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ON A NEW PRODUCT–TYPE OPERATOR ON THE UNIT BALL

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Abstract. Let $m \in \mathbb{N}$, u_j, $j=\overline{1,m}$, be holomorphic functions on the open unit ball $\mathbb{B} \subset \mathbb{C}^n$, φ be a holomorphic self-map of $\mathbb{B}$, and D_l be the partial derivative operator in the lth variable $l \in \{1,2,\dots,n\}$. We introduce here the following polynomial differentiation composition operator

$$P^m_{D,\varphi} f := \sum_{j=1}^m u_j C_\varphi D_{l_j} \cdots D_{l_1} f$$

and give some necessary and sufficient conditions for the boundedness and compactness of the operator from the logarithmic Bloch spaces to weighted-type spaces of holomorphic functions on $\mathbb{B}$.

1. Introduction

Before we present some basic facts, previous investigations in the topic of the paper, motivations, and the aim of the investigation, we give some notation and conventions.

1.1. Notation and conventions

Let $\mathbb{N}$ denote the set of positive integers, $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, and $\mathbb{C}$ the set of all complex numbers. If $k,l \in \mathbb{N}_0$, then we use the notation $j=\overline{k,l}$ instead of $j=k,\dots,l$, $k \leqslant l$. If $s,t \in \mathbb{N}_0$ and $s>t$, then we regard by convention that $\sum_{j=s}^t a_j = 0$ and $\prod_{j=s}^{s-1} a_j = 1$.

Let $\mathbb{B}^n = \mathbb{B}$ be the open unit ball in $\mathbb{C}^n$ and $\mathbb{D} = \mathbb{B}^1$. Let $z=(z_1,z_2,\dots,z_n)$, $w=(w_1,w_2,\dots,w_n) \in \mathbb{C}^n$, then the quantity $\langle z,w\rangle = z_1\overline{w}_1 + z_2\overline{w}_2 + \cdots + z_n\overline{w}_n$ is the standard inner product in $\mathbb{C}^n$, whereas $|z| = \langle z,z\rangle^{1/2}$ is the corresponding norm therein.

If $n>1$, then by D_j, where $j \in \{1,2,\dots,n\}$, we denote the partial derivative operator

$$D_j f = \frac{\partial f}{\partial z_j}, \quad j=\overline{1,n}, \tag{1}$$

(the differentiation in the jth variable).

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By Ω we denote a domain in $\mathbb{C}^n$ (an open and connected subset in $\mathbb{C}^n$), $H(\Omega)$ the linear space of holomorphic functions on Ω, and by $S(\Omega)$ the class of holomorphic self-maps of Ω. Some information on holomorphic functions in several variables can be found, for example, in [19] and [20].

1.2. Basic operators and some history

Let $\varphi \in S(\Omega)$. The composition operator C_φ induced by φ is defined by

$$C_\varphi f = f \circ \varphi, \quad f \in H(\Omega). \tag{2}$$

Let $u \in H(\Omega)$, then the multiplication operator M_u induced by function u is defined by

$$M_u f = uf, \quad f \in H(\Omega). \tag{3}$$

Let $n = 1$ and $m \in \mathbb{N}_0$, then the mth differentiation operator D^m on $H(\Omega)$ is defined by

$$D^m f(z) = f^{(m)}(z), \quad f \in H(\Omega),\ z \in \Omega,$$

where we regard that $D^0 = I$ (the identity operator, i.e., $If = f$), whereas $D^1 = D$ is the classical differentiation operator, that is,

$$Df = f', \quad f \in H(\Omega). \tag{4}$$

There have been a lot of investigations of products of operators (2), (3), (4), and other operators on spaces of holomorphic functions. Out of the ones containing the differentiation operator, the products DC_φ and $C_\varphi D$ seem studied first, usually on some subspaces of $H(\mathbb{D})$ (see, for instance, [6, 11, 12, 16, 17] and the related references therein). Some of the products of all three operators in (2)–(4) on subspaces of $H(\mathbb{D})$ were studied, for example, in [7, 10, 24]. The weighted differentiation composition operators

$$D^m_{\varphi,u} := M_u C_\varphi D^m$$

and their special cases (usually with $u \equiv 1$), have been also considerably studied on subspaces of $H(\mathbb{D})$ (see, e.g., [13, 28, 37, 38, 39, 42, 43, 44, 45]). The operator

$$\mathfrak{R}^m_{\varphi,u} := M_u C_\varphi \mathfrak{R}^m,$$

where $\mathfrak{R}$ is the radial differentiation operator

$$\mathfrak{R} f = \sum_{j=1}^n z_j D_j f,$$

which is a multidimensional relative of the operator $D^m_{\varphi,u}$, was introduced in [29], and studied later, for example, in [30, 32, 33]. Some other product-type operators containing differentiation type operators have been also studied in [14, 41]. In [34] and [35] were investigated sums of two operators of the form $M_u C_\varphi D^j$, and the investigation was continued, e.g., in [1, 4, 5, 8, 36]. For some other related operators, including those containing integral ones, see, e.g., [9, 18, 23, 25] and the related references therein.

1.3. A new operator

A natural problem is to investigate sums of related product-type operators on spaces of holomorphic functions on $\mathbb{B}$. I have introduced such an operator recently, and studied it, e.g., in [31]. Here we introduce and start studying properties of the following new polynomial differentiation composition operator

$$P^m_{D,\varphi}f := \sum_{j=1}^{m} u_j C_\varphi D_{l_j}\cdots D_{l_1} f, \quad f \in H(\mathbb{B}), \tag{5}$$

where $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{1,m}$, and $\varphi \in S(\mathbb{B})$.

REMARK 1. Since we deal with holomorphic functions, which on $\mathbb{B}$ have derivatives of all orders, note that the order of the differentiation operators appearing in the summands in (5) is not relevant. Namely, for any permutation σ of the entries in the vector $(1,2,\dots,j)$, $j \in \mathbb{N}\setminus\{1\}$, we have

$$D_{l_j}\cdots D_{l_1} f = D_{l_{\sigma(j)}}\cdots D_{l_{\sigma(1)}} f.$$

Since we have products of maximally m partial differentiation operators, the observation concerns the case $2 \leqslant j \leqslant m$.

1.4. The spaces used in the paper

The logarithmic Bloch space $\mathscr{B}_{\log}(\mathbb{B}) = \mathscr{B}_{\log}$ (see, e.g., [25]) consists of all $f \in H(\mathbb{B})$ such that

$$\beta_1(f) = \sup_{z\in\mathbb{B}}(1-|z|^2)\left(\ln\frac{e}{1-|z|^2}\right)|\nabla f(z)| < +\infty.$$

It becomes a Banach space with the norm $\|f\|_{\mathscr{B}_{\log},1} = |f(0)| + \beta_1(f)$.

