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GENERALIZED WEIGHTED SOBOLEV–MORREY ESTIMATES FOR HYPOELLIPTIC OPERATORS WITH DRIFT ON HOMOGENEOUS GROUPS

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Abstract. Let $\mathbb{G}=\left(\mathbb{R}^N,\circ,\delta_\lambda\right)$ be a homogeneous group, Q be the homogeneous dimension of $\mathbb{G}$, $X_0,X_1,\dots,X_m$ be left invariant real vector fields on $\mathbb{G}$ and satisfy Hörmander's rank condition on $\mathbb{R}^N$. Assume that $X_1,\dots,X_m$ $(m\leqslant N-1)$ are homogeneous of degree one and X_0 is homogeneous of degree two with respect to the family of dilations $\left(\delta_\lambda\right)_{\lambda>0}$. Consider the following hypoelliptic operator with drift on $\mathbb{G}$

$$\mathscr{L}=\sum_{i,j=1}^{m}a_{ij}X_iX_j+a_0X_0,$$

where (a_{ij}) is a constant matrix satisfying the elliptic condition in $\mathbb{R}^m$ and $a_0\neq 0$. In this paper, for this class of operators we obtain generalized weighted Sobolev-Morrey estimates by establishing boundedness of a large class of sublinear operators T_α, $\alpha\in[0,Q)$ generated by Calderón-Zygmund operators ($\alpha=0$) and generated by fractional integral operator ($\alpha>0$) on generalized weighted Morrey spaces and proving interpolation results in generalized weighted Sobolev-Morrey spaces on $\mathbb{G}$.

1. Introduction and statement of main results

Let $\mathbb{G}=\left(\mathbb{R}^N,\circ,\delta_\lambda\right)$ be a homogeneous group on $\mathbb{R}^N$ and $X_0,X_1,\dots,X_m$ $(m<N)$ be left invariant real vector fields on $\mathbb{G}$. Assume that $X_1,\dots,X_m$ are δ_λ-homogeneous of degree one and X_0 is δ_λ-homogeneous of degree two with respect to the family of dilations $\left(\delta_\lambda\right)_{\lambda>0}$ and satisfying Hörmander's condition

$$rank L(X_0,X_1,\dots,X_m)(x)=N,\ \ x\in\mathbb{G}, \tag{1.1}$$

where $L(X_0,X_1,\dots,X_m)$ denotes the Lie algebra generated by $X_0,X_1,\dots,X_m$.

Our aim is to check generalized weighted Sobolev-Morrey estimates of the hypoelliptic operator with drift

$$\mathscr{L}=\sum_{i,j=1}^{m}a_{ij}X_iX_j+a_0X_0, \tag{1.2}$$

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where $a_0 \neq 0$, $(a_{ij})_{i,j=1}^m$ is a constant coefficient matrix satisfying that for some $\mu > 0$,

$$\mu^{-1}|\xi|^2 \leqslant \sum_{i,j=1}^m a_{ij}\xi_i\xi_j \leqslant \mu|\xi|^2, \quad \xi \in \mathbb{R}^m.$$

The operator $\mathscr{L}$ has been studied extensively by many authors. Hörmander in [30] pointed out that (1.1) implies the hypoellipticity of (1.2). In [15], Folland proved that homogeneous hypoelliptic operators on nilpotent groups possess homogeneous fundamental solutions. For the further properties of the fundamental solutions, see Bramanti and Brandolini [6]. In the papers [6, 28, 32, 52] a priori estimates for the operator $\mathscr{L}$ are considered. The operator $\mathscr{L}$ contains many particular cases. When $X_0 = \sum_{i,j=1}^n b_{ij}x_i\partial_{x_j} - \partial_t$, $X_i = \partial_{x_i}$, $i = 1,2,\dots,m$, $\mathscr{L}$ is a Kolmogorov-Fokker-Planck ultraparabolic operator of the kind

$$\mathscr{L}_1 u = \sum_{i,j=1}^m a_{ij}\partial^2_{x_ix_j}u + \sum_{i,j=1}^n b_{ij}x_i\partial_{x_j}u - \partial_t u,$$

where $(x,t) \in \mathbb{R}^{n+1}$, $(a_{ij})_{i,j=1}^m$ is a positive definite matrix, $(b_{ij})_{i,j=1}^n$ is a constant coefficient matrix with a suitable upper triangular structure. It is clear that $\mathscr{L}_1$ is a heat operator, when $m = n$, $(b_{ij})_{i,j=1}^n = (0)_{i,j=1}^n$. For more details see [38, 39, 54, 55].

