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# A UNIFIED FRAMEWORK FOR SOLVING GENERALIZED VARIATIONAL INEQUALITIES

WENLONG SUN*, GANG LU, YUANFENG JIN* AND CHOONKIL PARK

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*Abstract.* In this paper, a generalized variational inequality and fixed points problem is presented. An iterative algorithm is introduced for finding a solution of the generalized variational inequalities and fixed point of two uniformly Lipschitzian asymptotically quasi-pseudo-contractive operators under a nonlinear transformation. Strong convergence of the suggested algorithm is demonstrated.

## 1. Introduction

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$. For the given two nonlinear operators $\mathscr{F}:\mathscr{C}\to\mathscr{H}$ and $\psi:\mathscr{C}\to\mathscr{C}$, recall that the generalized variational inequality ($GVI$) aims to find an element $x^\dagger\in\mathscr{C}$ such that

$$\langle \mathscr{F}x^\dagger, \psi(x)-\psi(x^\dagger)\rangle \geqslant 0,\ \ \forall x\in\mathscr{C}. \tag{1.1}$$

The solution set of Equation (1.1) is denoted by $GVI(\mathscr{F},\psi,\mathscr{C})$.

If $\psi\equiv\mathscr{I}$ (identity operator), then $GVI$(1.1) can be reduced to find an element $x^\dagger\in\mathscr{C}$ such that

$$\langle \mathscr{F}x^\dagger, x-x^\dagger\rangle \geqslant 0,\ \ \forall x\in\mathscr{C}. \tag{1.2}$$

The solution set of Equation (1.2) is denoted by $VI(\mathscr{F},\mathscr{C})$.

Stampacchia ([3]) introduced variational inequalities which provide a useful tool for researching a large variety of interesting problems arising in elasticity, optimization, network analysis, physics, economics, finance, water resources, structural analysis and medical images ([4]–[8]). For solving variational inequality, projection methods are very popular. For some related work, please refer to References ([1], [2], [9]–[13]).

In particular, very recently, a general class, in which the involved operators are quasi-pseudocontractive operators, was considered by Yao et al. ([15]), and the following iteration was introduced.

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* Corresponding author.


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ALGORITHM 1.1. Let $\mathscr{S},\mathscr{T}$ are two quasi-pseudocontractive operators and $\phi$ is a $L$-Lipschitzian operator. Let $\mathscr{A}:\mathscr{C}\to\mathscr{H}$ is a $\alpha$-inverse strongly $\psi$-monotone operator. Let $x_1\in\mathscr{C}$ be arbitrary. Assume $\{x_n\}$ has been constructed. Compute

$$\begin{aligned}
&u_n = Proj_C[\alpha_n \nu\phi(x_n)+(1-\alpha_n)(\psi(x_n)-\varsigma_n\mathscr{A}x_n)],\\
&y_n=(1-\sigma_n)u_n+\sigma_n(\mathscr{T}((1-\delta_n)I+\delta_n\mathscr{T}))u_n,\\
&z_n=(1-\zeta_n)y_n+\zeta_n(\mathscr{S}((1-\eta_n)I+\eta_n\mathscr{S}))y_n,\\
&\psi(x_{n+1})=\theta_n\psi(x_n)+(1-\theta_n)z_n,\ \ n\geqslant 1.
\end{aligned}\tag{1.3}$$

In this paper, motivited and inspired by Yao et al. ([15]), we consider the following generalized variational inequalities and fixed points problems for finding an element $\hat{x}$ such that

$$\hat{x}\in GVI(\mathscr{F},\psi,\mathscr{C}) \text{ and } \psi(\hat{x})\in Fix(\mathscr{S})\bigcap Fix(\mathscr{T}),\tag{1.4}$$

where S and T are two uniformly $L$-Lipschitzian asymptotically quasi-pseudocontractive operators.

In this paper, a unified framework for generalized variational inequality problems is given. We will extend the above results to the class of uniformly Lipschitzian asymptotically quasi-pseudocontractive operators. Based on the algorithm 3.1, we construct an iterative algorithm and demonstrate its strong convergence.

## 2. Preliminaries

Let $\mathscr{H}$ be a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, respectively. Let $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$. We use $Fix(\mathscr{T})$ to denote the set of fixed points of $\mathscr{T}$, that is, $Fix(\mathscr{T})=\{u|u=\mathscr{T}u,u\in\mathscr{C}\}$.

DEFINITION 2.1. An operator $\mathscr{F}:\mathscr{C}\longrightarrow\mathscr{H}$ is called to be

(1) $\delta$-strongly monotone if $\langle\mathscr{F}z^\dagger-\mathscr{F}z^\ddagger,z^\dagger-z^\ddagger\rangle\geqslant\delta\|z^\dagger-z^\ddagger\|^2$ for some constant $\delta>0$ and all $z^\dagger,z^\ddagger\in\mathscr{C}$;

(2) $\alpha$-inverse strongly monotone if $\langle\mathscr{F}z^\dagger-\mathscr{F}z^\ddagger,z^\dagger-z^\ddagger\rangle\geqslant\alpha\|\mathscr{F}z^\dagger-\mathscr{F}z^\ddagger\|^2$ for some constant $\alpha>0$ and all $z^\dagger,z^\ddagger\in\mathscr{C}$;

(3) $\delta$-strongly $\psi$-monotone if $\langle\mathscr{F}z^\dagger-\mathscr{F}z^\ddagger,\psi(z^\dagger)-\psi(z^\ddagger)\rangle\geqslant\delta\|\psi(z^\dagger)-\psi(z^\ddagger)\|^2$ for some constant $\delta>0$ and all $z^\dagger,z^\ddagger\in\mathscr{C}$;

(4) $\alpha$-inverse strongly $\psi$-monotone if $\langle\mathscr{F}z^\dagger-\mathscr{F}z^\ddagger,\psi(z^\dagger)-\psi(z^\ddagger)\rangle\geqslant\alpha\|\mathscr{F}z^\dagger-\mathscr{F}z^\ddagger\|^2$ for some constant $\alpha>0$ and all $z^\dagger,z^\ddagger\in\mathscr{C}$;

DEFINITION 2.2. A monotone operator $\mathscr{R}:\mathscr{H}\rightrightarrows 2^{\mathscr{H}}$ is maximal monotone if the graph of $\mathscr{R}$ is a maximal monotone set.

DEFINITION 2.3. An operator $\mathscr{T}:\mathscr{C}\longrightarrow\mathscr{C}$ is called to be

(i) $L$-Lipschitzian if there exists $L>0$ such that $\|\mathscr{T}z^\dagger-\mathscr{T}z^\ddagger\|\leqslant L\|z^\dagger-z^\ddagger\|$ for all $z^\dagger,z^\ddagger\in\mathscr{C}$;

(ii) uniformly $L$-Lipschitzian if there exists $L > 0$ such that $\|\mathscr{T}^n z^\dagger - \mathscr{T}^n z^\ddagger\| \leqslant L\|z^\dagger - z^\ddagger\|$ for all $z^\dagger, z^\ddagger \in \mathscr{C}$ and all $n \geqslant 1$;

(iii) $(L,\psi)$-Lipschitzian if there exists $L > 0$ such that $\|\mathscr{T} z^\dagger - \mathscr{T} z^\ddagger\| \leqslant L\|\psi(z^\dagger) - \psi(z^\ddagger)\|$ for all $z^\dagger, z^\ddagger \in \mathscr{C}$, where $\psi : \mathscr{C} \longrightarrow \mathscr{C}$ is a nonlinear operator. In particular, if $L = 1$, the operator $\mathscr{T}$ is said to be $\psi$-nonexpensive.

DEFINITION 2.4. An operator $\mathscr{T} : C \longrightarrow C$ is said to be asymptotically quasi-pseudocontractive if there exists a sequence $\{k_n\} \subset [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\|\mathscr{T}^n x - z^\dagger\|^2 \leqslant (2k_n - 1)\|x - z^\dagger\|^2 + \|\mathscr{T}^n x - x\|^2$$

for all $x \in C$, $z^\dagger \in Fix(\mathscr{T})$ and for all $n \geqslant 1$.

The weak and strong convergence problems of the iterative algorithms for such a class of mappings have been studied by a large number of authors (see, e.g., [16]–[25]).

In general, the convergence of fixed point algorithms requires some extra smoothness properties of the mapping $\mathscr{T}$ such as demi-closedness.

DEFINITION 2.5. An operator $\mathscr{T}$ is said to be demiclosed if, for any sequence $\{x_n\}$ which weakly converges to $x^\natural$, and $\mathscr{T} x_n \longrightarrow w$, then $\mathscr{T}(x^\natural) = w$.

Recall that the (nearest point or metric) projection from $\mathscr{H}$ onto $\mathscr{C}$, denoted $Proj_{\mathscr{C}}$, assigns to each $x \in \mathscr{H}$, the unique point $Proj_{\mathscr{C}} x \in \mathscr{C}$ with the property

$$\|x - Proj_{\mathscr{C}} x\| = inf\{\|x - z\| : z \in \mathscr{C}\}.$$

The metric projection $Proj_{\mathscr{C}}$ of $\mathscr{H}$ onto $\mathscr{C}$ is characterized by

$$\langle x - Proj_{\mathscr{C}} x, z - Proj_{\mathscr{C}} x\rangle \leqslant 0 \tag{2.1}$$

for all $x \in \mathscr{H}, z \in \mathscr{C}$. Recall that the metric projection $Proj_{\mathscr{C}} : \mathscr{H} \to \mathscr{C}$ is firmly nonexpansive, that is,

$$\begin{aligned} &\langle x - y, Proj_{\mathscr{C}} x - Proj_{\mathscr{C}} y\rangle \geqslant \|Proj_{\mathscr{C}} x - Proj_{\mathscr{C}} y\|^2 \\ &\quad \text{or} \quad \|Proj_{\mathscr{C}} x - Proj_{\mathscr{C}} y\|^2 \leqslant \|x - y\|^2 - \|(I - Proj_{\mathscr{C}})x - (I - Proj_{\mathscr{C}})y\|^2 \end{aligned} \tag{2.2}$$

for all $x, y \in \mathscr{H}$.

