &\geqslant \|A+iB\|^2 + \|A\|\|B\| + \omega(B^*A).\end{aligned}$$

Hence

$$\sqrt{\|A+iB\|^2 + \|A\|\|B\| + \omega(B^*A)} \leqslant \|A\| + \|B\|.$$

To some up, we can get inequality (2.2). □

THEOREM 2.8. *Let $A \in \mathbb{B}(\mathscr{H})$ and let f,g be nonnegative continuous functions on $[0,\infty)$ satisfying $f(t)g(t) = t(t \geqslant 0)$, then*

$$\begin{aligned}\omega^{2r}(A) &\leqslant \frac{1}{4}\left\|f^{4r}(|A|) + g^{4r}(|A^*|)\right\| + \frac{1}{4}\left\|f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\right\| \\ &\leqslant \frac{1}{4}\left\|f^{4r}(|A|) + g^{4r}(|A^*|)\right\| + \frac{1}{2}\omega(f^{2r}(|A|)g^{2r}(|A^*|)) \qquad (2.3)\end{aligned}$$

for all $r \geqslant 1$.

Proof. Let $x \in \mathscr{H}$ and $\|x\| = 1$, then we have

$$\begin{aligned}|\langle Ax,x\rangle|^{2r} &\leqslant \left\langle f^2(|A|)x,x\right\rangle^r \left\langle g^2(|A^*|)x,x\right\rangle^r \quad \text{(by Lemma 2.5)} \\ &\leqslant \left\langle f^{2r}(|A|)x,x\right\rangle \left\langle g^{2r}(|A^*|)x,x\right\rangle \quad (\text{since } \langle Ax,x\rangle^r \leqslant \langle A^r x,x\rangle (r \geqslant 1)) \\ &\leqslant \left(\frac{\left\langle f^{2r}(|A|)x,x\right\rangle + \left\langle g^{2r}(|A^*|)x,x\right\rangle}{2}\right)^2 \\ &= \frac{1}{4}\left\langle \left(f^{2r}(|A|) + g^{2r}(|A^*|)\right)x,x\right\rangle^2 \\ &\leqslant \frac{1}{4}\left\langle \left(f^{2r}(|A|) + g^{2r}(|A^*|)\right)^2 x,x\right\rangle \\ &= \frac{1}{4}\left\langle \left(f^{4r}(|A|) + g^{4r}(|A^*|) + f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\right)x,x\right\rangle \\ &\leqslant \frac{1}{4}\left\|f^{4r}(|A|) + g^{4r}(|A^*|) + f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\right\| \\ &\leqslant \frac{1}{4}\left(\left\|f^{4r}(|A|) + g^{4r}(|A^*|)\right\| + \left\|f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\right\|\right).\end{aligned}$$

Now, taking the supremum over $x \in \mathscr{H}$ with $\|x\| = 1$ in the above inequalities we get

$$\omega^{2r}(A) \leqslant \frac{1}{4}\left\|f^{4r}(|A|) + g^{4r}(|A^*|)\right\| + \frac{1}{4}\left\|f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\right\|.$$

In order to get the second inequality in (2.3), we need to prove the following inequalities

$$\begin{aligned}\omega(f^{2r}(|A|)g^{2r}(|A^*|)) &= \frac{1}{2}\omega(f^{2r}(|A|)g^{2r}(|A^*|)) + \frac{1}{2}\omega(g^{2r}(|A^*|)f^{2r}(|A|)) \\ &\qquad\qquad (\text{since } \omega(A) = \omega(A^*)) \\ &\geqslant \frac{1}{2}\omega\big(f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\big) \\ &\qquad\qquad (\text{since } \omega(A+B) \leqslant \omega(A) + \omega(B)) \\ &= \frac{1}{2}\|f^{2r}(|A|)g^{2r}(|A^*|) + g^{2r}(|A^*|)f^{2r}(|A|)\|. \quad \square\end{aligned}$$

