Journal of Mathematical Inequalities
Volume 16, Number 4 (2022), 1661–1674 doi:10.7153/jmi-2022-16-108

EMBEDDING THEOREM FOR BESOV–MORREY TYPE SPACES AND VOLTERRA INTEGRAL OPERATORS

DAN QU AND XIANGLING ZHU*

(*Communicated by T. Burić*)

Abstract. A family of Besov-Morrey type spaces in the open unit disc are introduced in this paper. The boundedness of the embedding from Besov-Morrey type spaces to a class tent spaces is studied. As an application, the boundedness, compactness and essential norm of the Volterra integral operator from Besov-Morrey type spaces to a general function space are investigated.

1. Introduction

Let $\mathbb{D}$ be the open unit disc in the complex plane and $H(\mathbb{D})$ be the set of all analytic functions in $\mathbb{D}$. Let $0 < s < 1 < p < \infty$. A function $f \in H(\mathbb{D})$ belongs to the Besov type space $B_p(s)$ if

$$\|f\|^p_{B_p(s)} = |f(0)|^p + \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty,$$

where dA denotes the normalized area measure on $\mathbb{D}$. The space $B_p(0)$ is just the classical Besov space, which always denoted by B_p.

Let $0 < p < \infty$, $-2 < q < \infty$ and $0 \leqslant s < \infty$. A function $f \in H(\mathbb{D})$ belongs to the general function space $F(p,q,s)$, if

$$\|f\|^p_{F(p,q,s)} = |f(0)|^p + \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) < \infty,$$

where $\sigma_a(z) = \frac{a-z}{1-\bar{a}z}$ is a Möbius map that interchanges 0 and a. The space $F(p,q,s)$ was introduced by Zhao in [23]. When $q+s > -1$, the space $F(p,q,s)$ is nontrivial. $F(2,0,1) = BMOA$, the space of analytic functions of bounded mean oscillation. When $s = 0$ and $q > -1$, $F(p,q,s)$ is called Dirichlet type space and always denoted by D^p_q. When $q = p-2$ and $s = 0$, $F(p,p-2,0)$ is the Besov space B_p. A function $f \in H(\mathbb{D})$ is said to belong to the space $F_0(p,q,s)$ if

$$\lim_{|a|\to 1} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^q (1-|\sigma_a(z)|^2)^s dA(z) = 0.$$

Mathematics subject classification (2020): 30H99, 47B38.
Keywords and phrases: Besov-Morrey type space, Carleson measure, Volterra integral operator.
* Corresponding author.

Paper JMI-16-108

Let $g \in H(\mathbb{D})$. The Volterra integral operator T_g induced by g is defined by

$$T_g f(z) = \int_0^z f(\zeta)g'(\zeta)d\zeta, \quad f \in H(\mathbb{D}), \quad z \in \mathbb{D}.$$

In [13], Pommerenke showed that T_g is bounded on H^2 if and only if $g \in BMOA$. In [1], the authors proved that T_g is bounded on $H^p (p \geqslant 1)$ if and only if $g \in BMOA$. The operator T_g has attracted the attention of many authors. See [1, 2, 4, 5, 6, 9, 8, 7, 10, 12, 14, 15, 16, 17, 18, 22] and the references therein for more results of the operator T_g.

Throughout this paper, let $K : [0,\infty) \to [0,\infty)$ be a nondecreasing and right-continuous function, not identically zero. Without losing generality, we assume that K satisfies (see [21]):

$$\int_0^1 \frac{\varphi_K(x)}{x} dx < \infty \tag{1}$$

and

$$\int_1^\infty \frac{\varphi_K(x)}{x^{1+\delta}} dx < \infty, \ \delta > 0, \tag{2}$$

where

$$\varphi_K(x) = \sup_{0<s\leqslant 1} \frac{K(sx)}{K(s)}, \ \ 0 < x < \infty.$$

In [18], Sun and Wulan defined a Dirichlet-Morrey type space $\mathscr{D}_K^s$ for $1 \leqslant s < \infty$, which consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{\mathscr{D}_K^s}^2 = |f(0)|^2 + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \|f \circ \sigma_a - f(a)\|_{D_s^2}^2 < \infty.$$

They found the sufficient and necessary conditions of the boundedness for the identity operator I_d from $\mathscr{D}_K^s$ to a tent space $\mathscr{T}_K^s(\mu)$ and characterized the boundedness and compactness of the operator T_g on $\mathscr{D}_K^s$.

Let $0 < s, \lambda < 1 < p < \infty$. In [22], Yang and Zhu introduced the Besov-Morrey space, denoted by $B_p^\lambda(s)$, which consisting of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{B_p^\lambda(s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} (1-|a|^2)^{s(1-\lambda)} \|f \circ \sigma_a - f(a)\|_{B_p(s)}^p < \infty.$$

They found the sufficient and necessary conditions for the identity operator I_d to be bounded from $B_p^\lambda(s)$ to a tent space and characterized the boundedness and essential norm of the operator T_g from $B_p^\lambda(s)$ to a function space.

Inspired by [18, 22], here we define a new Besov-Morrey type space $B_p^K(s)$ as follows. Let $0 < s < 1 < p < \infty$. The Besov-Morrey type space $B_p^K(s)$ is the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{B_p^K(s)}^p = |f(0)|^p + \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \|f \circ \sigma_a - f(a)\|_{B_p(s)}^p < \infty.$$

If $K(t) = t^{s\lambda}$ and $0 < s, \lambda < 1$, then $B_p^K(s) = B_p^\lambda(s)$, and if $p = 2$, then $B_p^K(s) = \mathscr{D}_K^s$.