By $\mathscr{B}_{\log,0}(\mathbb{B}) = \mathscr{B}_{\log,0}$ we denote the class of all $f \in \mathscr{B}_{\log}$ such that

$$\lim_{|z|\to 1}(1-|z|^2)\left(\ln\frac{e}{1-|z|^2}\right)|\nabla f(z)| = 0.$$

It is known that $\mathscr{B}_{\log,0}$ is a closed subspace of $\mathscr{B}_{\log}$ and that the set of all polynomials is dense in $\mathscr{B}_{\log,0}$ (see, e.g., [26]). For some results on the logarithmic Bloch-type spaces and operators from or to them see, e.g., [13, 14, 25, 26, 40].

Since

$$\beta(f) := \sup_{z\in\mathbb{B}}(1-|z|^2)\left(\ln\frac{e}{1-|z|^2}\right)|\Re f(z)| \leqslant \sup_{z\in\mathbb{B}}(1-|z|^2)\left(\ln\frac{e}{1-|z|^2}\right)|\nabla f(z)|,$$

for every $f \in H(\mathbb{B})$, and $\|f\|_{\mathscr{B}_{\log}} := |f(0)| + \beta(f)$ is another norm on space $\mathscr{B}_{\log}$, by the open mapping theorem (see, e.g., [3] or [21]), we have that the norms are equivalent on the space, that is, there is $C_0 \geqslant 1$ such that

$$C_0^{-1}\|f\|_{\mathscr{B}_{\log,1}} \leqslant \|f\|_{\mathscr{B}_{\log}} \leqslant C_0\|f\|_{\mathscr{B}_{\log,1}}, \tag{6}$$

for every $f \in \mathscr{B}_{\log}$. So, we can use either of the norms depending on the situation.

A positive and continuous function on $\mathbb{B}$ is called weight function. Let μ be a weight function. The weighted-type space $H_\mu^\infty(\mathbb{B}) = H_\mu^\infty$ consists of all $f \in H(\mathbb{B})$ such that

$$\|f\|_{H_\mu^\infty} := \sup_{z\in\mathbb{B}} \mu(z)|f(z)| < +\infty.$$

The little weighted-type space $H_{\mu,0}^\infty(\mathbb{B}) = H_{\mu,0}^\infty$ is a subspace of H_μ^∞ containing all $f \in H(\mathbb{B})$ such that

$$\lim_{|z|\to 1} \mu(z)|f(z)| = 0.$$

It is well known that $H_{\mu,0}^\infty$ is a closed subspace of H_μ^∞. If $\mu \equiv 1$, then H_μ^∞ becomes the space of bounded holomorphic function on $\mathbb{B}$, which is denoted by H^∞. For these and related weighted-type spaces and operators acting from or to them see, for example, [2, 7, 9, 13, 14, 27, 28, 32, 38, 43, 46] and the related references therein).

1.5. Our aim

Let X and Y be two Banach spaces, and $L : X \to Y$ be a linear operator. The operator is called bounded if there is $M \in [0, +\infty)$ such that

$$\|Lf\|_Y \leqslant M\|f\|_X,$$

for every $f \in X$. If the operator maps bounded subsets of X to relatively compact subsets of Y, then we say that it is compact. Many information on such linear operators can be found, for instance, in books [3] and [21]. If Z is a normed space, then by B_Z we denote the unit ball in Z.

Investigation of boundedness and compactness of concrete operators on spaces of holomorphic functions on domains in the complex plain $\mathbb{C}$ or the vector space $\mathbb{C}^n$, is a topic of a great recent interest. Some results in the topic can be found, e.g., in [1, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 37, 38, 39, 40, 41, 42, 43, 44, 45, 46], as well as in the related references therein.

Our aim here is to investigate the boundedness and compactness of the operator (5) from the logarithmic Bloch spaces to weighted-type spaces on $\mathbb{B}$.

1.6. On constants in the paper

As usual, the constants in this paper are denoted by C. They may vary from line to line. The notation $a \lesssim b$ (resp. $a \gtrsim b$) means that there is a constant $C > 0$, independent of essential variables, such that $a \leqslant Cb$ (resp. $a \geqslant Cb$). When $a \lesssim b$ and $b \lesssim a$, we write $a \asymp b$.

2. Preliminary results

In this section, we present several auxiliary results which are used in the proofs of the main results in this paper.

The first auxiliary result is a typical characterization for the compactness which is proved similarly to the proof of the corresponding result in [22], so we omit the proof.

LEMMA 1. *Let* $u_j \in H(\mathbb{B})$, $j=\overline{1,m}$, $\varphi \in S(\mathbb{B})$, μ *be a weight function on* $\mathbb{B}$, *and* $X=\mathscr{B}_{\log}(\mathbb{B})$ *(or* $\mathscr{B}_{\log,0}(\mathbb{B})$*). Then the bounded operator* $P^m_{D,\varphi}: X \to H^\infty_\mu(\mathbb{B})$ *is compact if and only if for any bounded sequence* $(f_k)_{k\in\mathbb{N}} \subset X$ *such that* $f_k \to 0$ *uniformly on compact subsets of* $\mathbb{B}$ *as* $k \to +\infty$, *it follows that*

$$\lim_{k\to+\infty} \|P^m_{D,\varphi} f_k\|_{H^\infty_\mu} = 0.$$

The following lemma is a technical auxiliary result, which is easily proved by differentiation.

LEMMA 2. *Let* $k \geqslant 1$ *and* $a \geqslant \sqrt[k]{e}$. *Then the function*

$$h_{k,a}(x) = x^k \ln\frac{a}{x},$$

is increasing on the interval $(0,1]$.

The following result gives some estimates for differentiation operators on $\mathscr{B}_{\log}(\mathbb{B})$.

LEMMA 3. *Let* $N \in \mathbb{N}$, *and for each* $k \in \{1,\dots,N\}$, D_{j_k} *be one of the operators in* (1). *Then there are* $C_N > 0$ *and* $E_N \geqslant 1$ *such that*

$$\left|D_{j_N}\cdots D_{j_2} D_{j_1} f(z)\right| \leqslant \frac{C_N\,\|f\|_{\mathscr{B}_{\log}}}{(1-|z|^2)^N \ln\frac{E_N e}{1-|z|^2}}, \tag{7}$$

for every $f \in \mathscr{B}_{\log}(\mathbb{B})$ *and* $z \in \mathbb{B}$.