REMARK 2.9. If we take $f(t) = t^{1-v}$ and $g(t) = t^v$ with $0 \leqslant v \leqslant 1$ in inequality (2.3), then

$$\begin{aligned}\omega^{2r}(A) &\leqslant \frac{1}{4}\left\||A|^{4r(1-v)} + |A^*|^{4rv}\right\| + \frac{1}{4}\left\||A|^{2r(1-v)}|A^*|^{2rv} + |A^*|^{2rv}|A|^{2r(1-v)}\right\| \\ &\leqslant \frac{1}{4}\left\||A|^{4r(1-v)} + |A^*|^{4rv}\right\| + \frac{1}{2}\omega(|A|^{2r(1-v)}|A^*|^{2rv}).\end{aligned}$$

THEOREM 2.10. *For any $A, B \in \mathbb{B}(\mathscr{H})$ and $r \geqslant 1$, we have*

$$\begin{aligned}\omega^{2r}(B^*A) &\leqslant \frac{1}{4}\left\||A|^{4r} + |B|^{4r}\right\| + \frac{1}{4}\left\||A|^{2r}|B|^{2r} + |A|^{2r}|B|^{2r}\right\| \\ &\leqslant \frac{1}{4}\left\||A|^{4r} + |B|^{4r}\right\| + \frac{1}{2}\omega\left(|A|^{2r}|B|^{2r}\right) \\ &\leqslant \frac{1}{2}\left\||A|^{4r} + |B|^{4r}\right\|.\end{aligned} \tag{2.4}$$

Proof. The following inequality can be obtained by Cauchy-Schwarz inequality

$$|\langle B^*Ax, x\rangle|^{2r} \leqslant \left\langle |A|^{2r}x, x\right\rangle \left\langle |B|^{2r}x, x\right\rangle.$$

We can get the first and second inequalities in (2.4) use the same method as Theorem 2.8, the last inequality in (2.4) is demonstrated by proving the following inequalities

$$\begin{aligned}\frac{1}{4}\left\||A|^{4r} + |B|^{4r}\right\| + \frac{1}{2}\omega\left(|A|^{2r}|B|^{2r}\right) &\leqslant \frac{1}{4}\left\||A|^{4r} + |B|^{4r}\right\| + \frac{1}{2}\left\||A|^{2r}|B|^{2r}\right\| \\ &\leqslant \frac{1}{4}\||A|^{4r} + |B|^{4r}\| + \frac{1}{2}\left\|\left(\frac{|A|^{4r} + |B|^{4r}}{2}\right)\right\| \\ &\leqslant \frac{1}{4}\left\||A|^{4r} + |B|^{4r}\right\| + \frac{1}{4}\left\||A|^{4r} + |B|^{4r}\right\| \\ &= \frac{1}{2}\left\||A|^{4r} + |B|^{4r}\right\|,\end{aligned} \tag{2.5}$$

where the second inequality in (2.5) follows from [3, Corollary 3.16]. It says that let $A,B \in \mathbb{B}(\mathscr{H})$, then $\|AB^*\| \leqslant \left\|\frac{|A|^2+|B|^2}{2}\right\|$. $\square$

COROLLARY 2.11. *Assume that $A \in \mathbb{B}(\mathscr{H})$ and $r \geqslant 1$ is arbitrary real number, then let $v = \frac{1}{2}$ in Remark 2.9, we can get*

$$\begin{aligned}\omega^{2r}(A) &\leqslant \frac{1}{4}\left\||A|^{2r}+|A^*|^{2r}\right\| + \frac{1}{4}\left\||A|^r|A^*|^r+|A^*|^r|A|^r\right\| \\ &\leqslant \frac{1}{4}\left\||A|^{2r}+|A^*|^{2r}\right\| + \frac{1}{2}\omega\left(|A|^r|A^*|^r\right) \\ &\leqslant \frac{1}{2}\left\||A|^{2r}+|A^*|^{2r}\right\|.\end{aligned}$$

The above inequality is an improvement of inequality (1.3).

In the following theorem, we get the improvement of the second inequality in Corollary 2.11.