For all $x, y \in \mathscr{H}$, the following conclusions hold:

$$\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \ t \in [0,1],$$

$$\|x + y\|^2 = \|x\|^2 + 2\langle x, y\rangle + \|y\|^2$$

and

$$\|x + y\|^2 \leqslant \|x\|^2 + 2\langle y, x + y\rangle.$$

LEMMA 2.6. ([27]) *Let* $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ *be an L-Lipschitzian operator with* $L\geqslant 1$. *Then*

$$Fix(((1-\delta)\mathscr{I}+\delta\mathscr{T})\mathscr{T})=Fix(\mathscr{T}((1-\delta)\mathscr{I}+\delta\mathscr{T}))=Fix(\mathscr{T}),$$

*where* $\delta\in(0,\frac{1}{L})$.

LEMMA 2.7. ([14]) *If* $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ *be a uniformly L-Lipschitzian asymptotically pseudo-contractive operator with* $L>1$ *and coefficient* $k_n$. *If* $0<\eta<\zeta<\frac{1}{\sqrt{k_n^2+L^2+k_n^2}}$ *for all* $n\geqslant 1$, *then we have*

$$\begin{aligned}\|(1-\eta)\,x\eta\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x-x^\natural\|^2&\\ \leqslant[1+2(k_n-1)\eta+2(k_n-1)(2k_n-1)\zeta\eta]\|x-x^\natural\|^2&\\ +\eta(\eta-\zeta)\|\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x-x\|^2&\end{aligned}\tag{2.3}$$

*for all* $x\in\mathscr{C}$ *and* $x^\natural\in Fix(\mathscr{T})$.

REMARK 2.8. It is readily seen that, in Lemma 2.7, if the operator is uniformly L-Lipschitzian asymptotically quasi-pseudocontractive, the conclusion still holds.

LEMMA 2.9. ([26]) *Assume that* $\{\alpha_n\}$ *is a sequence of nonnegative real numbers such that*

$$\alpha_{n+1}\leqslant(1-\gamma_n)\alpha_n+\delta_n,\ \ n\in N,$$

*where* $\{\gamma_n\}$ *is a sequence in* $(0,1)$ *and* $\{\delta_n\}$ *is a sequence such that*

*(1)* $\sum_{n=1}^{\infty}\gamma_n=\infty$;

*(2)* $\limsup_{n\to\infty}\frac{\delta_n}{\gamma_n}\leqslant 0$ *or* $\sum_{n=1}^{\infty}|\delta_n|<\infty$.

*Then* $\lim_{n\to\infty}\alpha_n=0$.

LEMMA 2.10. *Let* $\{\varpi_n\}$ *be a sequence of real numbers. Assume there exists at least a subsequence* $\{\varpi_{n_k}\}$ *of* $\{\varpi_n\}$ *such that* $\varpi_{n_k}\leqslant\varpi_{n_k+1}$ *for all* $k\geqslant 0$. *For every* $n\geqslant N_0$, *define an integer sequence* $\{\tau(n)\}$ *as*

$$\tau(n)=\max\{l\leqslant n:\varpi_l\leqslant\varpi_{l+1}\}.$$

*Then,* $\tau(n)\to\infty$ *as* $n\to\infty$ *and for all* $n\geqslant N_0$, *we have* $\max\{\varpi_{\tau(n)},\varpi_n\}\leqslant\varpi_{\tau(n)+1}$.

## 3. Main results

In this section, we first show the following crucial Lemma.

LEMMA 3.1. *Let* $\mathscr{H}$ *be a Hilbert space and* $\mathscr{C}(\neq\emptyset)\subset\mathscr{H}$ *be a closed convex set. Let* $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ *be uniformly L-Lipschitzian asymptotically quasi-pseudocontractive with* $L>1$, *coefficient* $k_n<K$ *and* $Fix(\mathscr{T})\neq\emptyset$. *Then* $Fix(\mathscr{T})$ *is a nonempty closed convex set.*

*Proof.* First, we prove that $Fix(\mathscr{T})$ is convex.
Assume $0<\eta<\zeta\leqslant\frac{1}{\sqrt{K^2+L^2}+K^2}$ for all $n\geqslant 1$. Since $k_n<K$,

$$\frac{1}{\sqrt{K^2+L^2}+K^2}<\frac{1}{\sqrt{k_n^2+L^2}+k_n^2}.$$

Then

$$\zeta L<\zeta(\sqrt{K^2+L^2}+K^2)\leqslant 1. \tag{3.1}$$

Let

$$\mathscr{T}_n=(1-\eta)\mathscr{I}+\eta\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n).$$

Assume $x_1^*,x_2^*\in Fix(\mathscr{T})$. It is obvious that $Fix(\mathscr{T})\subseteq Fix(\mathscr{T}_n)$. So $x_1^*,x_2^*\in Fix(\mathscr{T})\subseteq Fix(\mathscr{T}_n)$. Let $x_t^*=tx_1^*+(1-t)x_2^*$, where $t\in(0,1)$. According to Definition 2.7, Lemma 2.10 and Remark 2.11, we have

$$\begin{aligned}
&\|\mathscr{T}_nx_t^*-x_t^*\|^2\\
&\quad=\|t(x_1^*-\mathscr{T}_nx_t^*)+(1-t)(x_2^*-\mathscr{T}_nx_t^*)\|^2\\
&\quad=t\|x_1^*-\mathscr{T}_nx_t^*\|^2+(1-t)\|x_2^*-\mathscr{T}_nx_t^*\|^2\\
&\qquad-t(1-t)\|x_1^*-x_2^*\|^2\\
&\quad\leqslant t[1+2(k_n-1)\eta+2(k_n-1)(2k_n-1)\zeta\eta]\|x_1^*-x_t^*\|^2\\
&\qquad+(1-t)[1+2(k_n-1)\eta+2(k_n-1)(2k_n-1)\zeta\eta]\|x_2^*-x_t^*\|^2\\
&\qquad-t(1-t)\|x_1^*-x_2^*\|^2\\
&\quad\leqslant t(1-t)^2[1+2(k_n-1)+2(k_n-1)(2k_n-1)]\|x_1^*-x_2^*\|^2\\
&\qquad+t^2(1-t)[1+2(k_n-1)+2(k_n-1)(2k_n-1)]\|x_1^*-x_2^*\|^2\\
&\qquad-t(1-t)\|x_1^*-x_2^*\|^2\\
&\quad=4t(1-t)(k_n-1)^2\|x_1^*-x_2^*\|^2\to 0
\end{aligned} \tag{3.2}$$

which implies $\lim_{n\to\infty}\mathscr{T}_nx_t^*=x_t^*$. It follows that

$$\lim_{n\to\infty}\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x_t^*=x_t^*.$$

Since $\mathscr{T}$ is uniformly $L$-Lipschitzian, we obtain

$$\begin{aligned}
&\|\mathscr{T}^nx_t^*-x_t^*\|\\
&\quad\leqslant\|\mathscr{T}^nx_t^*-\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x_t^*\|\\
&\qquad+\|x_t^*-\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x_t^*\|\\
&\quad\leqslant\zeta L\|\mathscr{T}^nx_t^*-x_t^*\|\\
&\qquad+\|x_t^*-\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x_t^*\|.
\end{aligned} \tag{3.3}$$

By (3.1), we have

$$\|\mathscr{T}^nx_t^*-x_t^*\|\leqslant\frac{1}{1-\zeta L}\|x_t^*-\mathscr{T}^n((1-\zeta)\mathscr{I}+\zeta\mathscr{T}^n)x_t^*\|\to 0 \tag{3.4}$$

which implies $\lim_{n\to\infty}\mathscr{T}^n x_t^* = x_t^*$. Let $\hat{\mathscr{T}}x_t^* = \lim_{n\to\infty}\mathscr{T}^n x_t^*$, we obtain $\hat{\mathscr{T}}x_t^* = x_t^*$. Then,

$$\begin{aligned}\|\mathscr{T}x_t^* - x_t^*\| &= \|\mathscr{T}\hat{\mathscr{T}}x_t^* - \hat{\mathscr{T}}x_t^*\| \\ &= \|\mathscr{T}\hat{\mathscr{T}}x_t^* - \mathscr{T}^{n+1}x_t^* + \mathscr{T}^{n+1}x_t^* - \hat{\mathscr{T}}x_t^*\| \\ &\leqslant L\|\hat{\mathscr{T}}x_t^* - \mathscr{T}^n x_t^*\| + \|\mathscr{T}^{n+1}x_t^* - \hat{\mathscr{T}}x_t^*\| \\ &\to 0.\end{aligned} \tag{3.5}$$

So $\mathscr{T}x_t^* = x_t^*$. Therefore, $Fix(\mathscr{T})$ is convex.

For all $\{x_n\} \subset Fix(\mathscr{T})$ with $x_n \to x^*$, we have

$$\|x_n - \mathscr{T}x^*\| \leqslant L\|x_n - x^*\|$$

and hence $x^* = \mathscr{T}x^*$. That is to say $x^* \in Fix(\mathscr{T})$. Therefore, $Fix(\mathscr{T})$ is closed. This completes the proof. □

Next, we first present some properties for $\alpha$-inverse strongly $\psi$-monotone operator, $\delta$-strongly $\psi$-monotone operators and $(L,\psi)$-Lipschitzian operator. For our main theorem, these properties will be useful.