Proof. If $N=1$, then by using the fact

$$|D_j f(z)| \leqslant |\nabla f(z)|, \quad j=\overline{1,n},$$

the definition of the logarithmic space, and (6), we obtain immediately estimate (7), with $C_1 = C_0$ and $E_1 = 1$, where C_0 is defined in (6).

We prove that

$$\left|D_{j_N}\cdots D_{j_2} D_{j_1} f(z)\right| \leqslant \frac{\widehat{C}_N}{(1-|z|)^N \ln\frac{\widehat{E}_N e}{1-|z|}}\|f\|_{\mathscr{B}_{\log}}, \tag{8}$$

for every $f \in \mathscr{B}_{\log}(\mathbb{B})$ and $z \in \mathbb{B}$, and some $\widehat{C}_N > 0$ and $\widehat{E}_N \geqslant 1$, from which along with

$$(1-|z|)^N \ln\frac{E_N e}{1-|z|} \geqslant \frac{(1-|z|^2)^N}{2^N} \ln\frac{E_N e}{1-|z|^2}, \quad z \in \mathbb{B},$$

estimate (7) with $C_N = 2^N \widehat{C}_N$ and $E_N = \widehat{E}_N$, easily follows.

Assume that we have proved (8) for any such operator with $N = k-1$ for some $k \in \mathbb{N} \setminus \{1\}$, that is, there exist $\widehat{C}_{k-1} > 0$ and $\widehat{E}_{k-1} \geqslant 1$ such that

$$\left|D_{i_{k-1}} \cdots D_{i_2} D_{i_1} f(z)\right| \leqslant \frac{\widehat{C}_{k-1}}{(1-|z|)^{k-1} \ln \frac{\widehat{E}_{k-1} e}{1-|z|}} \|f\|_{\mathscr{B}_{\log}}, \tag{9}$$

for all $f \in \mathscr{B}_{\log}(\mathbb{B})$ and $z \in \mathbb{B}$, where $D_{i_{k-1}} \cdots D_{i_2} D_{i_1}$ is any product of $k-1$ partial differential operators.

Let $g \in H(\mathbb{B})$ and $q \in (0,1)$. By Chauchy's estimate, we have

$$|D_j g(z)| \leqslant \widetilde{C}_2 \frac{\sup_{w \in B(z, q(1-|z|))} |g(w)|}{1-|z|}, \tag{10}$$

for $j \in \{1, 2, \dots, n\}$, $z \in \mathbb{B}$ and some $\widetilde{C}_2 > 0$.

If $w \in B(z, q(1-|z|))$, then we have

$$(1-q)(1-|z|) \leqslant 1-|w| \leqslant (1+q)(1-|z|).$$

From this, and by Lemma 2 where k is replaced by $k-1$ and $a = \widehat{E}_{k-1} e$, we have

$$(1-q)^{k-1}(1-|z|)^{k-1} \ln \frac{\widehat{E}_{k-1} e}{(1-q)(1-|z|)} \leqslant (1-|w|)^{k-1} \ln \frac{\widehat{E}_{k-1} e}{1-|w|}, \tag{11}$$

for $w \in B(z, q(1-|z|))$ and $z \in \mathbb{B}$.

From (9) and (11), it follows that

$$\sup_{w \in B(z, q(1-|z|))} |D_{i_{k-1}} \cdots D_{i_2} D_{i_1} f(w)| \leqslant \frac{\widehat{C}_{k-1}}{(1-q)^{k-1}} \frac{\|f\|_{\mathscr{B}_{\log}}}{(1-|z|)^{k-1} \ln \frac{\frac{\widehat{E}_{k-1} e}{1-q}}{1-|z|}}. \tag{12}$$

From (10) with $g = D_{i_{k-1}} \cdots D_{i_2} D_{i_1} f$, and where j is replaced by i_k, we have

$$|D_{i_k} D_{i_{k-1}} \cdots D_{i_2} D_{i_1} f(z)| \leqslant \widetilde{C}_2 \frac{\sup_{w \in B(z, q(1-|z|))} |D_{i_{k-1}} \cdots D_{i_2} D_{i_1} f(w)|}{1-|z|}. \tag{13}$$

Combining (12) and (13) we obtain

$$\left|D_{i_k} \cdots D_{i_2} D_{i_1} f(z)\right| \leqslant \frac{\widehat{C}_k}{(1-|z|)^k \ln \frac{\widehat{E}_k e}{1-|z|}} \|f\|_{\mathscr{B}_{\log}}, \tag{14}$$

where $\widehat{C}_k = \widetilde{C}_2 (1-q)^{1-k} \widehat{C}_{k-1}$ and $\widehat{E}_k = \widehat{E}_{k-1}/(1-q)$, from which together with $\widehat{E}_1 = 1$ we have $\widehat{E}_k = (1-q)^{1-k} > 1$, $k \geqslant 2$. From this and the method of induction, relation (8), and consequently (7) follows. $\square$

The following result gives a useful class of functions belonging to $\mathscr{B}_{\log,0}(\mathbb{B})$.

LEMMA 4. *Let $w \in \mathbb{B}$, $t \geqslant 1$ and $E \geqslant 1$, then the following function*

$$f_{w,t}(z) = \frac{(1-|w|^2)^t}{(1-\langle z,w\rangle)^t \ln \frac{Ee}{1-|w|^2}}, \tag{15}$$

belongs to $\mathscr{B}_{\log,0}(\mathbb{B})$.

Moreover, we have

$$\sup_{w\in\mathbb{B}} \|f_{w,t}\|_{\mathscr{B}_{\log}} \lesssim 1. \tag{16}$$

Proof. By a direct calculation, we have

$$\Re f_{w,t}(z) = \frac{t(1-|w|^2)^t \langle z,w\rangle}{(1-\langle z,w\rangle)^{t+1} \ln \frac{Ee}{1-|w|^2}},$$

and consequently

$$|\Re f_{w,t}(z)| \leqslant \frac{t(1+|w|)^t |w|}{(1-|w|) \ln \frac{Ee}{1-|w|^2}}.$$

This shows that $\Re f_{w,t}$ is bounded on $\mathbb{B}$.

From this and since

$$\lim_{|z|\to 1} (1-|z|^2) \ln \frac{e}{1-|z|^2} = 0, \tag{17}$$

we obtain that $f_{w,t} \in \mathscr{B}_{\log,0}(\mathbb{B})$.