Let $0 \leqslant \alpha < \infty$, $0 < q < \infty$ and μ be a positive Borel measure on $\mathbb{D}$. The space $\mathscr{T}_K^{q,\alpha}(\mu)$ consists of all measure functions f for which

$$\|f\|_{\mathscr{T}_K^{q,\alpha}(\mu)}^q = \sup_{I \subset \partial\mathbb{D}} \frac{1}{(K(|I|))^\alpha} \int_{S(I)} |f(z)|^q d\mu(z) < \infty.$$

The paper is organized as follows. In Section 2, some basic properties of Besov-Morrey type spaces $B_p^K(s)$ are given. In Section 3, we study the boundedness of the embedding mapping I_d from $B_p^K(s)$ to $\mathscr{T}_K^{q,\frac{q}{p}}(\mu)$. As an application, the boundedness, compactness and the essential norm of the Volterra integral operator T_g from $B_p^K(s)$ into a general function space are discussed in Section 4.

In the whole paper, we say that $f \lesssim g$ if there exists a constant C such that $f \leqslant Cg$. The symbol $f \approx g$ means that $f \lesssim g \lesssim f$.

2. Some basic properties

In this section, some basic properties of the space $B_p^K(s)$ are given.

PROPOSITION 1. *Let* $0 < s < 1 < p < \infty$. *Then* $B_p^K(s) \subseteq B_p(s)$. *Moreover,* $B_p^K(s) = B_p(s)$ *if and only if* $K(0) > 0$.

Proof. Let $f \in B_p^K(s)$. Making a change of variable $w = \sigma_a(z)$, we get

$$\begin{aligned}
\infty &> \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \|f\circ\sigma_a - f(a)\|_{B_p(s)}^p \\
&= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \int_{\mathbb{D}} |(f\circ\sigma_a)'(z)|^p (1-|z|^2)^{p-2+s} dA(z) \\
&= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2} (1-|\sigma_a(w)|^2)^s dA(w) \\
&= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2+s} \frac{(1-|a|^2)^s}{|1-\bar{a}w|^{2s}} dA(w) \\
&\geqslant \frac{1}{K(1)} \int_{\mathbb{D}} |f'(w)|^p (1-|w|^2)^{p-2+s} dA(w).
\end{aligned}$$

So $f \in B_p(s)$, that is, $B_p^K(s) \subseteq B_p(s)$.

Next, we prove that $B_p^K(s) = B_p(s)$ if and only if $K(0) > 0$. First, we assume that $f \in B_p(s)$ and $K(0) > 0$. Using the monotonicity of K, we get

$$\begin{aligned}
\sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \|f\circ\sigma_a - f(a)\|_{B_p(s)}^p &\lesssim \sup_{a\in\mathbb{D}} \frac{1}{K(0)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} \frac{(1-|a|^2)^{2s}}{|1-\bar{a}z|^{2s}} dA(z) \\
&\lesssim \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty.
\end{aligned}$$

Therefore, $f \in B_p^K(s)$. Furthermore, $B_p^K(s) = B_p(s)$.

Conversely, assume that $B_p^K(s) = B_p(s)$. For any $\gamma \in \mathbb{D}$, consider the function

$$f_\gamma(z) = \int_0^z \frac{(1-|\gamma|^2)dw}{(1-\bar{\gamma}w)^{2+\frac{s}{p}}}, \ z \in \mathbb{D}.$$

Applying Lemma 3.10 in [24], we get

$$\begin{aligned}\|f_\gamma\|_{B_p(s)}^p &= \int_{\mathbb{D}} |f_\gamma'(z)|^p (1-|z|^2)^{p-2+s} dA(z) \\ &= \int_{\mathbb{D}} \frac{(1-|\gamma|^2)^p}{|1-\bar{\gamma}z|^{2p+s}} (1-|z|^2)^{p-2+s} dA(z) \lesssim 1.\end{aligned}$$

Thus, $f_\gamma \in B_p(s)$. For $a \in \mathbb{D}$ and $r > 0$, let $\mathbb{D}(a,r)$ denote the Bergman metric disk centered at a with radius r, i.e., $\mathbb{D}(a,r) = \{z \in \mathbb{D} : \beta(a,z) < r\}$. Then

$$\begin{aligned}\infty > \|f_\gamma\|_{B_p(s)}^p &= \|f_\gamma\|_{B_p^K(s)}^p \\ &= \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \int_{\mathbb{D}} |f_\gamma'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_a(z)|^2)^s dA(z) \\ &\gtrsim \frac{(1-|\gamma|^2)^s}{K(1-|\gamma|^2)} \int_{\mathbb{D}} |f_\gamma'(z)|^p (1-|z|^2)^{p-2} (1-|\sigma_\gamma(z)|^2)^s dA(z) \\ &\gtrsim \frac{(1-|\gamma|^2)^{p+2s}}{K(1-|\gamma|^2)} \int_{\mathbb{D}(\gamma,r)} \frac{(1-|z|^2)^{p-2+s}}{|1-\bar{\gamma}z|^{2p+3s}} dA(z) \\ &\approx \frac{1}{K(1-|\gamma|^2)},\end{aligned}$$

which implies that $K(0) > 0$. □

PROPOSITION 2. *Let* $0 < s < 1 < p < \infty$. *Then* $B_p^K(s) = F(p, p-2, s)$ *if and only if* $K(x) \approx x^s$.