PROPERTY 3.2. *Assume that $\mathscr{F}:\mathscr{C}\to\mathscr{H}$ is an $\alpha$-inverse strongly $\psi$-monotone operator and $\gamma>0$ is a constant. Then,*

$$\|\psi(x)-\gamma\mathscr{F}x-(\psi(y)-\gamma\mathscr{F}y)\|^2 \leqslant \|\psi(x)-\psi(y)\|^2+\gamma(\gamma-2\alpha)\|\mathscr{F}x-\mathscr{F}y\|^2,\ \forall x,y\in\mathscr{C}.$$

*Proof.* By a direct calculation, we have

$$\begin{aligned}\|\psi(x) &- \gamma\mathscr{F}x) - (\psi(y)-\gamma\mathscr{F}y))\|^2 \\ &= \|\psi(x)-\psi(y)\|^2 + \gamma^2\|\mathscr{F}x-\mathscr{F}y\|^2 - 2\gamma\langle\psi(x)-\psi(y),\mathscr{F}x-\mathscr{F}y\rangle \\ &\leqslant \|\psi(x)-\psi(y)\|^2 + \gamma^2\|\mathscr{F}x-\mathscr{F}y\|^2 - 2\alpha\gamma\|\mathscr{F}x-\mathscr{F}y\|^2 \\ &= \|\psi(x)-\psi(y)\|^2 + \gamma(\gamma-2\alpha)\|\mathscr{F}x-\mathscr{F}y\|^2.\end{aligned} \tag{3.6}$$

The proof is complete. □

PROPERTY 3.3. *Assume that $\mathscr{F}:\mathscr{C}\longrightarrow\mathscr{H}$ is $\delta$-strongly $\psi$-monotone and $(L,\psi)$-Lipschitzian operator. Let $\psi:\mathscr{C}\longrightarrow\mathscr{C}$ be a $\varsigma$-strongly monotone operator and $R(\psi)=\mathscr{C}$. Then the generalized variational inequality GVI(1.1) has a unique solution.*

*Proof.* By the $\varsigma$-strongly monotonicity of $\psi$, we get

$$\|\psi(x)-\psi(y)\| \geqslant \varsigma\|x-y\| \tag{3.7}$$

which implies that $\psi$ is injective. Owing to $R(\psi)=\mathscr{C}$, $\psi$ is bijective.

Let $\Psi = Proj_{\mathscr{C}}(\mathscr{I} - \gamma\mathscr{F}\psi^{-1})$, where $0<\gamma<2\delta/L^2$. In vurtue of $\mathscr{F}:\mathscr{C}\longrightarrow\mathscr{H}$ be a $(L,\psi)$-Lipschitzian $\delta$-strongly $\psi$-monotone operator, we deduce

$$\begin{aligned} &\| \Psi(\psi(x)) - \Psi(\psi(y))\|^2 \\ &\quad = \|Proj_{\mathscr{C}}(\psi(x) - \gamma\mathscr{F}x)) - Proj_{\mathscr{C}}(\psi(y) - \gamma\mathscr{F}y)\|^2 \\ &\quad \leqslant \|\psi(x) - \gamma\mathscr{F}x - (\psi(y) - \gamma\mathscr{F}y)\|^2 \\ &\quad = \|\psi(x) - \psi(y)\|^2 + \gamma^2\|\mathscr{F}x - \mathscr{F}y\|^2 - 2\gamma\langle\psi(x) - \psi(y), \mathscr{F}x - \mathscr{F}y\rangle \\ &\quad \leqslant \|\psi(x) - \psi(y)\|^2 + \gamma^2L^2\|\psi(x) - \psi(y)\|^2 - 2\gamma\delta\|\psi(x) - \psi(y)\|^2 \\ &\quad = (1-\gamma(2\delta - \gamma L^2))\|\psi(x) - \psi(y)\|^2. \end{aligned} \tag{3.8}$$

In view of $R(\psi) = \mathscr{C}$, $\Psi$ is a contraction on $\mathscr{C}$. Hence there exists a unique fixed point $\hat{x}\in\mathscr{C}$ satisfying $Proj_{\mathscr{C}}(\psi(\hat{x}) - \gamma\mathscr{F}\hat{x}) = \psi(\hat{x})$. Equivalently, there exists a unique $\hat{x}\in\mathscr{C}$ solving $GVI$ (1.1). $\square$

In the following paper, we present an algorithm and prove its strong convergence. A list of assumptions on the underlying spaces and involved operators are provided below.

$(R_1)$ $\mathscr{H}$ is a real Hilbert space and $\mathscr{C}(\neq\emptyset)\subset\mathscr{H}$ be a nonempty closed convex subset;

$(R_2)$ $\psi:\mathscr{C}\to\mathscr{C}$ is a $\delta$-strongly monotone and weakly continuous operator such that its rang $R(\psi)=\mathscr{C}$ and $\phi:\mathscr{C}\to\mathscr{H}$ is an $L$-Lipschitzian operator;

$(R_3)$ $\mathscr{T}:\mathscr{C}\to\mathscr{C}$ is a uniformly $L_1$-Lipschitzian asymptotically quasi-pseudocontractive operator with $L_1>1$ and coefficient $k_n$;

$(R_4)$ $\mathscr{S}:\mathscr{C}\to\mathscr{C}$ is a uniformly $L_2$-Lipschitzian asymptotically quasi-pseudocontractive operator with $L_2>1$ and coefficient $l_n$;

$(R_5)$ $\mathscr{F}:\mathscr{C}\to\mathscr{H}$ is an $\alpha$-inverse strongly $\psi$-monotone operator;

$(R_6)$ $\Omega = GVI(\mathscr{F},\psi,\mathscr{C})\bigcap\psi^{-1}(Fix(\mathscr{S})\bigcap Fix(\mathscr{T}))\neq\emptyset$.

Next we present the following iterative algorithm to find $\hat{x}\in\Omega$.

ALGORITHM 3.4. Choose an arbitrary initial value $x_1\in\mathscr{C}$. Assume $\{x_n\}$ has been constructed. Compute

$$\begin{aligned} u_n &= \mathscr{P}_C[\alpha_n\nu\phi(x_n) + (1-\alpha_n)(\psi(x_n) - \beta_n\mathscr{F}x_n)], \\ y_n &= (1-\sigma_n)u_n + \sigma_n(\mathscr{T}^n((1-\delta_n)\mathscr{I} + \delta_n\mathscr{T}^n))u_n, \\ z_n &= (1-\zeta_n)y_n + \zeta_n(\mathscr{S}^n((1-\eta_n)\mathscr{I} + \eta_n\mathscr{S}^n))y_n, \\ \psi(x_{n+1}) &= \theta_n\psi(x_n) + (1-\theta_n)z_n, \ \ n\geqslant 1, \end{aligned} \tag{3.9}$$

where $\nu>0$ is a constant, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$ are six sequences in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,\infty)$.

THEOREM 3.5. *Suppose that* $\mathscr{I}-\mathscr{T}$ *and* $\mathscr{I}-\mathscr{S}$ *are demiclosed at zero. If* $\Omega\neq\emptyset$ *and the following conditions are satisfied:*

$(C_1)$ $\lim_{n\to\infty}\alpha_n = 0$ *and* $\sum_{n=0}^{\infty}\alpha_n=\infty$*;*

$(C_2)$ $0 < a_1 < \sigma_n < c_1 < \hat{c}_1 < \delta_n < b_1 < \frac{1}{\sqrt{k_n^2+L_1^2+k_n^2}}$;

$(C_3)$ $0 < a_2 < \zeta_n < c_2 < \hat{c}_2 < \eta_n < b_2 < \frac{1}{\sqrt{l_n^2+L_2^2+l_n^2}}$;

$(C_4)$ $0 < \liminf_{n\to\infty} \theta_n \leqslant \limsup_{n\to\infty} \theta_n < 1$;

$(C_5)$ $L\nu < \delta < 2\alpha$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 2\alpha$;

$(C_6)$ $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$, $\sum_{n=1}^{\infty}(l_n - 1) < +\infty$;

$(C_7)$ $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = \lim_{n\to\infty} \frac{l_n-1}{\alpha_n} = 0$.

*Then, the iterative sequence* $\{x_n\}$ *defined by Equation (3.9) strongly converges to* $\hat{x} \in \Omega$ *which solves the generalized variational inequality*

$$\langle \nu\phi(\hat{x}) - \psi(\hat{x}), \psi(x) - \psi(\hat{x})\rangle \leqslant 0, \quad \forall x \in \Omega. \tag{3.10}$$

*In particular, if we take* $\phi \equiv 0$*, then* $\psi(\hat{x})$ *is minimum-norm.*

*Proof.* First, we prove that $\psi(x) - \nu\phi(x)$ is $(1 - \nu L/\delta)$-strongly $\psi$-monotone.

$$\begin{aligned}
&\langle \psi(x) - \nu\phi(x) - (\psi(y) - \nu\phi(y)), \psi(x) - \psi(y)\rangle \\
&\quad = \langle \psi(x) - \psi(y), \psi(x) - \psi(y)\rangle - \nu\langle \phi(x) - \phi(y), \psi(x) - \psi(y)\rangle \\
&\quad \geqslant \|\psi(x) - \psi(y)\|^2 - \nu L\|x - y\|\|\psi(x) - \psi(y)\| \\
(by(3.7))&\quad \geqslant \|\psi(x) - \psi(y)\|^2 - \nu L/\delta\|\psi(x) - \psi(y)\|^2 \\
&\quad = (1 - \nu L/\delta)\|\psi(x) - \psi(y)\|^2.
\end{aligned} \tag{3.11}$$