By using the following inequality

$$(1-|z|^2)\left(\ln \frac{e}{1-|z|^2}\right) \leqslant 2(1-|z|)\left(\ln \frac{e}{1-|z|}\right), \quad z \in \mathbb{B},$$

and Lemma 2 with $k=1$ and $a=e$, and $k=t$ and $a=e$, we have

$$\begin{aligned}
(1-|z|^2)\left(\ln \frac{e}{1-|z|^2}\right)|\Re f_{w,t}(z)| &= \frac{t(1-|z|^2)(1-|w|^2)^t |\langle z,w\rangle| \ln \frac{e}{1-|z|^2}}{|1-\langle z,w\rangle|^{t+1} \ln \frac{Ee}{1-|w|^2}} \\
&\leqslant \frac{2t(1-|z|)(1-|w|^2)^t \ln \frac{e}{1-|z|}}{(1-|z||w|)^{t+1} \ln \frac{Ee}{1-|w|^2}} \\
&= \frac{2t(1-|z|) \ln \frac{e}{1-|z|}}{(1-|z||w|) \ln \frac{e}{1-|z||w|}} \cdot \frac{(1-|w|^2)^t \ln \frac{e}{1-|z||w|}}{(1-|z||w|)^t \ln \frac{Ee}{1-|w|^2}} \\
&\leqslant \frac{t2^{t+1} \ln \frac{e}{1-|w|}}{\ln \frac{Ee}{1-|w|^2}} = t2^{t+1} \frac{\ln \frac{e}{1-|w|^2} + \ln(1+|w|)}{\ln \frac{e}{1-|w|^2} + \ln E} \\
&\leqslant t2^{t+1} \ln(2e),
\end{aligned}$$

from which (16) follows. □

The following lemma presents a family of test functions, which is employed in the investigation of the boundedness and compactness of the operators studied in this paper.

LEMMA 5. *Let* $m\in\mathbb{N}$, $w\in\mathbb{B}$ *and* $E\geqslant 1$. *Then for each* $s\in\{1,\dots,m\}$ *there exist constants* $c_k^{(s)}$, $k=\overline{1,m}$, *such that the function*

$$h_w^{(s)}(z)=\sum_{k=1}^m c_k^{(s)} f_{w,k}(z)$$

where $f_{w,t}$ *is defined in* (15), *satisfies*

$$D_{l_s}\cdots D_{l_1}h_w^{(s)}(w)=\frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_s}}{(1-|w|^2)^s\ln\frac{Ee}{1-|w|^2}} \tag{18}$$

and

$$D_{l_r}\cdots D_{l_1}h_w^{(s)}(w)=0, \tag{19}$$

for every $r\in\{1,\dots,m\}\setminus\{s\}$.

Moreover, $h_w^{(s)}\in\mathscr{B}_{\log,0}(\mathbb{B})$ *and*

$$\sup_{w\in\mathbb{B}}\|h_w^{(s)}\|_{\mathscr{B}_{\log}}\lesssim 1. \tag{20}$$

Proof. Let

$$h_w(z)=\sum_{k=1}^m c_k f_{w,k}(z).$$

By some standard calculations we get

$$D_{l_r}\cdots D_{l_1}h_w(z)=\sum_{k=1}^m c_k\frac{k(k+1)\cdots(k+r-1)\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_r}(1-|w|^2)^k}{(1-\langle z,w\rangle)^{k+r}\ln\frac{Ee}{1-|w|^2}},$$

for each $r\in\mathbb{N}_0$, from which it follows that

$$D_{l_r}\cdots D_{l_1}h_w(w)=\frac{\overline{w}_{l_1}\overline{w}_{l_2}\cdots\overline{w}_{l_r}}{(1-|w|^2)^r\ln\frac{Ee}{1-|w|^2}}\sum_{k=1}^m c_k\prod_{l=0}^{r-1}(k+l), \tag{21}$$

for each $r\in\mathbb{N}_0$.

Consider the linear system

$$\begin{bmatrix} 1 & 2 & \cdots & m\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=1}^{s}k & \prod_{k=1}^{s}(k+1) & \cdots & \prod_{k=1}^{s}(k+m-1)\\ \vdots & \vdots & \ddots & \vdots\\ \prod_{k=1}^{m}k & \prod_{k=1}^{m}(k+1) & \cdots & \prod_{k=1}^{m}(k+m-1)\end{bmatrix}\begin{bmatrix} c_1\\ \vdots\\ c_s\\ \vdots\\ c_m\end{bmatrix}=\begin{bmatrix}0\\ \vdots\\ 1\\ \vdots\\ 0\end{bmatrix}, \tag{22}$$

where the unit in the last vector is on the sth position.

The determinant D_m of system (22) is

$$D_m = m! \begin{vmatrix} 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=2}^{s} k & \prod_{k=2}^{s}(k+1) & \cdots & \prod_{k=2}^{s}(k+m-1) \\ \vdots & \vdots & \ddots & \vdots \\ \prod_{k=2}^{m} k & \prod_{k=2}^{m}(k+1) & \cdots & \prod_{k=2}^{m}(k+m-1) \end{vmatrix}$$

The last determinant is different from zero (see, e.g., [27, Lemma 5]), and consequently $D_m \neq 0$. Hence, for each $s \in \{1,\dots,m\}$, there is a unique solution to the system $c_k := c_k^{(s)}$, $k = \overline{1,m}$. Let $h_w^{(s)}(z) := \sum_{k=1}^{m} c_k^{(s)} f_{w,k}(z)$, then it satisfies (18) and (19). Lemma 4 implies $h_w^{(s)} \in \mathscr{B}_{\log,0}(\mathbb{B})$, and (16) implies (20), as desired. □

3. Main results

This section contains our main results. First, we consider the boundedness of the operator $P_{D,\varphi}^m : \mathscr{B}_{\log}(\mathbb{B})(\text{or } \mathscr{B}_{\log,0}(\mathbb{B})) \to H_\mu^\infty(\mathbb{B})$.

THEOREM 1. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{1,m}$, $\varphi = (\varphi_1,\dots,\varphi_n) \in S(\mathbb{B})$,*

$$\min_{j=\overline{1,n}} \inf_{z\in\mathbb{B}} |\varphi_j(z)| \geqslant \delta > 0, \tag{23}$$

and μ be a weight function on $\mathbb{B}$. Then the operator $P_{D,\varphi}^m : \mathscr{B}_{\log}(\mathbb{B})(\text{or } \mathscr{B}_{\log,0}(\mathbb{B})) \to H_\mu^\infty(\mathbb{B})$ is bounded if and only if

$$M_j := \sup_{z\in\mathbb{B}} \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j \ln \frac{E_j e}{1-|\varphi(z)|^2}} < +\infty, \quad j = \overline{1,m}, \tag{24}$$

where E_j, $j = \overline{1,m}$, are defined in Lemma 3.