Proof. Since

$$\|f\|_{F(p,p-2,s)}^p = \sup_{a\in\mathbb{D}} \|f\circ\sigma_a - f(a)\|_{B_p(s)}^p \lesssim \sup_{a\in\mathbb{D}} \frac{K(1-|a|^2)}{(1-|a|^2)^s} \|f\|_{B_p^K(s)}^p$$

and

$$\|f\|_{B_p^K(s)}^p \lesssim \sup_{a\in\mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \|f\|_{F(p,p-2,s)}^p,$$

the desired result follows immediately. □

LEMMA 1. [11] *Let* $s,t > 0$, $r > -1$ *and* $s+t-r > 2$. *If* $t < 2+r < s$, *then*

$$\int_{\mathbb{D}} \frac{(1-|z|^2)^r}{|1-\bar{a}z|^s |1-\bar{b}z|^t} dA(z) \lesssim \frac{1}{(1-|a|^2)^{s-r-2} |1-\bar{a}b|^t}, \quad a,b \in \mathbb{D}.$$

LEMMA 2. [18] *Let* $0<\alpha\leqslant\beta<\infty$ *and* K *satisfy* (2) *for some* $\delta>0$. *Then for sufficiently small positive constants* $c<\delta$,

$$\frac{K(\beta)}{K(\alpha)}\leqslant\left(\frac{\beta}{\alpha}\right)^{\delta-c}\leqslant\left(\frac{\beta}{\alpha}\right)^{\delta}.$$

PROPOSITION 3. *Let* $0<s<1<p<\infty$, $\gamma\in\mathbb{D}$ *and* K *satisfy* (2) *for some* $\delta\in(0,2s)$. *Then the function*

$$f_\gamma(z)=\left(\frac{(1-|\gamma|^2)^sK(1-|\gamma|^2)}{(1-\bar{\gamma}z)^{2s}}\right)^{\frac{1}{p}},\ z\in\mathbb{D},$$

belongs to $B_p^K(s)$.

Proof. Using Lemmas 1 and 2, we have that

$$\begin{aligned}\|f_\gamma\|^p_{B_p^K(s)}&=\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^s}{K(1-|a|^2)}\int_{\mathbb{D}}|f'_\gamma(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^s dA(z)\\&\lesssim\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2s}K(1-|\gamma|^2)(1-|\gamma|^2)^s}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{(1-|z|^2)^{p-2+s}}{|1-\bar{\gamma}z|^{2s+p}|1-\bar{a}z|^{2s}}dA(z)\\&\lesssim\sup_{a\in\mathbb{D}}\frac{(1-|a|^2)^{2s}K(1-|\gamma|^2)(1-|\gamma|^2)^s}{K(1-|a|^2)}\frac{1}{(1-|\gamma|^2)^s|1-\bar{a}\gamma|^{2s}}\\&\lesssim\sup_{a\in\mathbb{D}}\frac{K(|1-\bar{a}\gamma|)}{K(1-|a|^2)}\left(\frac{1-|a|^2}{|1-\bar{a}\gamma|}\right)^{2s}\\&\lesssim\sup_{a\in\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}\gamma|}\right)^{2s-\delta}\lesssim 1,\end{aligned}$$

which means that $f_\gamma\in B_p^K(s)$. □

PROPOSITION 4. *Let* $0<s<1<p<\infty$ *and* K *satisfy* (2) *for some* $\delta\in(0,s)$. *Then for any* $f\in B_p^K(s)$,

$$|f(a)-f(0)|\lesssim\left(\frac{K(1-|a|^2)}{(1-|a|^2)^s}\right)^{\frac{1}{p}}\|f\|_{B_p^K(s)},\ a\in\mathbb{D}.$$

Proof. For $a\in\mathbb{D}$ and $r>0$, from [24] we see that

$$\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}\approx\frac{1}{(1-|z|^2)^2}\approx\frac{1}{(1-|a|^2)^2},$$

when $z \in \mathbb{D}(a,r)$. Hence,

$$\begin{aligned}|f'(a)|^p &\lesssim \frac{1}{(1-|a|^2)^p}\int_{\mathbb{D}(a,r)}|f'(z)|^p dA_{p-2}(z)\\ &\lesssim \frac{1}{(1-|a|^2)^p}\int_{\mathbb{D}(a,r)}|f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^s dA(z)\\ &\lesssim \frac{K(1-|a|^2)}{(1-|a|^2)^{p+s}}\|f\|^p_{B_p^K(s)}.\end{aligned}$$

Therefore,

$$|f'(a)| \lesssim \frac{K^{\frac{1}{p}}(1-|a|^2)}{(1-|a|^2)^{\frac{s}{p}+1}}\|f\|_{B_p^K(s)}.$$

By Lemma 2, there exists a constant $c \in (0, s-\delta)$ such that

$$\begin{aligned}|f(a)-f(0)| = \left|a\int_0^1 f'(az)dz\right| &\lesssim \|f\|_{B_p^K(s)}\int_0^1 \frac{|a|K^{\frac{1}{p}}(1-|az|^2)}{(1-|az|^2)^{\frac{s}{p}+1}}dz\\ &\lesssim \|f\|_{B_p^K(s)}\frac{K^{\frac{1}{p}}(1-|a|^2)}{(1-|a|^2)^{\frac{\delta-c}{p}}}\int_0^1(1-|az|)^{\frac{\delta-c-s}{p}-1}|a|dz\\ &\lesssim \left(\frac{K(1-|a|^2)}{(1-|a|^2)^s}\right)^{\frac{1}{p}}\|f\|_{B_p^K(s)}.\end{aligned}$$