Since

$$\begin{aligned}
\|\psi(x) - \nu\phi(x) - (\psi(y) - \nu\phi(y))\| &\leqslant \|\psi(x) - \psi(y)\| + \nu\|\phi(x) - \phi(y)\| \\
&\leqslant \|\psi(x) - \psi(y)\| + L\nu\|x - y\| \\
&\leqslant (1 + L\nu/\delta)\|\psi(x) - \psi(y)\|,
\end{aligned} \tag{3.12}$$

$\psi - \nu\phi$ is $(1 + L\nu/\delta)$-Lipschitzian. Therefore, from Property 3.3, the $GVI(3.10)$ has a unique solution which is denoted by $\hat{x}$. Since $\hat{x} \in GVI(\mathscr{F}, \psi, \mathscr{C})$ and $\psi(\hat{x}) \in Fix(\mathscr{S})\bigcap Fix(\mathscr{T})$, by virtue of (2.1), we get $\psi(\hat{x}) = Proj_{\mathscr{C}}[\psi(\hat{x}) - \beta_n\mathscr{F}\hat{x}]$ for all $n \geqslant 1$. In view of Property 3.1, we obtain

$$\begin{aligned}
&\|\psi(x) - \beta_n\mathscr{F}x - (\psi(y) - \beta_n\mathscr{F}y)\|^2 \\
&\leqslant \|\psi(x) - \psi(y)\|^2 + \beta_n(\beta_n - 2\alpha)\|\mathscr{F}x - \mathscr{F}y\|^2.
\end{aligned} \tag{3.13}$$

From Equations (3.9) and (3.13), we have

$$\begin{aligned}
&\|u_n - \psi(\hat{x})\| \\
&\quad = \|Proj_{\mathscr{C}}[\alpha_n\nu\phi(x_n) + (1 - \alpha_n)(\psi(x_n) - \beta_n\mathscr{F}x_n)] - Proj_{\mathscr{C}}[\psi(\hat{x}) - \beta_n\mathscr{F}\hat{x}]\| \\
&\quad \leqslant \|\alpha_n\nu\phi(x_n) + (1 - \alpha_n)(\psi(x_n) - \beta_n\mathscr{F}x_n) - (\psi(\hat{x}) - \beta_n\mathscr{F}\hat{x})\| \\
&\quad \leqslant \alpha_n\nu\|\phi(x_n) - \phi(\hat{x})\| + \alpha_n\|\nu\phi(\hat{x}) - \psi(\hat{x}) + \beta_n\mathscr{F}\hat{x}\| \\
&\qquad + (1 - \alpha_n)\|(\psi(x_n) - \beta_n\mathscr{F}x_n) - (\psi(\hat{x}) - \beta_n\mathscr{F}\hat{x})\|
\end{aligned}$$

$$\begin{aligned}
&\leqslant \alpha_n \nu L\|x_n-\hat{x}\|+\alpha_n\|\nu\phi(\hat{x})-\psi(\hat{x})+\beta_n\mathscr{F}\hat{x}\| \\
&\quad +(1-\alpha_n)\|(\psi(x_n)-\beta_n\mathscr{F}x_n)-(\psi(\hat{x})-\beta_n\mathscr{F}\hat{x})\| \\
&\leqslant \alpha_n \nu L/\delta\|\psi(x_n)-\psi(\hat{x})\|+\alpha_n\|\nu\phi(\hat{x})-\psi(\hat{x})+\beta_n\mathscr{F}\hat{x}\| \\
&\quad +(1-\alpha_n)\|\psi(x_n)-\psi(\hat{x})\| \\
&=[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|+\alpha_n\|\nu\phi(\hat{x})-\psi(\hat{x})+\beta_n\mathscr{F}\hat{x}\| \\
&\leqslant [1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|+\alpha_n(\|\nu\phi(\hat{x})-\psi(\hat{x})\|+2\alpha\|\mathscr{F}\hat{x}\|) \\
&=[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\| \\
&\quad +\alpha_n(1-\nu L/\delta)\frac{\|\nu\phi(\hat{x})-\psi(\hat{x})\|+2\alpha\|\mathscr{F}\hat{x}\|}{1-\nu L/\delta}.
\end{aligned} \tag{3.14}$$

By (3.13) and (3.14), we get

$$\begin{aligned}
\|u_n-\psi(\hat{x})\|^2 &\leqslant \|\alpha_n\nu\phi(x_n)+(1-\alpha_n)(\psi(x_n)-\beta_n\mathscr{F}x_n)-(\psi(\hat{x})-\beta_n\mathscr{F}\hat{x})\|^2 \\
&\leqslant \alpha_n\|\nu\phi(x_n)-\psi(\hat{x})+\beta_n\mathscr{F}\hat{x}\|^2 \\
&\quad +(1-\alpha_n)\|(\psi(x_n)-\beta_n\mathscr{F}x_n)-(\psi(\hat{x})-\beta_n\mathscr{F}\hat{x})\|^2 \\
&\leqslant \alpha_n\|\nu\phi(x_n)-\psi(\hat{x})+\beta_n\mathscr{F}\hat{x}\|^2+(1-\alpha_n)\|\psi(x)-\psi(\hat{x})\|^2 \\
&\quad +(1-\alpha_n)\beta_n(\beta_n-2\alpha)\|\mathscr{F}x-\mathscr{F}\hat{x}\|^2.
\end{aligned} \tag{3.15}$$

By the condition $(C_6)$, without loss of generality, we may assume that $sup_n k_n \leqslant 2$ and $sup_n l_n \leqslant 2$ for all $n \geqslant 1$. By virtue of Lemma 2.10, we deduce

$$\begin{aligned}
\|y_n-\psi(\hat{x})\|^2 &= \|(1-\sigma_n)u_n+\sigma_n(\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n))u_n-\psi(\hat{x})\|^2 \\
&\leqslant [1+2(k_n-1)+2(k_n-1)(2k_n-1)]\|u_n-\psi(\hat{x})\|^2 \\
&\quad +\sigma_n(\sigma_n-\delta_n)\|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|^2 \\
&\leqslant [1+8(k_n-1)]\|u_n-\psi(\hat{x})\|^2 \\
&\quad +\sigma_n(\sigma_n-\delta_n)\|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|^2 \\
&\leqslant [1+8(k_n-1)]\|u_n-\psi(\hat{x})\|^2
\end{aligned} \tag{3.16}$$

and

$$\begin{aligned}
\|z_n-\psi(\hat{x})\|^2 &= \|(1-\zeta_n)y_n+\zeta_n(\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n))y_n-\psi(\hat{x})\|^2 \\
&\leqslant [1+2(l_n-1)+2(l_n-1)(2l_n-1)]\|y_n-\psi(\hat{x})\|^2 \\
&\quad +\zeta_n(\zeta_n-\eta_n)\|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|^2 \\
&\leqslant [1+8(l_n-1)]\|y_n-\psi(\hat{x})\|^2 \\
&\quad +\zeta_n(\zeta_n-\eta_n)\|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|^2 \\
&\leqslant [1+8(l_n-1)]\|y_n-\psi(\hat{x})\|^2.
\end{aligned} \tag{3.17}$$

Hence,

$$\|y_n-\psi(\hat{x})\| \leqslant [1+4(k_n-1)]\|u_n-\psi(\hat{x})\| \tag{3.18}$$

and

$$\|z_n - \psi(\hat{x})\| \leqslant [1+4(l_n-1)]\|y_n - \psi(\hat{x})\|. \tag{3.19}$$

Combining Equations (3.9), (3.18) and (3.19), we obtain

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(\hat{x})\| &\leqslant \theta_n\|\psi(x_n) - \psi(\hat{x})\| + (1-\theta_n)\|z_n - \psi(\hat{x})\| \\
&\leqslant (1-\theta_n)[1+4(k_n-1)][1+4(l_n-1)]\|u_n - \psi(\hat{x})\| \\
&\quad + \theta_n\|\psi(x_n) - \psi(\hat{x})\| \\
&\leqslant (1-\theta_n)[1+4(k_n-1)][1+4(l_n-1)]\|u_n - \psi(\hat{x})\| \\
&\quad + \theta_n\|\psi(x_n) - \psi(\hat{x})\|.
\end{aligned} \tag{3.20}$$

Applying Equation (3.14), we have

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(\hat{x})\| &\leqslant (1-\theta_n)[1+4(k_n-1)][1+4(l_n-1)]\|u_n - \psi(\hat{x})\| \\
&\quad + \theta_n\|\psi(x_n) - \psi(\hat{x})\| \\
&\leqslant [1+4(k_n-1)][1+4(l_n-1)](1-\theta_n) \\
&\quad \times \Big\{[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n) - \psi(\hat{x})\| \\
&\quad + \alpha_n(1-\nu L/\delta)\frac{\|\nu\phi(\hat{x}) - \psi(\hat{x})\| + 2\alpha\|\mathscr{F}\hat{x}\|}{1-\nu L/\delta}\Big\} \\
&\quad + \theta_n\|\psi(x_n) - \psi(\hat{x})\| \\
&\leqslant [1+4(k_n-1)][1+4(l_n-1)] \\
&\quad \times \Big\{[1-\alpha_n(1-\theta_n)(1-\nu L/\delta)]\|\psi(x_n) - \psi(\hat{x})\| \\
&\quad + \alpha_n(1-\theta_n)(1-\nu L/\delta)\frac{\|\nu\phi(\hat{x}) - \psi(\hat{x})\| + 2\alpha\|\mathscr{F}\hat{x}\|}{1-\nu L/\delta}\Big\} \\
&\leqslant [1+4(k_n-1)][1+4(l_n-1)] \\
&\quad \times \max\Big\{\|\psi(x_n) - \psi(\hat{x})\|, \frac{\|\nu\phi(\hat{x}) - \psi(\hat{x})\| + 2\alpha\|\mathscr{F}\hat{x}\|)\|}{(1-\nu L/\delta)}\Big\}.
\end{aligned} \tag{3.21}$$

By an inductive method, we derive

$$\begin{aligned}
\|\psi(x_n) - \psi(\hat{x})\| &\leqslant \prod_{i=1}^{n}[1+4(k_i-1)]\prod_{i=1}^{n}[1+4(l_i-1)] \\
&\quad \times \max\Big\{\|\psi(x_1) - \psi(\hat{x})\|, \frac{\|\nu\phi(\hat{x}) - \psi(\hat{x})\| + 2\alpha\|\mathscr{F}\hat{x}\|)\|}{(1-\nu L/\delta)}\Big\}.
\end{aligned} \tag{3.22}$$

It follows that

$$\begin{aligned}
\|x_n - \hat{x}\| \leqslant \frac{1}{\delta}\|\psi(x_n) - \psi(\hat{x})\| &\leqslant \prod_{i=1}^{n}[1+4(k_i-1)]\prod_{i=1}^{n}[1+4(l_i-1)] \\
&\quad \times \max\Big\{\|\psi(x_1) - \psi(\hat{x})\|, \frac{\|\nu\phi(\hat{x}) - \psi(\hat{x})\| + 2\alpha\|\mathscr{F}\hat{x}\|)\|}{(1-\nu L/\delta)}\Big\}.
\end{aligned} \tag{3.23}$$

By the conditions $(C_6)$, it is easy to see that the sequence $\{x_n\}$ and $\{\psi(x_n)\}$ are all bounded.