Moreover, if the operator $P_{D,\varphi}^m : \mathscr{B}_{\log}(\mathbb{B})(\text{or } \mathscr{B}_{\log,0}(\mathbb{B})) \to H_\mu^\infty(\mathbb{B})$ is bounded, then the following asymptotic relationships hold

$$\|P_{D,\varphi}^m\|_{\mathscr{B}_{\log}(\mathbb{B})\to H_\mu^\infty(\mathbb{B})} \asymp \|P_{D,\varphi}^m\|_{\mathscr{B}_{\log,0}(\mathbb{B})\to H_\mu^\infty(\mathbb{B})} \asymp \sum_{j=1}^{m} M_j. \tag{25}$$

Proof. Suppose that (24) holds. By using Lemma 3, we have

$$\begin{aligned} \mu(z)|P_{D,\varphi}^m f(z)| &= \mu(z)\left|\sum_{j=1}^{m} u_j(z) D_{l_j}\cdots D_{l_1} f(\varphi(z))\right| \\ &\lesssim \sum_{j=1}^{m} \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j \ln \frac{E_j e}{1-|\varphi(z)|^2}} \|f\|_{\mathscr{B}_{\log}}. \end{aligned} \tag{26}$$

By taking the supremum in (26) over $\mathbb{B}$, then over the ball $B_{\mathscr{B}_{\log}(\mathbb{B})}$ (or $B_{\mathscr{B}_{\log,0}(\mathbb{B})}$), and using (24), we have that the operator $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})(\text{or } \mathscr{B}_{\log,0}(\mathbb{B}))\to H^\infty_\mu(\mathbb{B})$ is bounded. From this and since $B_{\mathscr{B}_{\log,0}(\mathbb{B})}\subset B_{\mathscr{B}_{\log}(\mathbb{B})}$, we have

$$\|P^m_{D,\varphi}\|_{\mathscr{B}_{\log,0}(\mathbb{B})\to H^\infty_\mu(\mathbb{B})}\leqslant \|P^m_{D,\varphi}\|_{\mathscr{B}_{\log}(\mathbb{B})\to H^\infty_\mu(\mathbb{B})}\lesssim \sum_{j=1}^m M_j. \tag{27}$$

Now, suppose that $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})(\text{or } \mathscr{B}_{\log,0}(\mathbb{B}))\to H^\infty_\mu(\mathbb{B})$ is bounded. Then there exists a positive constant C such that

$$\|P^m_{D,\varphi}f\|_{H^\infty_\mu}\leqslant C\|f\|_{\mathscr{B}_{\log}} \tag{28}$$

for every $f\in\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$).

By Lemma 5, for each $s\in\{1,\dots,m\}$, $\varphi(w)\in\mathbb{B}$ and $E=E_s$, there is a function $h^{(s)}_{\varphi(w)}\in\mathscr{B}_{\log,0}(\mathbb{B})$ such that

$$D_{l_s}\cdots D_{l_1}h^{(s)}_{\varphi(w)}(\varphi(w))=\frac{\overline{\varphi_{l_1}(w)\varphi_{l_2}(w)}\cdots\overline{\varphi_{l_s}(w)}}{(1-|\varphi(w)|^2)^s\ln\frac{E_se}{1-|\varphi(w)|^2}}, \tag{29}$$

$$D_{l_t}\cdots D_{l_1}h^{(s)}_{\varphi(w)}(\varphi(w))=0, \tag{30}$$

for every $t\in\{1,\dots,m\}\setminus\{s\}$, and $L_s:=\sup_{w\in\mathbb{B}}\|h^{(s)}_{\varphi(w)}\|_{\mathscr{B}_{\log}}<+\infty$.

From this, the relations in (28)–(30), and employing condition (23), it follows that

$$\begin{aligned}
L_s\|P^m_{D,\varphi}\|_{X\to H^\infty_\mu(\mathbb{B})}\geqslant&\|P^m_{D,\varphi}h^{(s)}_{\varphi(w)}\|_{H^\infty_\mu}\\
=&\sup_{z\in\mathbb{B}}\mu(z)\Big|\sum_{j=1}^m u_j(z)D_{l_j}\cdots D_{l_1}h^{(s)}_{\varphi(w)}(\varphi(z))\Big|\\
\geqslant&\mu(w)\Big|\sum_{j=1}^m u_j(w)D_{l_j}\cdots D_{l_1}h^{(s)}_{\varphi(w)}(\varphi(w))\Big|\\
=&\mu(w)|u_s(w)|\frac{|\overline{\varphi_{l_1}(w)}|\cdots|\overline{\varphi_{l_s}(w)}|}{(1-|\varphi(w)|^2)^s\ln\frac{E_se}{1-|\varphi(w)|^2}}\\
\geqslant&\delta^s\frac{\mu(w)|u_s(w)|}{(1-|\varphi(w)|^2)^s\ln\frac{E_se}{1-|\varphi(w)|^2}},
\end{aligned} \tag{31}$$

for every $w\in\mathbb{B}$, where $X\in\{\mathscr{B}_{\log}(\mathbb{B}),\mathscr{B}_{\log,0}(\mathbb{B})\}$.

By taking the supremum in (31) over $\mathbb{B}$, we obtain $M_s<+\infty$, for each $s\in\{1,\dots,m\}$. Moreover, we have

$$L_s\|P^m_{D,\varphi}\|_{X\to H^\infty_\mu(\mathbb{B})}\geqslant\delta^sM_s,\quad s=\overline{1,m},$$

from which it follows that

$$\sum_{j=1}^m M_j\lesssim\|P^m_{D,\varphi}\|_{\mathscr{B}_{\log,0}(\mathbb{B})\to H^\infty_\mu(\mathbb{B})}. \tag{32}$$

Combining asymptotic relations (27) and (32), we get (25), finishing the proof. □

REMARK 3. From the proof of Theorem 1 we see that if we do not pose condition (23), then we can only conclude that

$$\sup_{z\in\mathbb{B}} \frac{\mu(z)|u_s(z)||\varphi_{l_1}(z)|\cdots|\varphi_{l_s}(z)|}{(1-|\varphi(z)|^2)^s \ln\frac{Ee}{1-|\varphi(z)|^2}} < +\infty, \quad s=\overline{1,m}.$$

The following theorem deals with the boundedness of the operator $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})\to H^\infty_{\mu,0}(\mathbb{B})$.