THEOREM 2.12. *Let $A \in \mathbb{B}(\mathscr{H})$, $r \geqslant 1$. Then*

$$\omega^{2r}(A) \leqslant \frac{1}{4\mu\gamma}\left\||A|^{2r}+|A^*|^{2r}\right\| + \frac{1}{2\mu\gamma}w\left(|A|^r|A^*|^r\right),$$

where

$$\mu = \inf\left\{1+2(r-1)\left(1-\frac{\langle |A|^{\frac{1}{2}}x,x\rangle}{\langle |A|x,x\rangle^{\frac{1}{2}}}\right) : \|x\|=1\right\},$$

$$\gamma = \inf\left\{1+2(r-1)\left(1-\frac{\langle |A^*|^{\frac{1}{2}}x,x\rangle}{\langle |A^*|x,x\rangle^{\frac{1}{2}}}\right) : \|x\|=1\right\}.$$

Proof. Let $x \in \mathscr{H}$ with $\|x\|=1$. Considering $f(t)=g(t)=t^{\frac{1}{2}}$ in Lemma 2.5, we have

$$|\langle Ax,x\rangle|^2 \leqslant \langle |A|x,x\rangle \langle |A^*|x,x\rangle.$$

So we get

$$\begin{aligned}|\langle Ax,x\rangle|^{2r} &\leqslant \langle |A|x,x\rangle^r \langle |A^*|x,x\rangle^r \\ &\leqslant \frac{1}{\mu\gamma}\langle |A^*|^r x,x\rangle\langle x,|A|^r x\rangle \qquad \text{(by Lemma 2.4)} \\ &\leqslant \frac{1}{2\mu\gamma}\||A|^r x\|\,\||A^*|^r x\| + \frac{1}{2\mu\gamma}|\langle |A^*|^r x,|A|^r x\rangle| \qquad \text{(by Lemma 2.1)} \\ &\leqslant \frac{1}{4\mu\gamma}\left(\||A|^r x\|^2+\||A^*|^r x\|^2\right) + \frac{1}{2\mu\gamma}|\langle |A|^r|A^*|^r x,x\rangle| \\ &= \frac{1}{4\mu\gamma}\left(\langle |A|^{2r}x,x\rangle + \left\langle |A^*|^{2r}x,x\right\rangle\right) + \frac{1}{2\mu\gamma}|\langle |A|^r|A^*|^r x,x\rangle|\end{aligned}$$

$$= \frac{1}{4\mu\gamma}\left\langle \left(|A|^{2r}+|A^*|^{2r}\right)x,x\right\rangle + \frac{1}{2\mu\gamma}\left|\langle |A|^r|A^*|^r x,x\rangle\right|$$
$$\leqslant \frac{1}{4\mu\gamma}\left\| |A|^{2r}+|A^*|^{2r}\right\| + \frac{1}{2\mu\gamma}\omega\left(|A|^r|A^*|^r\right).$$

Therefore, taking supremum over $\|x\|=1$ in the above inequalities, we get

$$\omega^{2r}(A) \leqslant \frac{1}{4\mu\gamma}\left\| |A|^{2r}+|A^*|^{2r}\right\| + \frac{1}{2\mu\gamma}\omega\left(|A|^r|A^*|^r\right). \quad \square$$

3. Application of operator convex function in numerical radius inequalities

The main goal of this section is to present new interpolation inequalities of some known inequalities for the numerical radius by using the properties of operator convex functions. Now, we recall the definition of operator convex functions. It says that: A real-valued continuous function f on an interval J is said to be operator convex if

$$f((1-t)A+tD) \leqslant (1-t)f(A)+tf(D)$$

in the operator order for all $t\in[0,1]$ and for every self-adjoint operator A and D on a Hilbert space $\mathscr{H}$ whose spectra are contained in J. If either $1\leqslant r\leqslant 2$ or $-1\leqslant r\leqslant 0$, the function $f(t)=t^r$ is operator convex.

LEMMA 3.1. ([8]) *Let $A\in\mathbb{B}(\mathscr{H})$ be a self-adjoint operator and let $x\in\mathscr{H}$ be a unit vector. If f is a convex function on an interval containing the spectrum of A, then*

$$f(\langle Ax,x\rangle) \leqslant \langle f(A)x,x\rangle. \tag{3.1}$$

If f is concave, then inequality (3.1) holds in the reverse direction.