By Equation (3.9), we get

$$\psi(x_{n+1}) - \psi(x_n) = (1-\theta_n)[z_n - \psi(x_n)], \quad n \geqslant 1. \tag{3.24}$$

Combining Equations (3.9), (3.18)with (3.19), we obtain

$$\begin{aligned}
\|\psi(x_{n+1}) - \psi(\hat{x})\|^2 &= \|\theta_n \psi(x_n) + (1-\theta_n)z_n - \psi(\hat{x})\|^2 \\
&\leqslant \theta_n \|\psi(x_n) - \psi(\hat{x})\|^2 + (1-\theta_n)\|z_n - \psi(\hat{x})\|^2 \\
&\quad - \theta_n(1-\theta_n)\|z_n - \psi(x_n)\|^2 \\
&\leqslant \theta_n \|\psi(x_n) - \psi(\hat{x})\|^2 - \theta_n(1-\theta_n)\|z_n - \psi(x_n)\|^2 \\
&\quad + (1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\|u_n - \psi(\hat{x})\|^2.
\end{aligned} \tag{3.25}$$

According to Equation (3.15), we deduce

$$\begin{aligned}
&\|\psi(x_{n+1}) \ - \psi(\hat{x})\|^2 \\
&\quad \leqslant \theta_n \|\psi(x_n) - \psi(\hat{x})\|^2 - \theta_n(1-\theta_n)\|z_n - \psi(x_n)\|^2 \\
&\qquad + (1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\|u_n - \psi(\hat{x})\|^2 \\
&\quad \leqslant \theta_n \|\psi(x_n) - \psi(\hat{x})\|^2 - \theta_n(1-\theta_n)\|z_n - \psi(x_n)\|^2 \\
&\qquad + (1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)] \\
&\qquad \times \{\alpha_n \|\nu\phi(x_n) - \psi(\hat{x}) + \beta_n \mathscr{F}\hat{x}\|^2 + (1-\alpha_n)\|\psi(x) - \psi(\hat{x})\|^2 \\
&\qquad + (1-\alpha_n)\beta_n(\beta_n - 2\alpha)\|\mathscr{F}x_n - \mathscr{F}\hat{x}\|^2\}.
\end{aligned} \tag{3.26}$$

In the sequel, we take into account two possible cases.

*Case* $\mathbb{I}$. There exists $m > 0$ such that

$$\{\|\psi(x_n) - \psi(\hat{x})\|\}$$

is decreasing when $n \geqslant m$. Thus, $\lim_{n\to\infty} \|\psi(x_n) - \psi(\hat{x})\|$ exists. According to Equations (3.26), we have

$$\begin{aligned}
&(1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)](1-\alpha_n)\beta_n(2\alpha - \beta_n)\|\mathscr{F}x - \mathscr{F}\hat{x}\|^2 \\
&\quad + \theta_n(1-\theta_n)\|z_n - \psi(x_n)\|^2 \\
&\leqslant \|\psi(x_n) - \psi(\hat{x})\|^2 - \|\psi(x_{n+1}) - \psi(\hat{x})\|^2 \\
&\quad + (1-\theta_n)\{[1+8(k_n-1)][1+8(l_n-1)] - 1\}\|\psi(x_n) - \psi(\hat{x})\|^2 \\
&\quad + \alpha_n(1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)] \\
&\quad \times (\|\nu\phi(x_n) - \psi(\hat{x}) + \beta_n \mathscr{F}\hat{x}\|^2 - \|\psi(x_n) - \psi(\hat{x})\|^2).
\end{aligned} \tag{3.27}$$

This together with assumptions $(C_1)$, $(C_4)$, $(C_5)$ and $(C_7)$ implies that

$$\lim_{n\to\infty} \|z_n - \psi(x_n)\| = 0 \tag{3.28}$$

and

$$\lim_{n\to\infty} \|\mathscr{F}x_n - \mathscr{F}\hat{x}\| = 0. \tag{3.29}$$

By Equation (3.9), we get

$$\psi(x_{n+1}) - \psi(x_n) = (1-\theta_n)[z_n - \psi(x_n)], \ \ n \geqslant 1. \tag{3.30}$$

Furthermore, it follows from Equation (3.28) and (3.30) that

$$\lim_{n\to\infty} \|\psi(x_{n+1}) - \psi(x_n)\| = 0. \tag{3.31}$$

Set $v_n = \psi(x_n) - \beta_n \mathscr{F}x_n - (\psi(\hat{x}) - \beta_n \mathscr{F}\hat{x})$ for all $n \geqslant 1$. In virtue of Equation (2.2), the continuity of the norm, and the boundedness of the sequence $\{x_n\}$, $\{\psi(x_n)\}$ and $\{\mathscr{F}x_n\}$, we deduce

$$\begin{aligned}
&\|u_n - \psi(\hat{x})\|^2 \\
&\leqslant \|Proj_{\mathscr{C}}[\alpha_n \nu\phi(x_n) + (1-\alpha_n)(\psi(x_n) - \beta_n\mathscr{F}x_n)] - Proj_C[\psi(\hat{x}) - \beta_n\mathscr{F}\hat{x}]\| \\
&\leqslant \langle \alpha_n(\nu\phi(x_n) - \psi(\hat{x}) + \beta_n\mathscr{F}\hat{x}) + (1-\alpha_n)v_n, u_n - \psi(\hat{x})\rangle \\
&\leqslant \frac{1}{2}\{\|\alpha_n(\nu\phi(x_n) - \psi(\hat{x}) + \beta_n\mathscr{F}\hat{x}) + (1-\alpha_n)v_n\|^2 + \|u_n - \psi(\hat{x})\|^2 \\
&\quad - \|\alpha_n(\nu\phi(x_n) - \psi(\hat{x}) + \beta_n\mathscr{F}\hat{x}) + (1-\alpha_n)v_n - u_n + \psi(\hat{x})\|^2\} \\
&\leqslant \frac{1}{2}\{\|\alpha_n(\nu\phi(x_n) - \psi(\hat{x}) + \beta_n\mathscr{F}\hat{x} - v_n) + v_n\|^2 + \|u_n - \psi(\hat{x})\|^2 \\
&\quad - \|\alpha_n(\nu\phi(x_n) - \psi(\hat{x}) + \beta_n\mathscr{F}\hat{x} - v_n) + v_n - u_n + \psi(\hat{x})\|^2\} \\
&\leqslant \frac{1}{2}\{\|v_n\|^2 + \|u_n - \psi(\hat{x})\|^2 - \|v_n - u_n + \psi(\hat{x})\|^2\} + \varepsilon_n \\
&= \frac{1}{2}\{\|\psi(x_n) - \beta_n\mathscr{F}x_n - (\psi(\hat{x}) - \beta_n\mathscr{F}\hat{x})\|^2 + \|u_n - \psi(\hat{x})\|^2 \\
&\quad - \|\psi(x_n) - u_n - \beta_n(\mathscr{F}x_n - \mathscr{F}\hat{x})\|^2\} + \varepsilon_n,
\end{aligned} \tag{3.32}$$

where $\varepsilon_n > 0$ and $\lim_{n\to\infty} \varepsilon_n = 0$. From Equation (3.29) and (3.32), we obtain

$$\begin{aligned}
\|u_n - \psi(\hat{x})\|^2 \leqslant \frac{1}{2}\{&\|\psi(x_n) - \psi(\hat{x})\|^2 + \|u_n - \psi(\hat{x})\|^2 \\
&- \|\psi(x_n) - u_n\|^2\} + \widehat{\varepsilon}_n,
\end{aligned} \tag{3.33}$$

where $\widehat{\varepsilon}_n > 0$ and $\lim_{n\to\infty} \widehat{\varepsilon}_n = 0$. Hence,

$$\|u_n - \psi(\hat{x})\|^2 \leqslant \|\psi(x_n) - \psi(\hat{x})\|^2 - \|\psi(x_n) - u_n\|^2 + \widehat{\varepsilon}_n. \tag{3.34}$$