THEOREM 2. *Let $m\in\mathbb{N}$, $u_j\in H(\mathbb{B})$, $j=\overline{1,m}$, $\varphi\in S(\mathbb{B})$, and μ be a weight function on $\mathbb{B}$. Then the operator $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu,0}(\mathbb{B})$ is bounded if and only if $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu}(\mathbb{B})$ is bounded and*

$$\lim_{|z|\to 1}\mu(z)|u_j(z)|=0, \quad j=\overline{1,m}. \tag{33}$$

Proof. Assume that $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu}(\mathbb{B})$ is bounded and (33) holds. Then, for each polynomial p, we have

$$\begin{aligned}\mu(z)\Big|\sum_{j=1}^m u_j(z)D_{l_j}\cdots D_{l_1}p(\varphi(z))\Big| &\leqslant \sum_{j=1}^m \mu(z)|u_j(z)||D_{l_j}\cdots D_{l_1}p(\varphi(z))| \\ &\leqslant \sum_{j=1}^m \mu(z)|u_j(z)|\|D_{l_j}\cdots D_{l_1}p\|_{H^\infty}. \end{aligned} \tag{34}$$

By letting $|z|\to 1$ in (34), using the fact

$$\|D_{l_j}\cdots D_{l_1}p\|_{H^\infty} := \sup_{z\in\mathbb{B}}|D_{l_j}\cdots D_{l_1}p(z)| < +\infty, \quad j=\overline{1,m},$$

and the conditions in (33), it follows that $P^m_{D,\varphi}p\in H^\infty_{\mu,0}(\mathbb{B})$, for each polynomial p.

Since the set of all polynomials is dense in $\mathscr{B}_{\log}(\mathbb{B})$ and $\mathscr{B}_{\log,0}(\mathbb{B})$, we have that for every $f\in\mathscr{B}_{\log}(\mathbb{B})$ there is a sequence of polynomials $(p_k)_{k\in\mathbb{N}}$ such that

$$\lim_{k\to+\infty}\|f-p_k\|_{\mathscr{B}_{\log}}=0.$$

From this and the boundedness of $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu}(\mathbb{B})$ we have

$$\|P^m_{D,\varphi}f-P^m_{D,\varphi}p_k\|_{H^\infty_\mu} \leqslant \|P^m_{D,\varphi}\|_{\mathscr{B}_{\log}(\mathbb{B})\to H^\infty_\mu(\mathbb{B})}\|f-p_k\|_{\mathscr{B}_{\log}}\to 0$$

as $k\to+\infty$.

From this and since $H^\infty_{\mu,0}(\mathbb{B})$ is a closed subspace of $H^\infty_{\mu}(\mathbb{B})$, we have $P^m_{D,\varphi}f\in H^\infty_{\mu,0}(\mathbb{B})$. Thus, we have

$$P^m_{D,\varphi}(\mathscr{B}_{\log}(\mathbb{B}))\subseteq H^\infty_{\mu,0}(\mathbb{B}),$$

from which it follows that the operator $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu,0}(\mathbb{B})$ is bounded.

Now assume that the operator $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu,0}(\mathbb{B})$ is bounded. Then the operator $P^m_{D,\varphi}:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H^\infty_{\mu}(\mathbb{B})$ is bounded.

Let

$$f_1(z)=z_{l_1}. \tag{35}$$

Since $f_1\in\mathscr{B}_{\log,0}(\mathbb{B})$, we have $P^m_{D,\varphi}(f_1)\in H^\infty_{\mu,0}(\mathbb{B})$, that is,

$$\lim_{|z|\to 1}\mu(z)|P^m_{D,\varphi}f_1(z)|=\lim_{|z|\to 1}\mu(z)|u_1(z)|=0. \tag{36}$$

that is, $u_1\in H^\infty_{\mu,0}(\mathbb{B})$.

Let

$$f_2(z)=z_{l_1}z_{l_2}. \tag{37}$$

Since $f_2\in\mathscr{B}_{\log,0}(\mathbb{B})$, we have $P^m_{D,\varphi}(f_2)\in H^\infty_{\mu,0}(\mathbb{B})$. There are two cases to be considered.

Case $l_1\neq l_2$. Since $l_1\neq l_2$, we have

$$\lim_{|z|\to 1}\mu(z)|P^m_{D,\varphi}f_2(z)|=\lim_{|z|\to 1}\mu(z)|u_1(z)\varphi_{l_2}(z)+u_2(z)|=0. \tag{38}$$

From (38), since $|\varphi_{l_2}(z)|<1$, and using (36), it follows that

$$\lim_{|z|\to 1}\mu(z)|u_2(z)|=0, \tag{39}$$

that is, $u_2\in H^\infty_{\mu,0}(\mathbb{B})$, in this case.

Case $l_1=l_2$. Since $l_1=l_2$, we have

$$\lim_{|z|\to 1}\mu(z)|P^m_{D,\varphi}f_2(z)|=\lim_{|z|\to 1}\mu(z)|2u_1(z)\varphi_{l_1}(z)+2u_2(z)|=0. \tag{40}$$

From (40), since $|\varphi_{l_1}(z)|<1$, and using relation (36), it follows that (39) also holds in this case.

Now assume that we have proved (33) for $1\leqslant j\leqslant s$, for an s such that $2\leqslant s<m$. Let

$$f_{s+1}(z)=z_{l_1}z_{l_2}\cdots z_{l_{s+1}}.$$

Since $f_{s+1}\in\mathscr{B}_{\log,0}(\mathbb{B})$, we have $P^m_{D,\varphi}(f_{s+1})\in H^\infty_{\mu,0}(\mathbb{B})$. Note that f_{s+1} can be written in the form

$$f_{s+1}(z)=z_1^{\alpha_1}\cdots z_n^{\alpha_n},$$

for some $\alpha_j\in\mathbb{N}_0$, $j=\overline{1,n}$, such that $\sum_{j=1}^n\alpha_j=s+1$.

Since f_{s+1} is a homogeneous polynomial, it is easy to see that for each $t\in\mathbb{N}_0$ such that $0\leqslant t\leqslant s+1$, we have

$$D_{j_t}\cdots D_{j_1}f_{s+1}(z)=c_t z_1^{\alpha_1-k_1(t)}\cdots z_n^{\alpha_n-k_n(t)},$$

for some $c_t \in \mathbb{N}$, where $k_i(t)$ is the number of appearance the operators D_i in the product operator $D_{j_t} \cdots D_{j_1}$.

Note that

$$\sum_{j=1}^{n} k_i(t) = t$$

and

$$D_{j_{s+1}} \cdots D_{j_1} f_{s+1}(z) = c_{s+1}, \tag{41}$$

for some $c_{s+1} \in \mathbb{N}$.