The classical Morrey space was first introduced by Morrey [41] to study the partial differential equations, which characterized the regularity of the solutions to the second order elliptic partial differential equations. Since then, many studies have been focused on Morrey spaces; see, for instance, [1, 2, 12] and the references therein. In [1, 12] the authors showed the boundedness in Morrey spaces for some important operators in harmonic analysis such as Hardy-Littlewood operators, Calderón-Zygmund singular integral operators and fractional integral operators. Moreover, various Morrey spaces are defined in the process of study. The author, Mizuhara and Nakai [17, 42, 47] introduced generalized Morrey spaces $M_{p,\varphi}(\mathbb{R}^n)$ (see, also [18, 19, 22, 56]). Komori and Shirai [36] defined weighted Morrey spaces $L_{p,\kappa}(w)$. In [20] the author gave a concept of the generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{R}^n,w)$ which could be viewed as extension of both $M_{p,\varphi}(\mathbb{R}^n)$ and $L_{p,\kappa}(w)$. In [20], the boundedness of the classical operators and their commutators in spaces $M_{p,\varphi}(\mathbb{R}^n,w)$ was studied, see also [24, 25, 29, 35, 48, 49, 50, 51].

In this paper motivated by these articles, we will establish the boundedness of sublinear integral operators on generalized weighted Morrey spaces in the framework of homogeneous groups. The class of sublinear operators under consideration contains integral operators of harmonic analysis such as Hardy-Littlewood and fractional maximal operators, Calderón-Zygmund operators, potential operators on homogeneous groups, etc. Homogeneous groups include the Euclidean space, the Heisenberg group, and the Carnot groups, see [5, 11, 16]. Furthermore, we give applications to generalized weighted Sobolev - Morrey estimates for hypoelliptic operators with drift on homogeneous groups. Also, we obtain generalized weighted Morrey estimates for the sublinear operators generated by fractional integral operators on the homogeneous group and an application.

Let us state the following three main results of the paper.

THEOREM 1.1. (Generalized weighted Sobolev-Morrey estimate) *Let* $1 < p < \infty$, $Q > 4$, $w \in A_p(\mathbb{G})$ *and* φ *satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi(x,s)\, w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \lesssim \varphi(x,r), \tag{1.3}$$

Let also $u \in S_{2,p,\varphi}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$. *Then there exists a constant* $C > 0$ *such that*

$$\|u\|_{S_{2,p,\varphi}(\mathbb{G},w)} \leqslant C\Big(\|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}\Big), \tag{1.4}$$

where $A_p(\mathbb{G})$ *is Muckenhoupt class,*

$$\|u\|_{S_{1,p}(\mathbb{G},w)} = \|u\|_{L_p(\mathbb{G},w)} + \sum_{i=1}^{m} \|X_i u\|_{L_p(\mathbb{G},w)},$$

$$\|u\|_{S_{2,p,\varphi}(\mathbb{G},w)} = \|u\|_{M_{p,\varphi}(\mathbb{G},w)} + \sum_{i=1}^{m} \|X_i u\|_{M_{p,\varphi}(\mathbb{G},w)} + \sum_{i,j=1}^{m} \|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|X_0 u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

The space $S_{2,p,\varphi}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$ *consists of all functions* $u \in S_{2,p}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$ *with* $D^2 u \in M_{p,\varphi}(\mathbb{G},w)$, *and is endowed by the same norm (see Definition 2.2). Recall that* $S^0_{1,p}(\mathbb{G},w)$ *is the closure of* $C_0^{\infty}(\mathbb{G})$ *with respect to the norm in* $S_{1,p}(\mathbb{G},w)$.