This finishes the proof. □

For any arc $I \subset \partial\mathbb{D}$, let $|I| = \frac{1}{\pi}\int_I |d\xi|$ be the normalized arc length of I and

$$S(I) = \{z = re^{i\theta} \in \mathbb{D} : 1-|I| \leqslant r < 1, e^{i\theta} \in I\}$$

be the Carleson box based on I. We say that a positive Borel measure μ on $\mathbb{D}$ is a K-Carleson measure if (see [18])

$$\|\mu\|_K = \sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{K(|I|)} < \infty.$$

Let $0 < s < \infty$. If we choose $K(t) = t^s$, then μ is an s-Carleson measure and

$$\|\mu\|_s = \sup_{I\subset\partial\mathbb{D}}\frac{\mu(S(I))}{|I|^s}.$$

LEMMA 3. [18] *Suppose K satisfies (2) for some $\delta \in (0,2)$. μ is a K-Carleson measure if and only if*

$$\sup_{a\in\mathbb{D}}\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}\left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^t d\mu(z) < \infty, \quad \delta \leqslant t < \infty.$$

PROPOSITION 5. *Let* $0 < s < 1 < p < \infty$, $f \in H(\mathbb{D})$, K *satisfy* (2) *for some* $\delta \in (0,2s)$. *Then* $f \in B_p^K(s)$ *if and only if*

$$\sup_{I \subset \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z) < \infty.$$

Proof. Given any arc $I \subset \partial\mathbb{D}$, let $a = (1-|I|)\xi$, where ξ is the centre of I. We have

$$|1-\bar{a}z| \approx 1-|a|^2 \approx |I|, \;\; z \in S(I).$$

Let $d\mu_f(z) = |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z)$. By Lemma 3,

$$\begin{aligned}
\|f\|_{B_p^K(s)}^p &\approx \sup_{a \in \mathbb{D}} \frac{(1-|a|^2)^s}{K(1-|a|^2)} \|f \circ \sigma_a - f(a)\|_{B_p(s)}^p \\
&= \sup_{a \in \mathbb{D}} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} |f'(z)|^p (1-|z|^2)^{p-2+s} \left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^{2s} dA(z). \\
&= \sup_{a \in \mathbb{D}} \frac{1}{K(1-|a|^2)} \int_{\mathbb{D}} \left(\frac{1-|a|^2}{|1-\bar{a}z|}\right)^{2s} d\mu_f(z) \\
&\approx \sup_{I \subset \partial\mathbb{D}} \frac{\mu_f(S(I))}{K(|I|)} \\
&= \sup_{I \subset \partial\mathbb{D}} \frac{1}{K(|I|)} \int_{S(I)} |f'(z)|^p (1-|z|^2)^{p-2+s} dA(z).
\end{aligned}$$

Then the desired result immediately follows. □

3. Embedding map from $B_p^K(s)$ to $\mathcal{T}_K^{q,\frac{q}{p}}(\mu)$

In this section, we will consider the boundedness of the identity operator $I_d : B_p^K(s) \to \mathcal{T}_K^{q,\frac{q}{p}}(\mu)$.

LEMMA 4. [3] *Let* $1 < p < \infty$, $s > -1, t \geqslant 0$ *such that* $t < 2+s$. *If* $f \in H(\mathbb{D})$, *then*

$$\int_{\mathbb{D}} |f(z)-f(0)|^p \frac{(1-|z|^2)^s}{|1-\bar{w}z|^t} dA(z) \lesssim \int_{\mathbb{D}} |f'(z)|^p \frac{(1-|z|^2)^{s+p}}{|1-\bar{w}z|^t} dA(z), \;\; w \in \mathbb{D}.$$

THEOREM 1. *Let* $0 < s < 1 < p < q < \infty$, μ *be a positive Borel measure on* $\mathbb{D}$, *and* K *satisfy* (2) *for some* $\delta \in (0,s)$. *Then the identity operator* $I_d : B_p^K(s) \to \mathcal{T}_K^{q,\frac{q}{p}}(\mu)$ *is bounded if and only if* μ *is a* $\frac{qs}{p}$*-Carleson measure.*

Proof. First we assume that $I_d : B_p^K(s) \to \mathcal{T}_K^{q,\frac{q}{p}}(\mu)$ is bounded. For $I \subset \partial\mathbb{D}$, let ξ be the center of I and $\gamma = (1-|I|)\xi$. It is known that

$$|1-\bar{\gamma}z| \approx 1-|\gamma|^2 \approx |I|, \quad z \in S(I).$$

Using the function f_γ, given in Proposition 3, we get

$$\frac{\mu(S(I))}{|I|^{\frac{qs}{p}}} \approx \frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}|f_\gamma(z)|^q d\mu(z) \lesssim \|f_\gamma\|^q_{\mathscr{T}_K^{q,\frac{q}{p}}(\mu)} \lesssim \|f_\gamma\|^q_{B_p^K(s)} < \infty,$$

which implies that μ is a $\frac{qs}{p}$-Carleson measure.