In the light of Equations (3.25) and (3.34), we have

$$
\begin{aligned}
\|\psi(x_{n+1}) & - \psi(\hat{x})\|^2 \\
& \leqslant (1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\|u_n-\psi(\hat{x})\|^2 \\
& \quad + \theta_n\|\psi(x_n)-\psi(\hat{x})\|^2 \\
& \leqslant (1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)] \\
& \quad \times \{\|\psi(x_n)-\psi(\hat{x})\|^2-\|\psi(x_n)-u_n\|^2+\widehat{\varepsilon}_n\} \\
& \quad + \theta_n\|\psi(x_n)-\psi(\hat{x})\|^2 \\
& \leqslant (1-\theta_n)\{[1+8(k_n-1)][1+8(l_n-1)]-1\}\|\psi(x_n)-\psi(\hat{x})\|^2 \\
& \quad -(1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\{\|\psi(x_n)-u_n\|^2-\widehat{\varepsilon}_n\} \\
& \quad + \|\psi(x_n)-\psi(\hat{x})\|^2.
\end{aligned}
\tag{3.35}
$$

Hence,

$$
\begin{aligned}
(1-\theta_n) & \ [1+8(k_n-1)][1+8(l_n-1)]\|\psi(x_n)-u_n\|^2 \\
& \leqslant (1-\theta_n)\{[1+8(k_n-1)][1+8(l_n-1)]-1\}\|\psi(x_n)-\psi(\hat{x})\|^2+\tilde{\varepsilon_n} \\
& \quad + \|\psi(x_n)-\psi(\hat{x})\|^2-\|\psi(x_{n+1})-\psi(\hat{x})\|^2,
\end{aligned}
\tag{3.36}
$$

where $\tilde{\varepsilon_n}=(1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\widehat{\varepsilon}_n \to 0$. According to $(C_6)$, we easily deduce

$$
\lim_{n\to\infty}\|\psi(x_n)-u_n\|=0. \tag{3.37}
$$

In view of Equations (3.16) and (3.17), we get

$$
\begin{aligned}
\|z_n-\psi(\hat{x})\|^2 \leqslant & [1+8(l_n-1)][1+8(k_n-1)]\|u_n-\psi(\hat{x})\|^2 \\
& +[1+8(l_n-1)]\sigma_n(\sigma_n-\delta_n)\|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|^2 \\
& +\zeta_n(\zeta_n-\eta_n)\|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|^2.
\end{aligned}
\tag{3.38}
$$

Therefore,

$$
\begin{aligned}
[1+ & 8(l_n-1)]\sigma_n(\sigma_n-\delta_n)\|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|^2 \\
& +\zeta_n(\zeta_n-\eta_n)\|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|^2 \\
& \leqslant \{[1+8(l_n-1)][1+8(k_n-1)]-1\}\|u_n-\psi(\hat{x})\|^2 \\
& \quad +\|u_n-\psi(\hat{x})\|^2-\|z_n-\psi(\hat{x})\|^2 \\
& \leqslant \{[1+8(l_n-1)][1+8(k_n-1)]-1\}\|u_n-\psi(\hat{x})\|^2 \\
& \quad +\|u_n-z_n\|(\|u_n-\psi(\hat{x})\|^2+\|z_n-\psi(\hat{x})\|^2).
\end{aligned}
\tag{3.39}
$$

From (3.28) and (3.37), we obtain

$$
\lim_{n\to\infty}\|z_n-u_n\|=0. \tag{3.40}
$$

It follows from Equations (3.39), (3.40), $(C_2)$ and $(C_3)$ that

$$\lim_{n\to\infty} \|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|=0 \tag{3.41}$$

and

$$\lim_{n\to\infty} \|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|=0. \tag{3.42}$$

Note that

$$y_n-u_n=\sigma_n[\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n]$$

and

$$z_n-y_n=\zeta_n[\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n].$$

Therefore,

$$\lim_{n\to\infty} \|y_n-u_n\|=0 \tag{3.43}$$

and

$$\lim_{n\to\infty} \|z_n-y_n\|=0. \tag{3.44}$$

Observe that

$$\begin{aligned}\|u_n-\mathscr{T}^n u_n\| &\leqslant \|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|\\ &\quad+\|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-\mathscr{T}^n u_n\|\\ &\leqslant \|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\|\\ &\quad+L_1\delta_n\|\mathscr{T}^n u_n-u_n\|\end{aligned} \tag{3.45}$$

and

$$\begin{aligned}\|y_n-\mathscr{S}^n y_n\| &\leqslant \|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|\\ &\quad+\|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-\mathscr{S}^n y_n\|\\ &\leqslant \|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|\\ &\quad+L_2\eta_n\|\mathscr{S}^n y_n-y_n\|.\end{aligned} \tag{3.46}$$

It follows that

$$\|u_n-\mathscr{T}^n u_n\| \leqslant \frac{1}{1-L_1\delta_n}\|\mathscr{T}^n((1-\delta_n)I+\delta_n\mathscr{T}^n)u_n-u_n\| \tag{3.47}$$

and

$$\|y_n-\mathscr{S}^n y_n\| \leqslant \frac{1}{1-L_2\eta_n}\|\mathscr{S}^n((1-\eta_n)I+\eta_n\mathscr{S}^n)y_n-y_n\|. \tag{3.48}$$

This together with (3.41) and (3.42) implies that

$$\lim_{n\to\infty} \|u_n-\mathscr{T}^n u_n\|=0 \tag{3.49}$$

and

$$\lim_{n\to\infty}\|y_n-\mathscr{S}^n y_n\|=0. \tag{3.50}$$

From Equation (3.31), (3.37) and (3.43), we have

$$\lim_{n\to\infty}\|u_n-u_{n+1}\|=0 \tag{3.51}$$

and

$$\lim_{n\to\infty}\|y_n-y_{n+1}\|=0. \tag{3.52}$$

Since $\mathscr{T}$ and $\mathscr{S}$ are uniformly $L_1$-Lipschitzian and $L_2$-Lipschitzian, respectively, we can derive

$$\begin{aligned}\|u_{n+1}-\mathscr{T}u_{n+1}\| &\leqslant \|u_{n+1}-\mathscr{T}^{n+1}u_{n+1}\|+\|\mathscr{T}^{n+1}u_{n+1}-\mathscr{T}^{n+1}u_n\| \\ &\quad +\|\mathscr{T}^{n+1}u_n-\mathscr{T}u_{n+1}\| \\ &\leqslant \|u_{n+1}-\mathscr{T}^{n+1}u_{n+1}\|+2L_1\|u_{n+1}-u_n\| \\ &\quad +L_1\|u_n-\mathscr{T}^n u_n\|\end{aligned} \tag{3.53}$$

and

$$\begin{aligned}\|y_{n+1}-\mathscr{S}y_{n+1}\| &\leqslant \|y_{n+1}-\mathscr{S}^{n+1}y_{n+1}\|+\|\mathscr{S}^{n+1}y_{n+1}-\mathscr{S}^{n+1}y_n\| \\ &\quad +\|\mathscr{S}^{n+1}y_n-\mathscr{S}y_{n+1}\| \\ &\leqslant \|y_{n+1}-\mathscr{S}^{n+1}y_{n+1}\|+2L_2\|y_{n+1}-y_n\| \\ &\quad +L_2\|y_n-\mathscr{S}^n y_n\|.\end{aligned} \tag{3.54}$$

By (3.49), (3.50), (3.51), (3.52), (3.53) and (3.54), we have immediately that

$$\lim_{n\to\infty}\|u_n-\mathscr{T}u_n\|=0 \tag{3.55}$$

and

$$\lim_{n\to\infty}\|y_n-\mathscr{S}y_n\|=0. \tag{3.56}$$

Next, we prove $\limsup_{n\to\infty}\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\leqslant 0$. Let $\{u_{n_i}\}$ be a subsequence of $\{u_n\}$ such that

$$\begin{aligned}&\limsup_{n\to\infty}\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle \\ &\quad=\lim_{i\to\infty}\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_{n_i}-\psi(\hat{x})\rangle \\ &\quad=\lim_{i\to\infty}\langle\nu\phi(\hat{x})-\psi(\hat{x}),\psi(x_{n_i})-\psi(\hat{x})\rangle.\end{aligned} \tag{3.57}$$

Note that $x_{n_i}$ is bounded. We can select a subsequence $\{x_{n_{i_j}}\}$ of $x_{n_i}$ such that $x_{n_{i_j}}\rightharpoonup z\in\mathscr{C}$. Without loss of generality, assume $x_{n_i}\rightharpoonup z$. Owing to the weak continuity of $\psi$, this indicates that $\psi(x_{n_i})\rightharpoonup\psi(z)$. Therefore, $u_{n_i}\rightharpoonup\psi(z)$ and $y_{n_i}\rightharpoonup\psi(z)$ .

In view of Equation (3.55), (3.56) and the demi-closedness of $\mathscr{I}-\mathscr{T}$ and $\mathscr{I}-\mathscr{S}$, we deduce $\psi(z)\in Fix(\mathscr{T})$ and $\psi(z)\in Fix(\mathscr{S})$, respectively. That is, $\psi(z)\in Fix(\mathscr{T})$ $\bigcap Fix(\mathscr{S})$.