Hence, we have

$$\lim_{|z|\to 1} \mu(z)|P^m_{D,\varphi} f_{s+1}(z)| = \lim_{|z|\to 1} \mu(z)\left|\sum_{j=1}^{s+1} u_j(z) c_j \prod_{i=1}^{n} (\varphi_i(z))^{\alpha_i - k_i(j)}\right| = 0. \tag{42}$$

From (42), using the facts $|\varphi_i(z)| < 1$, $i = \overline{1,n}$, $\alpha_i \geqslant k_i(j)$, for $i = \overline{1,n}$, $j = \overline{1,s+1}$, the induction hypothesis $u_j \in H^\infty_{\mu,0}(\mathbb{B})$, $j = \overline{1,s}$, and (41), it follows that

$$\lim_{|z|\to 1} c_{s+1}\mu(z)|u_{s+1}(z)| = 0,$$

from which together with the fact $c_{s+1} \neq 0$, we obtain $u_{s+1} \in H^\infty_{\mu,0}(\mathbb{B})$. This inductive argument shows that (33) holds for $j = \overline{1,m}$, as claimed. $\square$

Now we give a characterization for the compactness of the operator $P^m_{D,\varphi} : \mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H^\infty_\mu(\mathbb{B})$.

THEOREM 3. *Let $m \in \mathbb{N}$, $u_j \in H(\mathbb{B})$, $j = \overline{1,m}$, $\varphi \in S(\mathbb{B})$, μ be a weight function on $\mathbb{B}$, and condition (23) holds. Then the operator $P^m_{D,\varphi} : \mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H^\infty_\mu(\mathbb{B})$ is compact if and only if the operator is bounded and the following conditions hold*

$$\lim_{|\varphi(z)|\to 1} \frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j \ln \frac{E_j e}{1-|\varphi(z)|^2}} = 0, \quad j = \overline{1,m}, \tag{43}$$

where E_j, $j = \overline{1,m}$, are defined in Lemma 3.

Proof. Assume that the operator $P^m_{D,\varphi} : \mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H^\infty_\mu(\mathbb{B})$ is compact. Then, the operator is obviously bounded. If we suppose that $\|\varphi\|_\infty < 1$, then the conditions in (43) vacuously hold. Assume now that $\|\varphi\|_\infty = 1$. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{B}$ such that $|\varphi(z_k)| \to 1$ as $k \to +\infty$, and

$$h^{(s)}_k := h^{(s)}_{\varphi(z_k)}, \quad s = \overline{1,m},$$

where $h^{(s)}_w$, $s = \overline{1,m}$, are defined in Lemma 5. Then we have $h^{(s)}_k \in \mathscr{B}_{\log,0}(\mathbb{B})$ and

$$\sup_{k\in\mathbb{N}} \|h^{(s)}_k\|_{\mathscr{B}_{\log}} < +\infty, \tag{44}$$

for $s=\overline{1,m}$.

From the definition of the functions it is easy to see that for each $s\in\{1,\dots,m\}$, $h_k^{(s)}\to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k\to+\infty$. From this, (44), and by using Lemma 1 it follows that

$$\lim_{k\to+\infty}\|P_{D,\varphi}^m h_k^{(s)}\|_{H_\mu^\infty}=0, \tag{45}$$

for $s=\overline{1,m}$.

On the other hand, by the proof of Theorem 1 (see (31)), we have

$$\frac{\mu(z_k)|u_s(z_k)|}{(1-|\varphi(z_k)|^2)^j\ln\frac{E_je}{1-|\varphi(z_k)|^2}}\leqslant C\|P_{D,\varphi}^m h_k^{(s)}\|_{H_\mu^\infty}, \tag{46}$$

for $s=\overline{1,m}$.

By letting $k\to+\infty$ in (46), and using (45), we get (43).

Now assume that the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H_\mu^\infty(\mathbb{B})$ is bounded and that (43) holds. From (43) we have that for arbitrary $\varepsilon>0$, there is $\delta\in(0,1)$ such that

$$\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j\ln\frac{E_je}{1-|\varphi(z)|^2}}<\varepsilon,\quad j=\overline{1,m}, \tag{47}$$

for every $z\in\mathbb{B}$ such that $|\varphi(z)|>\delta$.

Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathscr{B}_{\log}(\mathbb{B})$ or $\mathscr{B}_{\log,0}(\mathbb{B})$ such that

$$\sup_{k\in\mathbb{N}}\|f_k\|_{\mathscr{B}_{\log}}\leqslant M \tag{48}$$

and $f_k\to 0$ uniformly on compact subsets of $\mathbb{B}$ as $k\to+\infty$. Let $K_\delta=\{z\in\mathbb{B}:|\varphi(z)|>\delta\}$.

Then by Lemma 3, (47) and (48), we have

$$\begin{aligned}
\|P_{D,\varphi}^m f_k\|_{H_\mu^\infty}&=\sup_{z\in\mathbb{B}}\mu(z)\Big|\sum_{j=1}^m u_j(z)D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\Big|\\
&=\sup_{z\in K_\delta}\mu(z)\Big|\sum_{j=1}^m u_j(z)D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\Big|\\
&\quad+\sup_{z\in\mathbb{B}\setminus K_\delta}\mu(z)\Big|\sum_{j=1}^m u_j(z)D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\Big|\\
&\leqslant C\sum_{j=1}^m\sup_{z\in K_\delta}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j\ln\frac{E_je}{1-|\varphi(z)|^2}}\|f_k\|_{\mathscr{B}_{\log}}\\
&\quad+C\sum_{j=1}^m\sup_{z\in\mathbb{B}\setminus K_\delta}\mu(z)|u_j(z)||D_{l_j}\cdots D_{l_1}f_k(\varphi(z))|
\end{aligned}$$

$$\begin{aligned}&\leqslant mC\varepsilon\|f_k\|_{\mathscr{B}_{\log}}\\&\quad+C\sum_{j=1}^{m}\sup_{z\in\mathbb{B}\setminus K_\delta}\mu(z)|u_j(z)|\sup_{|\varphi(z)|\leqslant\delta}\left|D_{l_j}\cdots D_{l_1}f_k(\varphi(z))\right|\\&\leqslant mMC\varepsilon+C\sum_{j=1}^{m}\|u_j\|_{H_\mu^\infty}\sup_{|w|\leqslant\delta}\left|D_{l_j}\cdots D_{l_1}f_k(w)\right|.\end{aligned} \tag{49}$$

Since $f_k\to 0$ uniformly on compacts of $\mathbb{B}$ as $k\to+\infty$, by Cauchy's estimates, we also have that the sequences

$$D_{l_j}\cdots D_{l_1}f_k\to 0,\quad j=\overline{1,m}, \tag{50}$$

uniformly on compacts of $\mathbb{B}$ as $k\to+\infty$.