REMARK 1.1. For $\varphi \in \mathscr{G}^p_w$ (see Remark 2.2) the condition (1.3) stays the following form

$$\int_r^{\infty} \frac{\varphi(t)}{t}\, dt \lesssim \varphi(r). \tag{1.5}$$

COROLLARY 1.1. *Let* $1 < p < \infty$, $Q > 4$, $w \in A_p(\mathbb{G})$ *and* $\varphi \in \mathscr{G}^p_w$ *satisfy the condition* (1.5). *If* $u \in S_{2,p,\varphi}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$, *then inequality* (1.4) *is valid.*

COROLLARY 1.2. (Weighted Sobolev-Morrey estimate) [31] *If* $1 < p < \infty$, $Q > 4$, $w \in A_p$, *and* $0 < k < 1$. *If* $u \in S_{2,p,\kappa}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$, *then there exists a constant* $C > 0$ *such that*

$$\|u\|_{S_{2,p,\kappa}(\mathbb{G},w)} \leqslant C\Big(\|\mathscr{L}u\|_{L_{p,\kappa}(\mathbb{G},w)} + \|u\|_{L_{p,\kappa}(\mathbb{G},w)}\Big),$$

where

$$\|u\|_{S_{2,p,\kappa}(\mathbb{G},w)} = \|u\|_{L_{p,\kappa}(\mathbb{G},w)} + \sum_{i=1}^{m} \|X_i u\|_{L_{p,\kappa}(\mathbb{G},w)} + \sum_{i,j=1}^{m} \|X_i X_j u\|_{L_{p,\kappa}(\mathbb{G},w)} + \|X_0 u\|_{L_{p,\kappa}(\mathbb{G},w)}.$$

THEOREM 1.2. (Higher order generalized weighted Sobolev-Morrey estimate) *Let* $1<p<\infty$, $Q>4$, $w\in A_p(\mathbb{G})$ *and* φ *satisfy the condition* (1.3)*. If* k *is a positive integer and* $u\in S_{2k,p,\varphi}(\mathbb{G},w)\cap S^0_{1,p}(\mathbb{G},w)$, *then there exists a constant* $C>0$ *such that*

$$\|u\|_{S_{2k+2,p,\varphi}(\mathbb{G},w)}\leqslant C\Big(\|\mathscr{L}u\|_{S_{2k,p,\varphi}(\mathbb{G},w)}+\|u\|_{M_{p,\varphi}(\mathbb{G},w)}\Big), \tag{1.6}$$

where $\|u\|_{S_{2k,p,\varphi}(\mathbb{G},w)}=\sum_{h=1}^{2k}\|D^hu\|_{M_{p,\varphi}(\mathbb{G},w)}$ *(see Definition* 2.2*),*

$$\|D^hu\|_{M_{p,\varphi}(\mathbb{G},w)}=\sum\|X_{ji}\dots X_{jl}u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

where $X_{ji}\dots X_{jl}$ *is homogeneous of degree* h *(let us note that* X_0 *is homogeneous of degree two while the remaining* $X_1,\dots,X_m$ *are homogeneous of degree one).*

COROLLARY 1.3. *Let* $1<p<\infty$, $Q>4$, $w\in A_p(\mathbb{G})$, k *be a positive integer and* $\varphi\in\mathscr{G}^p_w$ *satisfy the condition* (1.5)*. If* $u\in S_{2k,p,\varphi}(\mathbb{G},w)\cap S^0_{1,p}(\mathbb{G},w)$, *then inequality* (1.6) *is valid.*