Next, we show $z\in GVI(\mathscr{F},\psi,\mathscr{C})$. By Remark 2.2, we obtain that $\mathscr{F}\psi^{-1}$ is monotone. Let

$$\mathscr{R}(v)=\begin{cases}\mathscr{F}\psi^{-1}v+\mathscr{N}_{\mathscr{C}}(v), & v\in\mathscr{C},\\ \emptyset & v\notin\mathscr{C},\end{cases} \tag{3.58}$$

where $\mathscr{N}_{\mathscr{C}}(v)$ is the normal cone to $\mathscr{C}$ at $v$. According to Reference [28], we can easily derive that $\mathscr{R}$ is maximal monotone. Let $(\psi(v),w)\in G(\mathscr{R})$. Since $w-Av\in\mathscr{N}_{\mathscr{C}}(\psi(v))$ and $x_n\in\mathscr{C}$, we have

$$\langle\psi(v)-\psi(x_n),w-Av\rangle\geqslant 0.$$

Noting that

$$u_n=Proj_{\mathscr{C}}[\alpha_n\nu\phi(x_n)+(\mathscr{I}-\alpha_n)(\psi(x_n)-\beta_n\mathscr{F}x_n)],$$

we get

$$\langle\psi(v)-u_n,u_n-[\alpha_n\nu\phi(x_n)+(\mathscr{I}-\alpha_n)(\psi(x_n)-\beta_n\mathscr{F}x_n)]\rangle\geqslant 0.$$

It follows that

$$\left\langle\psi(v)-u_n,\frac{u_n-\psi(x_n)}{\beta_n}+\mathscr{F}x_n-\frac{\alpha_n}{\beta_n}(\nu\phi(x_n)-\psi(x_n)+\beta_n\mathscr{F}x_n)\right\rangle\geqslant 0.$$

Thus

$$\begin{aligned}
&\langle\psi(v)-\psi(x_{n_i}),w\rangle\\
&\quad\geqslant\langle\psi(v)-\psi(x_{n_i}),\mathscr{F}v\rangle\\
&\quad\geqslant\left\langle\psi(v)-\psi(x_{n_i}),\mathscr{F}v\rangle-\langle\psi(v)-u_{n_i},\frac{u_{n_i}-\psi(x_{n_i})}{\beta_{n_i}}\right\rangle\\
&\qquad+\frac{\alpha_{n_i}}{\beta_{n_i}}\langle\psi(v)-u_{n_i},\nu\phi(x_{n_i})-\psi(x_n)+\beta_{n_i}\mathscr{F}x_{n_i}\rangle\\
&\qquad-\langle\psi(v)-u_{n_i},\mathscr{F}x_{n_i}\rangle\\
&\quad\geqslant\langle\psi(v)-\psi(x_{n_i}),Av-\mathscr{F}x_{n_i}\rangle-\left\langle\psi(v)-u_{n_i},\frac{u_{n_i}-\psi(x_{n_i})}{\beta_{n_i}}\right\rangle\\
&\qquad+\frac{\alpha_{n_i}}{\beta_{n_i}}\langle\psi(v)-u_{n_i},\nu\phi(x_{n_i})-\psi(x_{n_i})+\beta_{n_i}\mathscr{F}x_{n_i}\rangle\\
&\qquad-\langle\psi(v)-u_{n_i},\mathscr{F}x_{n_i}\rangle+\langle\psi(v)-\psi(x_{n_i}),\mathscr{F}x_{n_i}\rangle\\
&\quad\geqslant-\langle\psi(x_{n_i})-u_{n_i},\mathscr{F}x_{n_i}\rangle-\left\langle\psi(v)-u_{n_i},\frac{u_{n_i}-\psi(x_{n_i})}{\beta_{n_i}}\right\rangle\\
&\qquad+\frac{\alpha_{n_i}}{\beta_{n_i}}\langle\psi(v)-u_{n_i},\nu\phi(x_{n_i})-\psi(x_{n_i})+\beta_{n_i}\mathscr{F}x_{n_i}\rangle.
\end{aligned} \tag{3.59}$$

By virtue of Equation (3.59), we deduce that $\langle \psi(v)-\psi(z),w\rangle \geqslant 0$ owing to (3.37) and $\psi(x_{n_i}) \rightharpoonup \psi(z)$. By the maximal monotonicity of $\mathscr{R}$, $\psi(z)\in\mathscr{R}^{-1}0$. So, $z\in GVI(\mathscr{F},\psi,\mathscr{C})$. Therefore, $z\in\Omega$. From Equation (3.57), we obtain

$$\begin{aligned}\limsup_{n\to\infty}&\langle \nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\\ &=\lim_{i\to\infty}\langle \nu\phi(\hat{x})-\psi(\hat{x}),\psi(x_{n_i})-\psi(\hat{x})\rangle\\ &=\lim_{i\to\infty}\langle \nu\phi(\hat{x})-\psi(\hat{x}),\psi(z)-\psi(\hat{x})\rangle\leqslant 0.\end{aligned}\tag{3.60}$$

Applying Equation (2.2), we obtain

$$\begin{aligned}&\|u_n-\psi(\hat{x})\|^2\\ &=\|\mathscr{P}_{\mathscr{C}}[\alpha_n\nu\phi(x_n)+(1-\alpha_n)(\psi(x_n)-\beta_n\mathscr{F}x_n)]-Proj_{\mathscr{C}}[\psi(\hat{x})-(1-\alpha_n)\beta_n\mathscr{F}\hat{x}]\|^2\\ &\leqslant\langle\alpha_n(\nu\phi(x_n)-\psi(\hat{x}))+(1-\alpha_n)[(\psi(x_n)-\beta_n\mathscr{F}x_n)-(\psi(\hat{x})-\beta_n\mathscr{F}\hat{x})],u_n-\psi(\hat{x})\rangle\\ &\leqslant\alpha_n\nu\langle\phi(x_n)-\phi(\hat{x}),u_n-\psi(\hat{x})\rangle+\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\\ &\quad+(1-\alpha_n)\langle[(\psi(x_n)-\beta_n\mathscr{F}x_n)-(\psi(\hat{x})-\beta_n\mathscr{F}\hat{x})],u_n-\psi(\hat{x})\rangle\\ &\leqslant\alpha_n\nu L\|x_n-\hat{x}\|\times\|u_n-\psi(\hat{x})\|+\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\\ &\quad+(1-\alpha_n)\|\psi(x_n)-\psi(\hat{x})\|\times\|u_n-\psi(\hat{x})\|\\ &=[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|\times\|u_n-\psi(\hat{x})\|+\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\\ &\leqslant\frac{1-\alpha_n(1-\nu L/\delta)}{2}\|\psi(x_n)-\psi(\hat{x})\|^2+\frac{1}{2}\|u_n-\psi(\hat{x})\|^2\\ &\quad+\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle.\end{aligned}\tag{3.61}$$

It follows that

$$\|u_n-\psi(\hat{x})\|^2\leqslant[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|^2+2\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle.$$

This, together with Equation (3.35) implies that

$$\begin{aligned}&\|\psi(x_{n+1})-\psi(\hat{x})\|^2\\ &\leqslant(1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\|u_n-\psi(\hat{x})\|^2+\theta_n\|\psi(x_n)-\psi(\hat{x})\|^2\\ &\leqslant(1-\theta_n)[1+8(k_n-1)][1+8(l_n-1)]\\ &\quad\times\{[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|^2+2\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\}\\ &\quad+\theta_n\|\psi(x_n)-\psi(\hat{x})\|^2\\ &\leqslant(1-\theta_n)\{[1+8(k_n-1)][1+8(l_n-1)]-1\}\\ &\quad\times\{[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|^2+2\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\}\\ &\quad+[1-\alpha_n(1-\theta_n)(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|^2\\ &\quad+2(1-\theta_n)\alpha_n\langle\nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle.\end{aligned}\tag{3.62}$$

In the light of $(C_7)$ and (3.60), it can be seen easily that

$$(1-\theta_n)\{[1+8(k_n-1)][1+8(l_n-1)]-1\} \times \frac{\{[1-\alpha_n(1-\nu L/\delta)]\|\psi(x_n)-\psi(\hat{x})\|^2+2\alpha_n\langle \nu\phi(\hat{x})-\psi(\hat{x}),u_n-\psi(\hat{x})\rangle\}}{\alpha_n(1-\theta_n)(1-\nu L/\delta)} \to 0. \tag{3.63}$$

Therefore, applying Lemma 2.13 to Equation (3.62), we can conclude that $\psi(x_n) \to \psi(\hat{x})$ and so $x_n \to \hat{x}$.

*Case* III. There exists $n_0$ such that $\|\psi(x_{n_0})-\psi(\hat{x})\| \leqslant \|\psi(x_{n_0+1})-\psi(\hat{x})\|$. At this case, we set $\varpi_n = \|\psi(x_n)-\psi(\hat{x})\|$. Then, we have $\varpi_{n_0} \leqslant \varpi_{n_0+1}$. For $n \geqslant n_0$, we define a sequence $\{\tau_n\}$ by

$$\tau(n) = \max\{l \in \mathbb{N} | n_0 \leqslant l \leqslant n, \varpi_l \leqslant \varpi_{l+1}\}.$$

it is easy to show that $\tau(n)$ is a non-decreasing sequence such that

$$\lim_{n\to\infty} \tau(n) = +\infty$$

and

$$\varpi_{\tau(n)} \leqslant \varpi_{\tau(n)+1}.$$

Employing techniques similar to Equations (3.60) and (3.62), we have

$$\limsup_{n\to\infty} \langle \nu\phi(\hat{x})-\psi(\hat{x}),u_{\tau(n)}-\psi(\hat{x})\rangle \leqslant 0. \tag{3.64}$$

and

$$\begin{aligned}
\varpi^2_{\tau(n)+1} \leqslant{}& (1-\theta_{\tau(n)})\{[1+8(k_{\tau(n)}-1)][1+8(l_{\tau(n)}-1)]-1\} \\
&\times \{[1-\alpha_{\tau(n)}(1-\nu L/\delta)]\varpi^2_{\tau(n)}+2\alpha_{\tau(n)}\langle \nu\phi(\hat{x})-\psi(\hat{x}),u_{\tau(n)}-\psi(\hat{x})\rangle\} \\
&+[1-\alpha_{\tau(n)}(1-\theta_{\tau(n)})(1-\nu L/\delta)]\varpi^2_{\tau(n)} \\
&+2(1-\theta_{\tau(n)})\alpha_{\tau(n)}\langle \nu\phi(\hat{x})-\psi(\hat{x}),u_{\tau(n)}-\psi(\hat{x})\rangle.
\end{aligned} \tag{3.65}$$