LEMMA 3.2. ([11]) *Let $A,D\in\mathbb{B}(\mathscr{H})$ be positive. Then $\|A+D\|=\|A\|+\|D\|$ if and only if $\|AD\|=\|A\|\|D\|$.*

LEMMA 3.3. ([5]) *Let $f: J\to\mathbb{R}$ be an operator convex function on an interval J. Let A and D be two selfadjoint operators with spectra in J. Then*

$$f\left(\frac{A+D}{2}\right) \leqslant \int_0^1 f((1-t)A+tD)dt \leqslant \frac{1}{2}\left(f(A)+f(D)\right). \tag{3.2}$$

If f is non-negative, then the operator inequality (3.2) can be reduced to the following norm inequality

$$\left\|f\left(\frac{A+D}{2}\right)\right\| \leqslant \left\|\int_0^1 f((1-t)A+tD)dt\right\| \leqslant \frac{1}{2}\left\|f(A)+f(D)\right\|. \tag{3.3}$$

LEMMA 3.4. ([10]) *If $A \in \mathbb{B}(\mathscr{H})$ is hyponormal, then*

$$\omega(A) \leqslant \frac{1}{\zeta}\frac{\||A|+|A^*|\|}{2}, \tag{3.4}$$

where $\zeta \geqslant 1$, $\zeta = \inf\left\{K\left(\frac{\langle |A|x,x\rangle}{\langle |A^*|x,x\rangle},2\right)^r : \|x\|=1\right\}$, $r=\min\{\lambda,1-\lambda\}$, $0\leqslant\lambda\leqslant 1$ *and* $K(h,2)=\frac{(h+1)^2}{4h}$.

LEMMA 3.5. *Let $A \in \mathbb{B}(\mathscr{H})$ be hyponormal and f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$f(\omega(A)) \leqslant \left\| f\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right)\right\|. \tag{3.5}$$

Proof. It follows from Lemma 3.4 that

$$\omega(A) \leqslant \left\|\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right\|. \tag{3.6}$$

Therefore,

$$f(\omega(A)) \leqslant f\left(\left\|\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right\|\right)$$

Let $B \in \mathbb{B}(\mathscr{H})$ and B be a selfadjoint operator, f be non-negative increasing operator convex function on $[0,\infty)$. Then

$$\begin{aligned} f(\|B\|) &= f(\sup_{\|x\|=1}\langle Bx,x\rangle) = \sup_{\|x\|=1} f(\langle Bx,x\rangle) \\ &\leqslant \sup_{\|x\|=1}\langle f(B)x,x\rangle = \|f(B)\|. \end{aligned}$$

Since $\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|$ is a selfadjoint operator, then we can get

$$f\left(\left\|\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right\|\right) \leqslant \left\| f\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right)\right\|.$$

This completes the proof. □

THEOREM 3.6. *Let $A \in \mathbb{B}(\mathscr{H})$ be hyponormal and f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned} f(\omega(A)) &\leqslant \left\|\int_0^1 f\left((1-t)\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right) + t\omega(A)I\right)dt\right\| \\ &\leqslant \left\| f\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*|\right)\right\|. \end{aligned}$$

Proof. Since

$$\left\| \left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right) \omega(A) I \right\| = \left\| \frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right\| \|\omega(A)I\|,$$

It follows from Lemma 3.2 that

$$\left\| \frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| + \omega(A)I \right\| = \left\| \frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right\| + \omega(A). \tag{3.7}$$

By using inequality (3.6) and equality (3.7) we can get,

$$2\omega(A) \leqslant \left\| \frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| + \omega(A)I \right\|.$$

Then we have

$$\begin{aligned}
f(\omega(A)) &\leqslant f\left(\frac{1}{2} \left\| \frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| + \omega(A)I \right\| \right) \\
&\leqslant \left\| f\left(\frac{\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| + \omega(A)I}{2} \right) \right\| \\
&\leqslant \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right) + t\omega(A)I \right) dt \right\| \quad \text{(by inequality (3.3))} \\
&\leqslant \frac{1}{2} \left\| f\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right) + f(\omega(A))I \right\| \quad \text{(by inequality (3.3))} \\
&= \frac{1}{2} \left\| f\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right) \right\| + \frac{1}{2} f(\omega(A)) \quad \text{(by Lemma 3.2)} \\
&\leqslant \left\| f\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right) \right\| \quad \text{(by Lemma 3.5).} \quad \square
\end{aligned}$$

The following corollary is an immediate consequence of Theorem 3.6.