Conversely, let μ be a $\frac{qs}{p}$-Carleson measure and $f \in B_p^K(s)$. For any $I \subset \partial\mathbb{D}$, let ξ be the center of I and $a = (1-|I|)\xi$, we have

$$\begin{aligned}
&\frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}|f(z)|^q d\mu(z)\\
\lesssim&\frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}|f(a)|^q d\mu(z) + \frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}|f(z)-f(a)|^q d\mu(z)\\
=&G_1+G_2.
\end{aligned}$$

By Proposition 4, $|f(a)| \leqslant \left(\frac{K(1-|a|^2)}{(1-|a|^2)^s}\right)^{\frac{1}{p}}\|f\|_{B_p^K(s)}$. Hence

$$\begin{aligned}
G_1 &= \frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}|f(a)|^q d\mu(z)\\
&\leqslant \frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}\left(\frac{K(|I|)}{|I|^s}\right)^{\frac{q}{p}}\|f\|^q_{B_p^K(s)} d\mu(z) \lesssim \|f\|^q_{B_p^K(s)}.
\end{aligned}$$

By the assumption that μ is a $\frac{qs}{p}$-Carleson measure, we know that the identity operator $I_d : B_p(s) \to L^q(\mu)$ is bounded (see [5]). According the Proposition 1, we see that $I_d : B_p^K(s) \to L^q(\mu)$ is also bounded. We have

$$\begin{aligned}
G_2 &= \frac{1}{K^{\frac{q}{p}}(|I|)}\int_{S(I)}|f(z)-f(a)|^q d\mu(z)\\
&\lesssim \frac{(1-|a|^2)^{\frac{2qs}{p}}}{K^{\frac{q}{p}}(1-|a|^2)}\int_{S(I)}\left|\frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{2s}{p}}}\right|^q d\mu(z)\\
&\lesssim \left(\frac{(1-|a|^2)^{2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\left|\frac{d}{dz}\frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{2s}{p}}}\right|^p (1-|z|^2)^{p-2+s} dA(z)\right)^{\frac{q}{p}}.
\end{aligned}$$

Since

$$\frac{d}{dz}\frac{f(z)-f(a)}{(1-\bar{a}z)^{\frac{2s}{p}}} = \frac{f'(z)(1-\bar{a}z)^{\frac{2s}{p}} + \bar{a}(\frac{2s}{p})(f(z)-f(a))(1-\bar{a}z)^{\frac{2s}{p}-1}}{(1-\bar{a}z)^{\frac{4s}{p}}},$$

we deduce that $G_2 \lesssim (Q+J)^{\frac{q}{p}}$, where

$$Q = \frac{(1-|a|^2)^{2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|f'(z)|^p}{|1-\bar{a}z|^{2s}}(1-|z|^2)^{p-2+s} dA(z)$$

and

$$J=\frac{(1-|a|^2)^{2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^{2s+p}}(1-|z|^2)^{p-2+s}dA(z).$$

Clearly,

$$Q=\frac{(1-|a|^2)^s}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^s dA(z)\lesssim\|f\|^p_{B^K_p(s)}.$$

Making the change of variable $w=\sigma_a(z)$, by Lemma 4 we obtain

$$\begin{aligned}
J&=\frac{(1-|a|^2)^{2s}}{K(1-|a|^2)}\int_{\mathbb{D}}\frac{|f(z)-f(a)|^p}{|1-\bar{a}z|^{2s+p}}(1-|z|^2)^{p-2+s}dA(z)\\
&=\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}|f\circ\sigma_a(w)-f\circ\sigma_a(0)|^p\frac{(1-|w|^2)^{p-2+s}(1-|a|^2)^s}{|1-\bar{a}w|^p}dA(w)\\
&\lesssim\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}|(f\circ\sigma_a)'(w)|^p\frac{(1-|w|^2)^{2p-2+s}(1-|a|^2)^s}{|1-\bar{a}w|^p}dA(w)\\
&=\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(\sigma_a(w))|^p(1-|\sigma_a(w)|^2)^p\frac{(1-|w|^2)^{p-2+s}(1-|a|^2)^s}{|1-\bar{a}w|^p}dA(w)\\
&=\frac{1}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^p\frac{(1-|\sigma_a(z)|^2)^{p-2+s}(1-|a|^2)^s}{|1-\bar{a}\sigma_a(z)|^p}\frac{(1-|a|^2)^2}{|1-\bar{a}z|^4}dA(z)\\
&=\frac{(1-|a|^2)^s}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|^2)^{2p-2+s}(1-|a|^2)^s}{|1-\bar{a}z|^{p+2s}}dA(z)\\
&=\frac{(1-|a|^2)^s}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^p\frac{(1-|z|^2)^{2p-2+s}(1-|a|^2)^s}{|1-\bar{a}z|^{p+2s}}dA(z)\\
&\lesssim\frac{(1-|a|^2)^s}{K(1-|a|^2)}\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^{p-2}(1-|\sigma_a(z)|^2)^s dA(z)\\
&\lesssim\|f\|^p_{B^K_p(s)}.
\end{aligned}$$

Hence, $G_2\lesssim\|f\|^q_{B^K_p(s)}$. Therefore,

$$|f\|_{\mathcal{T}_K^{q,\frac{q}{p}}(\mu)}\lesssim\|f\|_{B^K_p(s)}$$

for all $f\in B^K_p(s)$, which implies the desired result. □

4. Integral operator T_g

Let $0<p<\infty$, $-2<q<\infty$, $0\leqslant s,t,r<\infty$ and $f\in H(\mathbb{D})$. We say that f belongs to the space $F_K(p,q,s,t,r)$ if

$$\|f\|^p_{F_K(p,q,s,t,r)}=|f(0)|^p+\sup_{a\in\mathbb{D}}\left(\frac{(1-|a|^2)^t}{K(1-|a|^2)}\right)^r\int_{\mathbb{D}}|f'(z)|^p(1-|z|^2)^q(1-|\sigma_a(z)|^2)^s dA(z)$$
$$<\infty.$$

It is easy to see that $F_K(p,q,s,t,r)$ is a Banach space when $p \geqslant 1$ under the above norm. Moreover, $F_K(p,q,s,t,r) = F(p,q,s)$ when $k(\alpha) = \alpha^t$.