Since $\varpi_{\tau(n)} \leqslant \varpi_{\tau(n)+1}$, we have

$$\begin{aligned}
\varpi^2_{\tau(n)} \leqslant{}& \frac{[1+8(k_{\tau(n)}-1)][1+8(l_{\tau(n)}-1)]-1}{\alpha_{\tau(n)}(1-\nu L/\delta)} \times [1-\alpha_{\tau(n)}(1-\nu L/\delta)]\varpi^2_{\tau(n)} \\
&+\frac{2[1+8(k_{\tau(n)}-1)][1+8(l_{\tau(n)}-1)]-1}{(1-\theta_{\tau(n)})(1-\nu L/\delta)}\langle \nu\phi(\hat{x})-\psi(\hat{x}),u_{\tau(n)}-\psi(\hat{x})\rangle \\
&+\frac{2}{1-\nu L/\delta}\langle \nu\phi(\hat{x})-\psi(\hat{x}),u_{\tau(n)}-\psi(\hat{x})\rangle.
\end{aligned} \tag{3.66}$$

By $(C_7)$ and (3.64), we obtain

$$\limsup_{n\to\infty} \varpi_{\tau(n)} \leqslant 0$$

and so

$$\lim_{n\to\infty} \varpi_{\tau(n)} = 0. \tag{3.67}$$

By Equations (3.64) and (3.65), we also obtain

$$\limsup_{n\to\infty} \varpi_{\tau(n)+1} \leqslant \limsup_{n\to\infty} \varpi_{\tau(n)}.$$

By the last inequality and Equation (3.67), we derive that

$$\lim_{n\to\infty} \varpi_{\tau(n)+1} = 0.$$

Applying Lemma 2.14 to get

$$\varpi_n \leqslant \varpi_{\tau(n)+1}.$$

Therefore, $\varpi_n \to 0$, i.e., $\psi(x_n) \to \psi(\hat{x})$ which implies $x_n \to \hat{x}$.

Finally, if we take $\phi \equiv 0$, we get

$$\langle -\psi(\hat{x}), \psi(x) - \psi(\hat{x})\rangle \leqslant 0, \quad \forall x \in \Omega. \tag{3.68}$$

Equivalently,

$$\|\psi(\hat{x})\|^2 \leqslant \langle \psi(x), \psi(\hat{x})\rangle, \quad \forall x \in \Omega.$$

This clealy implies that

$$\|\psi(\hat{x})\| \leqslant \|\psi(x)\|, \quad \forall x \in \Omega.$$

The proof is completed. □

ALGORITHM 3.6. Choose an arbitrary initial value $x_1 \in \mathscr{C}$. Assume $\{x_n\}$ has been constructed. Compute

$$\begin{aligned}
u_n &= Proj_{\mathscr{C}}[\alpha_n \nu \phi(x_n) + (1-\alpha_n)(x_n - \beta_n \mathscr{F} x_n)],\\
y_n &= (1-\sigma_n)\, u_n + \sigma_n \left(\mathscr{T}^n \left((1-\delta_n)\,\mathscr{I} + \delta_n \mathscr{T}^n\right)\right) u_n,\\
z_n &= (1-\zeta_n)\, y_n + \zeta_n \left(\mathscr{S}^n \left((1-\eta_n)\,\mathscr{I} + \eta_n \mathscr{S}^n\right)\right) y_n,\\
x_{n+1} &= \theta_n x_n + (1-\theta_n) z_n, \quad n \geqslant 1,
\end{aligned} \tag{3.69}$$

where $\nu > 0$ is a constant, $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$ are six sequences in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,\infty)$.

COROLLARY 3.7. *Suppose that $\mathscr{I} - \mathscr{T}$ and $\mathscr{I} - \mathscr{S}$ are demiclosed at zero. If*

$$\tilde{\Omega} = VI(\mathscr{F}, \mathscr{C}) \bigcap Fix(\mathscr{S}) \bigcap Fix(\mathscr{T}) \neq \emptyset$$

*and the following conditions are satisfied:*

$(C_1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;

$(C_2)$ $0 < a_1 < \sigma_n < c_1 < \hat{c}_1 < \delta_n < b_1 < \frac{1}{\sqrt{k_n^2 + L_1^2 + k_n^2}}$;

$(C_3)$ $0 < a_2 < \zeta_n < c_2 < \hat{c}_2 < \eta_n < b_2 < \frac{1}{\sqrt{l_n^2 + L_2^2 + l_n^2}}$;

$(C_4)$ $0 < \liminf_{n\to\infty} \theta_n \leqslant \limsup_{n\to\infty} \theta_n < 1$;
$(C_5)$ $L\nu < 1 < 2\alpha$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 2\alpha$;
$(C_6)$ $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$, $\sum_{n=1}^{\infty}(l_n - 1) < +\infty$;
$(C_7)$ $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = \lim_{n\to\infty} \frac{l_n-1}{\alpha_n} = 0$.

*Then, the iterative sequence* $\{x_n\}$ *defined by Equation (3.69) strongly converges to* $\hat{x} \in \tilde{\Omega}$ *which solves the generalized variational inequality*

$$\langle \nu\phi(\hat{x}) - \hat{x}, x - \hat{x}\rangle \leqslant 0, \;\; \forall x \in \tilde{\Omega}. \tag{3.70}$$

ALGORITHM 3.8. Choose an arbitrary initial value $x_1 \in \mathscr{C}$. Assume $\{x_n\}$ has been constructed. Compute

$$\begin{aligned} u_n &= Proj_{\mathscr{C}}[(1-\alpha_n)(x_n - \beta_n \mathscr{F} x_n)],\\ y_n &= (1-\sigma_n)\, u_n + \sigma_n \left(\mathscr{T}^n \left((1-\delta_n)\,\mathscr{I} + \delta_n \mathscr{T}^n\right)\right) u_n,\\ z_n &= (1-\zeta_n)\, y_n + \zeta_n \left(\mathscr{S}^n \left((1-\eta_n)\,\mathscr{I} + \eta_n \mathscr{S}^n\right)\right) y_n,\\ x_{n+1} &= \theta_n x_n + (1-\theta_n) z_n, \;\; n \geqslant 1, \end{aligned} \tag{3.71}$$

where $\{\alpha_n\}$, $\{\sigma_n\}$, $\{\delta_n\}$, $\{\zeta_n\}$, $\{\eta_n\}$ and $\{\theta_n\}$ are six sequences in $(0,1)$ and $\{\beta_n\}$ is a sequence in $(0,\infty)$.

COROLLARY 3.9. *Suppose that* $\mathscr{I} - \mathscr{T}$ *and* $\mathscr{I} - \mathscr{S}$ *are demiclosed at zero. If*

$$\tilde{\Omega} = VI(\mathscr{F},\mathscr{C}) \bigcap Fix(\mathscr{S}) \bigcap Fix(\mathscr{T}) \neq \emptyset$$

*and the following conditions are satisfied:*
$(C_1)$ $\lim_{n\to\infty} \alpha_n = 0$ *and* $\sum_{n=0}^{\infty} \alpha_n = \infty$;
$(C_2)$ $0 < a_1 < \sigma_n < c_1 < \hat{c}_1 < \delta_n < b_1 < \frac{1}{\sqrt{k_n^2 + L_1^2 + k_n^2}}$;
$(C_3)$ $0 < a_2 < \zeta_n < c_2 < \hat{c}_2 < \eta_n < b_2 < \frac{1}{\sqrt{l_n^2 + L_2^2 + l_n^2}}$;
$(C_4)$ $0 < \liminf_{n\to\infty} \theta_n \leqslant \limsup_{n\to\infty} \theta_n < 1$;
$(C_5)$ $\alpha > \frac{1}{2}$ *and* $0 < \liminf_{n\to\infty} \beta_n \leqslant \limsup_{n\to\infty} \beta_n < 2\alpha$;
$(C_6)$ $\sum_{n=1}^{\infty}(k_n - 1) < +\infty$, $\sum_{n=1}^{\infty}(l_n - 1) < +\infty$;
$(C_7)$ $\lim_{n\to\infty} \frac{k_n-1}{\alpha_n} = \lim_{n\to\infty} \frac{l_n-1}{\alpha_n} = 0$.

*Then, the iterative sequence* $\{x_n\}$ *defined by Equation (3.71) strongly converges to* $\hat{x} \in \tilde{\Omega}$ *which is minimum-norm solution, i.e.,* $\hat{x} = \mathscr{P}_{\tilde{\Omega}}\theta$.

## 4. Conclusion

In this paper, we investigated a generalized variational inequality and fixed points problems. We presented an iterative algorithm for finding a solution of the generalized variational inequality and fixed point of two uniformly Lipschitzian asymptotically quasi-pseudocontractive operators under a nonlinear transformation. Under some mild conditions, we demonstrated the strong convergence of the suggested algorithm.

## Declarations

*Availablity of data and materials.* Not applicable.

*Competing interests.* The authors declare that they have no competing interests.

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*Authors' contributions.* The authors equally conceived of the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript.

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*Wenlong Sun*
*Department of Mathematics, School of Science*
*Shenyang University of Technology*
*Shenyang 110870, P. R. China*
*e-mail:* sun_math@sut.edu.cn

*Gang Lu*
*Division of Foundational Teaching*
*Guangzhou College of Technology and Business*
*Guangzhou 510850, P. R. China*
*e-mail:* lvgang1234@hanmail.net

*Yuanfeng Jin*
*Department of Mathematics*
*Yanbian University*
*Yanji 133001, P. R. China*
*e-mail:* yfkim@ybu.edu.cn

*Choonkil Park*
*Research Institute for Natural Sciences*
*Hanyang University*
*Seoul 04763, Korea*
*e-mail:* baak@hanyang.ac.kr