COROLLARY 3.7. *Let $A \in \mathbb{B}(\mathscr{H})$ be hyponormal. Then*

$$\begin{aligned}
\omega(A) &\leqslant \left\| \int_0^1 \left((1-t)\left(\frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right) + t\omega(A)I \right)^2 dt \right\|^{\frac{1}{2}} \\
&\leqslant \left\| \frac{1}{2\zeta}|A| + \frac{1}{2\zeta}|A^*| \right\|.
\end{aligned} \tag{3.8}$$

THEOREM 3.8. *Let $A \in \mathbb{B}(\mathscr{H})$ and f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned}
f(\omega^r(A)) &\leqslant \left\| \int_0^1 f\left((1-t)\left(\frac{1}{2}|A|^{2\alpha r} + \frac{1}{2}|A^*|^{2(1-\alpha)r} \right) + t\omega^r(A)I \right) dt \right\| \\
&\leqslant \left\| f\left(\frac{1}{2}|A|^{2\alpha r} + \frac{1}{2}|A^*|^{2(1-\alpha)r} \right) \right\|
\end{aligned}$$

for all $\alpha \in (0,1)$, $r \geqslant 1$.

Proof. From [7, Corollary 2.15], we have

$$\omega^r(A) \leqslant \frac{1}{2}\left\| |A|^{2\alpha r} + |A^*|^{2(1-\alpha)r} \right\|, \quad \forall \alpha \in (0,1), \quad r \geqslant 1.$$

Similar to the proof in Theorem 3.6, we get the required inequalities. □

Considering $f(t) = t^2$ in Theorem 3.8, we get the following corollary.

COROLLARY 3.9. *Let $A \in \mathbb{B}(\mathscr{H})$. Then*

$$\begin{aligned}\omega^r(A) &\leqslant \left\| \int_0^1 \left((1-t)\left(\frac{1}{2}|A|^{2\alpha r} + \frac{1}{2}|A^*|^{2(1-\alpha)r}\right) + t\omega^r(A)I \right)^2 dt \right\|^{\frac{1}{2}} \\ &\leqslant \frac{1}{2}\left\| |A|^{2\alpha r} + |A^*|^{2(1-\alpha)r} \right\|\end{aligned}$$

for all $\alpha \in (0,1)$, $r \geqslant 1$.

THEOREM 3.10. *Let $A, B \in \mathbb{B}(\mathscr{H})$ and f be non-negative increasing operator convex function on $[0,\infty)$. Then*

$$\begin{aligned}f\left(\omega^{2r}(B^*A)\right) &\leqslant \left\| \int_0^1 f\left((1-t)\left(\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right) + t\omega^{2r}(B^*A)I \right) dt \right\| \\ &\leqslant \left\| f\left(\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right) \right\|\end{aligned}$$

for all $\alpha \in (0,1)$, $r \geqslant 1$.

Proof. From [6, Cor. 2.15], we have

$$\omega^{2r}(B^*A) \leqslant \left\| \alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}} \right\|, \quad \forall \alpha \in (0,1), \quad r \geqslant 1. \tag{3.9}$$

Using inequality (3.9) and proceeding similarly as in Theorem 3.6 we can get the required inequalities. □

In particular, if $f(t) = t^2$, we obtain the following interpolation inequalities of (3.9).

COROLLARY 3.11. *Let $A, B \in \mathbb{B}(\mathscr{H})$. Then*

$$\begin{aligned}\omega^{2r}(B^*A) &\leqslant \left\| \int_0^1 \left((1-t)\left(\alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}}\right) + t\omega^{2r}(B^*A)I \right)^2 dt \right\|^{\frac{1}{2}} \\ &\leqslant \left\| \alpha|A|^{\frac{2r}{\alpha}} + (1-\alpha)|B|^{\frac{2r}{1-\alpha}} \right\|\end{aligned}$$

for all $\alpha \in (0,1)$, $r \geqslant 1$.

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