In this section, using Theorem 1, we characterize the boundedness, compactness and essential norm of the Volterra operator $T_g : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$.

THEOREM 2. *Let $g \in H(\mathbb{D})$, $0 < s < 1 < p < q < \infty$ and K satisfy (2) for some $\delta \in (0,s)$. Then the operator $T_g : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ is bounded if and only if*

$$g \in F(q,q-2,qs/p).$$

Proof. Assume that $T_g : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ is bounded. For any $I \subset \partial\mathbb{D}$, let ξ be the midpoint of I and $\gamma = (1-|I|)\xi$. Set

$$f_\gamma(z) = \left(\frac{(1-|\gamma|^2)^s K(1-|\gamma|^2)}{(1-\overline{\gamma}z)^{2s}}\right)^{\frac{1}{p}}, \quad z \in \mathbb{D}.$$

Then by Proposition 4, we have that $f_\gamma \in B_p^K(s)$ and $\|f_\gamma\|_{B_p^K(s)} \lesssim 1$. Thus,

$$\|T_g f_\gamma\|_{F_K(q,q-2,qs/p,s,q/p)} \lesssim \|T_g\| \|f_\gamma\|_{B_p^K(s)} \lesssim \|T_g\|.$$

We have

$$\begin{aligned}
\infty &> \|T_g f_\gamma\|^q_{F_K(q,q-2,qs/p,s,q/p)} \\
&= \sup_{a\in\mathbb{D}} \left(\frac{(1-|a|^2)^s}{K(1-|a|^2)}\right)^{\frac{q}{p}} \int_{\mathbb{D}} |(T_g f_\gamma)'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^{\frac{qs}{p}} dA(z) \\
&\geqslant \left(\frac{(1-|\gamma|^2)^s}{K(1-|\gamma|^2)}\right)^{\frac{q}{p}} \int_{\mathbb{D}} |f_\gamma(z)|^q |g'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_\gamma(z)|^2)^{\frac{qs}{p}} dA(z) \\
&\gtrsim \frac{1}{(K(|I|))^{\frac{q}{p}}} \int_{S(I)} \frac{(K(|I|))^{\frac{q}{p}}}{|I|^{\frac{qs}{p}}} |g'(z)|^q (1-|z|^2)^{q-2+\frac{qs}{p}} dA(z) \\
&= \frac{1}{|I|^{\frac{qs}{p}}} \int_{S(I)} |g'(z)|^q (1-|z|^2)^{q-2+\frac{qs}{p}} dA(z),
\end{aligned}$$

which implies that $g \in F(q,q-2,qs/p)$.

Conversely, suppose that $g \in F(q,q-2,qs/p)$. From [23] we obtain

$$\begin{aligned}
\|g\|^q_{F(q,q-2,\frac{qs}{p})} &\approx \sup_{a\in\mathbb{D}} \int_{\mathbb{D}} |g'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_a(z)|^2)^{\frac{qs}{p}} dA(z) \\
&\approx \sup_{I\subset\partial\mathbb{D}} \frac{1}{|I|^{\frac{qs}{p}}} \int_{S(I)} |g'(z)|^q (1-|z|^2)^{q-2+\frac{qs}{p}} dA(z) \\
&\approx \sup_{I\subset\partial\mathbb{D}} \frac{\mu_g(S(I))}{|I|^{\frac{qs}{p}}},
\end{aligned}$$

which means that μ_g is a $\frac{qs}{p}$-Carleson measure. Here $\mu_g = |g'(z)|^q(1-|z|^2)^{q-2+\frac{qs}{p}}dA(z)$. By Theorem 1, the identity operator $I_d : B_p^K(s) \to \mathscr{T}_K^{q,\frac{q}{p}}(\mu_g)$ is bounded. Let $f \in B_p^K(s)$. We get

$$
\begin{aligned}
&\|T_g f\|^q_{F_K(q,q-2,qs/p,s,q/p)}\\
&=\sup_{a\in\mathbb{D}}\left(\frac{(1-|a|^2)^s}{K(1-|a|^2)}\right)^{\frac{q}{p}}\int_{\mathbb{D}}|f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\frac{qs}{p}}dA(z)\\
&=\sup_{a\in\mathbb{D}}\left(\frac{(1-|a|^2)^s}{K(1-|a|^2)}\right)^{\frac{q}{p}}\int_{\mathbb{D}}|f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+\frac{qs}{p}}\frac{(1-|a|^2)^{\frac{qs}{p}}}{|1-\bar{a}z|^{\frac{2qs}{p}}}dA(z)\\
&\approx \sup_{I\subset\partial\mathbb{D}}\frac{1}{(K(|I|))^{\frac{q}{p}}}\int_{S(I)}|f(z)|^q|g'(z)|^q(1-|z|^2)^{q-2+\frac{qs}{p}}dA(z)\\
&\approx \sup_{I\subset\partial\mathbb{D}}\frac{1}{(K(|I|))^{\frac{q}{p}}}\int_{S(I)}|f(z)|^q d\mu_g(z)\\
&=\|f\|^q_{\mathscr{T}_K^{q,\frac{q}{p}}(\mu_g)} \lesssim \|f\|^q_{B_p^K(s)}\|g\|^q_{F(q,q-2,\frac{qs}{p})}<\infty.
\end{aligned}
$$

Therefore $T_g : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ is bounded. $\square$

Finally, we give an estimation for the essential norm of T_g. Recall that the essential norm of $T : X \to Y$ is defined by

$$\|T\|_{e,X\to Y} = \inf_K\{\|T-K\|_{X\to Y} : \quad K \text{ is a compact operator from } X \text{ to } Y\},$$

where $(X, \|\cdot\|_X)$ and $(Y, \|\cdot\|_Y)$ are Banach spaces and $T : X \to Y$ is a bounded linear operator. It is clear that $T : X \to Y$ is compact if and only if $\|T\|_{e,X\to Y} = 0$. For a closed subspaces A of X, given $f \in X$, the distance from f to A, denoted by $dist_X(f,A)$, is defined by $dist_X(f,A) = \inf_{g\in A}\|f-g\|_X$.