By using the test functions

$$f_s(z)=\prod_{j=1}^{s}z_{l_j},\quad s=\overline{1,m},$$

and a similar inductive argument as in the proof of Theorem 2 we obtain $u_j\in H_\mu^\infty(\mathbb{B})$, $j=\overline{1,m}$, that is, $\|u_j\|_{H_\mu^\infty}<+\infty$, $j=\overline{1,m}$.

Using this fact, (50), and the fact that $|w|\leqslant\delta$ is a compact subset of $\mathbb{B}$, by letting $k\to+\infty$ in (49), we obtain

$$\limsup_{k\to+\infty}\|P_{D,\varphi}^m f_k\|_{H_\mu^\infty}\leqslant mMC\varepsilon.$$

Since the last inequality holds for each $\varepsilon>0$, it follows that

$$\lim_{k\to+\infty}\|P_{D,\varphi}^m f_k\|_{H_\mu^\infty}=0. \tag{51}$$

From (51) and Lemma 1, we have that the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H_\mu^\infty(\mathbb{B})$ is compact. □

Now we give a characterization for the compactness of the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H_{\mu,0}^\infty(\mathbb{B})$. Before it we need a known auxiliary result, which is a natural generalization of Lemma 1 in [15].

LEMMA 6. *A closed set K in $H_{\mu,0}^\infty(\mathbb{B})$ is compact if and only if it is bounded and*

$$\lim_{|z|\to 1}\sup_{f\in K}\mu(z)|f(z)|=0.$$

THEOREM 4. *Let $m\in\mathbb{N}$, $u_j\in H(\mathbb{B})$, $j=\overline{1,m}$, $\varphi\in S(\mathbb{B})$, μ be a weight function on $\mathbb{B}$, and condition (23) holds. Then the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$) $\to H_{\mu,0}^\infty(\mathbb{B})$ is compact if and only if the operator is bounded and*

$$\lim_{|z|\to 1}\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j\ln\frac{E_je}{1-|\varphi(z)|^2}}=0,\quad j=\overline{1,m}, \tag{52}$$

where $E_j,\ j=\overline{1,m}$, *are defined in Lemma 3.*

Proof. Suppose that the conditions in (52) hold, then it is easy to see that (24) hold. From this and Theorem 1 we have that the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H_\mu^\infty(\mathbb{B})$ is bounded. By letting $|z|\to 1$ in (26) and employing (52), we obtain

$$\lim_{|z|\to 1}\mu(z)|P_{D,\varphi}^m f(z)|=0$$

for each $f\in\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})$). So, $P_{D,\varphi}^m(\mathscr{B}_{\log,0}(\mathbb{B}))\subset P_{D,\varphi}^m(\mathscr{B}_{\log}(\mathbb{B}))\subset H_{\mu,0}^\infty(\mathbb{B})$. Thus, the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H_{\mu,0}^\infty(\mathbb{B})$ is bounded.

By taking the supremum in (26) over $\mathbb{B}$ and the balls $B_{\mathscr{B}_{\log}(\mathbb{B})}$ and $B_{\mathscr{B}_{\log,0}(\mathbb{B})}$, and using (24), it follows that

$$\sup_{f\in B_{\mathscr{B}_{\log,0}(\mathbb{B})}}\sup_{z\in\mathbb{B}}\mu(z)|P_{D,\varphi}^m f(z)|\leqslant \sup_{f\in B_{\mathscr{B}_{\log}(\mathbb{B})}}\sup_{z\in\mathbb{B}}\mu(z)|P_{D,\varphi}^m f(z)|\lesssim\sum_{j=1}^m M_j<+\infty, \tag{53}$$

where $M_j,\ j=\overline{1,m}$, are as in (24). So, the sets $K:=\{P_{D,\varphi}^m f: f\in B_{\mathscr{B}_{\log,0}(\mathbb{B})}\}$ and $\widehat{K}:=\{P_{D,\varphi}^m f: f\in B_{\mathscr{B}_{\log}(\mathbb{B})}\}$ are bounded in $H_{\mu,0}^\infty$. Taking the supremum in (26) over $B_{\mathscr{B}_{\log}(\mathbb{B})}$ and $B_{\mathscr{B}_{\log,0}(\mathbb{B})}$ and letting $|z|\to 1$ in such obtained inequalities we get

$$\lim_{|z|\to 1}\sup_{f\in B_{\mathscr{B}_{\log,0}(\mathbb{B})}}\mu(z)\big|P_{D,\varphi}^m f(z)\big|=\lim_{|z|\to 1}\sup_{f\in B_{\mathscr{B}_{\log}(\mathbb{B})}}\mu(z)\big|P_{D,\varphi}^m f(z)\big|=0.$$

This fact together with Lemma 6 implies compactness of the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H_{\mu,0}^\infty(\mathbb{B})$.

Now we assume that the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H_{\mu,0}^\infty(\mathbb{B})$ is compact. This implies that the operator $P_{D,\varphi}^m:\mathscr{B}_{\log}(\mathbb{B})$ (or $\mathscr{B}_{\log,0}(\mathbb{B})) \to H_{\mu}^\infty(\mathbb{B})$ is compact. This fact together with Theorem 3 implies that (47) holds, whereas from Theorem 2 it follows that (33) holds.

From (33) it follows that there is $\eta\in(0,1)$ such that

$$\mu(z)|u_j(z)|<\varepsilon(1-\delta^2)^j\ln\frac{E_je}{1-\delta^2},\quad j=\overline{1,m}, \tag{54}$$

when $\eta<|z|<1$, where ε and δ are the numbers in (47).

From (54) and by Lemma 2 with $k=j$ and $a=E_je$, it follows that

$$\frac{\mu(z)|u_j(z)|}{(1-|\varphi(z)|^2)^j\ln\frac{E_je}{1-|\varphi(z)|^2}}\leqslant\frac{\mu(z)|u_j(z)|}{(1-\delta^2)^j\ln\frac{E_je}{1-\delta^2}}<\varepsilon,\quad j=\overline{1,m}, \tag{55}$$

when $|\varphi(z)|\leqslant\delta$ and $\eta<|z|<1$.

From (47) and (55) we get (52) for $j=\overline{1,m}$. $\square$

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