LEMMA 5. [12] *Let $1 < q < \infty$, $0 < \alpha < \infty$. If $g \in F(q,q-2,\alpha)$, then*

$$
\begin{aligned}
dist_{F(q,q-2,\alpha)}(g, F_0(q,q-2,\alpha)) &\approx \limsup_{r\to 1^-}\|g-g_r\|_{F(q,q-2,\alpha)}\\
&\approx \limsup_{|a|\to 1}\left(\int_{\mathbb{D}}|g'(z)|^q(1-|z|^2)^{q-2}(1-|\sigma_a(z)|^2)^{\alpha}dA(z)\right)^{\frac{1}{q}}.
\end{aligned}
$$

Here $g_r(z) = g(rz)$, $0 < r < 1$, $z \in \mathbb{D}$.

Similarly to the proof of Lemma 5 in [22], we have the following result.

LEMMA 6. *Let $0 < s < 1 < p < q < \infty$ and K satisfy (2) for some $\delta \in (0,s)$. If $0 < r < 1$ and $g \in F(q,q-2,qs/p)$, then $T_{g_r} : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ is compact.*

THEOREM 3. *Let* $g \in H(\mathbb{D})$, $0 < s < 1 < p < q < \infty$ *and* K *satisfy* (2) *for some* $\delta \in (0,s)$. *If* $T_g : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ *is bounded, then*

$$\|T_g\|_{e,B_p^K(s)\to F_K(q,q-2,qs/p,s,q/p)} \approx dist_{F(q,q-2,qs/p)}(g, F_0(q,q-2,qs/p)).$$

Proof. Let $\{I_n\} \subset \partial\mathbb{D}$ and $\lim_{n\to\infty}|I_n| = 0$. Suppose $e^{i\theta_n}$ is the center of I_n and $c_n = (1-|I_n|)e^{i\theta_n}$. For each n, let

$$f_n(z) = \left(\frac{(1-|c_n|^2)^s K(1-|c_n|^2)}{(1-\overline{c_n}z)^{2s}}\right)^{\frac{1}{p}}.$$

Then f_n is bounded in $B_p^K(s)$ and $\{f_n\}$ converges to zero uniformly on every compact subsets of $\mathbb{D}$. Given a compact operator $K : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$. Using Lemma 2.10 in [19], we have $\lim_{n\to\infty}\|Kf_n\|_{F_K(q,q-2,qs/p,s,q/p)} = 0$. So

$$\begin{aligned}
\|T_g - K\| &\gtrsim \limsup_{n\to\infty} \|(T_g-K)f_n\|_{F_K(q,q-2,qs/p,s,q/p)} \\
&\gtrsim \limsup_{n\to\infty} \left(\|T_g f_n\|_{F_K(q,q-2,qs/p,s,q/p)} - \|Kf_n\|_{F_K(q,q-2,qs/p,s,q/p)}\right) \\
&= \limsup_{n\to\infty} \|T_g f_n\|_{F_K(q,q-2,qs/p,s,q/p)} \\
&\gtrsim \limsup_{n\to\infty} \left(\left(\frac{(1-|c_n|^2)^s}{K(1-|c_n|^2)}\right)^{\frac{q}{p}} \int_{\mathbb{D}} |f_n(z)|^q |g'(z)|^q (1-|z|^2)^{q-2} (1-|\sigma_{c_n}(z)|^2)^{\frac{qs}{p}} dA(z)\right)^{\frac{1}{q}} \\
&\gtrsim \limsup_{n\to\infty} \left(\frac{1}{|I_n|^{\frac{qs}{p}}} \int_{S(I_n)} |g'(z)|^q (1-|z|^2)^{q-2+\frac{qs}{p}} dA(z)\right)^{\frac{1}{q}},
\end{aligned}$$

which implies that

$$\|T_g\|_{e,B_p^K(s)\to F_K(q,q-2,qs/p,s,q/p)} \gtrsim \limsup_{n\to\infty} \left(\frac{1}{|I_n|^{\frac{qs}{p}}} \int_{S(I_n)} |g'(z)|^q (1-|z|^2)^{q-2+\frac{qs}{p}} dA(z)\right)^{\frac{1}{q}}.$$

By Lemma 5 and the arbitrariness of n, we have

$$\|T_g\|_{e,B_p^K(s)\to F_K(q,q-2,qs/p,s,q/p)} \gtrsim dist_{F(q,q-2,qs/p)}(g, F_0(q,q-2,qs/p)).$$

On the other hand, by Lemma 6, we see that $T_{g_r} : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ is compact. Then

$$\begin{aligned}
\|T_g\|_{e,B_p^K(s)\to F_K(q,q-2,qs/p,s,q/p)} &\leqslant \|T_g - T_{g_r}\| = \|T_{g-g_r}\| \\
&\approx \|g-g_r\|_{F(q,q-2,qs/p)}.
\end{aligned}$$

Using Lemma 5 again, we obtain

$$\begin{aligned}
\|T_g\|_{e,B_p^K(s)\to F_K(q,q-2,qs/p,s,q/p)} &\lesssim \limsup_{r\to 1^-} \|g-g_r\|_{F(q,q-2,\frac{qs}{p})} \\
&\approx dist_{F(q,q-2,qs/p)}(g, F_0(q,q-2,qs/p)).
\end{aligned}$$

The proof is complete. $\square$

The following result can be directly obtained by Theorem 3.

COROLLARY 1. *Let* $g \in H(\mathbb{D})$, $0 < s < 1 < p < q < \infty$ *and* K *satisfy* (2) *for some* $\delta \in (0,s)$. *Then the operator* $T_g : B_p^K(s) \to F_K(q,q-2,qs/p,s,q/p)$ *is compact if and only if* $g \in F_0(q,q-2,qs/p)$.

Acknowledgements. The second author is supported by Guang-Dong Basic and Applied Basic Research Foundation (no. 2022A1515010317) and the Foundation for Scientific and Technological Innovation in Higher Education of Guangdong (no. 2021 KTSCX182).

REFERENCES

[1] A. ALEMAN AND J. CIMA, *An integral operator on H^p and Hardy's inequality*, J. Anal. Math. **85** (2001), 157–176.

[2] A. ALEMAN AND A. SISKAKIS, *Integration operators on Bergman spaces*, Indiana Univ. Math. J. **46** (1997), 337–356.

[3] D. BLASI AND J. PAU, *A characterization of Besov type spaces and applications to Hankle type operators*, Michigan Math. J. **56** (2008), 401–417.

[4] P. GALANOPOULOS, N. MERCHÁN AND A. SISKAKIS, *A family of Dirichlet-Morrey spaces*, Complex Var. Elliptic Equ. **64** (2019), 1686–1702.

[5] D. GIRELA AND J. PELÁEZ, *Carleson measures, multipliers and integration operators for spaces of Dirichlet type*, J. Funct. Anal. **241** (2006), 334–358.

[6] L. HU, R. YANG AND S. LI, *Dirichlet-Morrey type spaces and Volterra integral operators*, J. Nonlinear Var. Anal. **5** (2021), 477–491.

[7] P. LI, J. LIU AND Z. LOU, *Integral operators on analytic Morrey spaces*, Sci. China Math. **57** (2014), 1961–1974.

[8] S. LI, J. LIU AND C. YUAN, *Embedding theorem for Dirichlet type spaces*, Canad. Math. Bull. **63** (2020), 106–17.

[9] J. LIU AND Z. LOU, *Carleson measure for analytic Morrey spaces*, Nonlinear Anal. **125** (2015), 423–432.

[10] J. LIU, Z. LOU AND K. ZHU, *Embedding of Möbius invariant function spaces into tent spaces*, J. Geom. Anal. **27** (2017), 1013–1028.

[11] J. ORTEGA AND J. FÀBREGA, *Pointwise multipliers and corona type decomposition in BMOA*, Ann. Inst. Fourier (Grenoble) **46** (1996), 111–137.

[12] J. PAU AND R. ZHAO, *Carleson measures, Riemann-Stieltjes and multiplication operators on a general family of function spaces*, Integral Equ. Operator Theory **78** (2014), 483–514.

[13] C. POMMERENKE, *Schlichte Funktionen und analytische Funktionen von beschrankter mittlerer Oszillation*, Comment. Math. Helv. **52** (1997), 591–602.

[14] R. QIAN AND S. LI, *Volterra type operators on Morrey type spaces*, Math. Inequal. Appl. **18** (2015), 1589–1599.

[15] R. QIAN AND X. ZHU, *Embedding of Q_p spaces into tent spaces and Volterra integral operator*, AIMS Math. **6** (2020), 698–711.

[16] C. SHEN, Z. LOU AND S. LI, *Embedding of $BMOA_{\log}$ into tent spaces and Volterra integral operators*, Comput. Meth. Funct. Theory 20 (2020), 217–234.

[17] Y. SHI AND S. LI, *Essential norm of integral operators on Morrey type spaces*, Math. Inequal. Appl. **19** (2016), 385–393.

[18] F. SUN AND H. WULAN, *Characterizations of Morrey type spaces*, Canad. Math. Bull. **65** (2022), 328–344.

[19] M. TJANI, *Compact composition operators on some Möbius invariant Banach spaces*, Michigan State University, Department of Mathematics (1996).

[20] H. WULAN AND J. ZHOU, *Q_K and Morrey type spaces*, Ann. Acad. Sci. Fenn. Math. **38** (2013), 193–207.
[21] H. WULAN AND K. ZHU, *Möbius Invariant Q_K Spaces*, Springer, Cham, 2017.
[22] R. YANG AND X. ZHU, *Besov-Morrey spaces and Volterra integral operator*, Math. Inequal. Appl. **24** (2021), 857–871.
[23] R. ZHAO, *On a general family of function spaces*, Ann. Acad. Sci. Fenn. Math. Diss. **105** (1996), 56.
[24] K. ZHU, *Operator Theory in Function Spaces*, 2nd edn, American Mathematical Society, Providence (2007).

(Received April 8, 2022)

Dan Qu
Faculty of Innovation Engineering
Macau University of Science and Technology
Avenida Wai Long, Taipa, Macau
and
School of Mathematics and Statistics
Hanshan Normal University
Chaozhou 521041, China
e-mail: 16983147@qq.com

Xiangling Zhu
University of Electronic Science and Technology of China
Zhongshan Institute
528402, Zhongshan, Guangdong, P. R. China
e-mail: jyuzxl@163.com