COROLLARY 1.4. (Higher order weighted Sobolev-Morrey estimate) [31] *Let* $1<p<\infty$, $Q>4$, $w\in A_p$, $0<k<1$ *and* k *be a positive integer. If* $u\in S_{2,p,\kappa}(\mathbb{G},w)\cap S^0_{1,p}(\mathbb{G},w)$, *then there exists a constant* $C>0$ *such that*

$$\|u\|_{S_{2k+2,p,\kappa}(\mathbb{G},w)}\leqslant C\Big(\|\mathscr{L}u\|_{S_{2k,p,\kappa}(\mathbb{G},w)}+\|u\|_{L_{p,\kappa}(\mathbb{G},w)}\Big).$$

To inspect two theorems, we first prove the boundedness of sublinear operators generated by Calderón-Zygmund operators T_0 in generalized weighted Morrey space on $\mathbb{G}$ by applying the representation formulas of functions. These formulas depend on the fundamental solution of $\mathscr{L}$. Next generalized weighted Sobolev-Morrey interpolations on the first order derivatives and higher order derivatives of vector fields are derived. Then based on these results, we obtain generalized weighted Sobolev-Morrey estimates for $\mathscr{L}$. Instead, we shall apply representation formulas of higher order derivatives [6] to prove interpolations desired.

THEOREM 1.3. (Generalized weighted Morrey estimate) *Let* $1<p<q<\infty$, $\frac{1}{q}=\frac{1}{p}-\frac{1}{Q}$, $w\in A_{p,q}(\mathbb{G})$, *and* (φ_1,φ_2) *satisfy the condition*

$$\int_r^\infty\frac{\operatorname*{ess\,inf}_{t<s<\infty}\varphi_1(x,s)w^p(B(x,s))^{\frac{1}{p}}}{w^q(B(x,t))^{\frac{1}{q}}}\frac{dt}{t}\lesssim\varphi_2(x,r). \tag{1.7}$$

Then there exists a constant $C>0$ *such that for every* $\mathscr{L}u\in M_{p,\varphi_1}(\mathbb{G},w^p)$, *we have*

$$\|X_iu\|_{M_{q,\varphi_2}(\mathbb{G},w^q)}\leqslant C\|\mathscr{L}u\|_{M_{p,\varphi_1}(\mathbb{G},w^p)},\quad i=1,2,\dots,m.$$

If in Theorem 1.3 take $\varphi_1\equiv\varphi\in\mathscr{G}^p_w$, $\varphi_2(r)=r\varphi(r)$, then we get the following new corollary.

COROLLARY 1.5. *Let* $1 < p < q < \infty$, $\frac{1}{q} = \frac{1}{p} - \frac{1}{Q}$, $w \in A_{p,q}(\mathbb{G})$, *and* $\varphi \in \mathcal{G}_w^p$ *satisfy the condition*

$$\int_r^\infty \varphi(t)\,dt \lesssim r\varphi(r). \tag{1.8}$$

Then there exists a constant $C > 0$ *such that for every* $\mathcal{L}u \in M_{p,\varphi}(\mathbb{G}, w^p)$, *we have*

$$\|X_i u\|_{M_{q,r\varphi(r)}(\mathbb{G},w^p)} \leqslant C\|\mathcal{L}u\|_{M_{p,\varphi}(\mathbb{G},w^p)}, \quad i = 1,2,\ldots,m.$$

COROLLARY 1.6. (Morrey estimate with two weights) [31] *If* $1 < p < q < \infty$, $1/q = 1/p - 1/Q$, $w \in A_{p,q}(\mathbb{G})$, *and* $0 < \kappa < p/q$, *then there exists a constant* $C > 0$ *such that for every* $\mathcal{L}u \in L_{p,\kappa}(\mathbb{G}, w^p, w^q)$, *we have*

$$\|X_i u\|_{L_{q,\kappa q/p}(\mathbb{G},w^q)} \leqslant C\,\|\mathcal{L}u\|_{L_{p,\kappa}(\mathbb{G},w^p,w^q)}, \quad i = 1,2,\ldots,m.$$

The proof uses the extension of generalized weighted Morrey estimates for the sublinear operators generated by fractional integral operators T_α, $0 < \alpha < Q$ in the Euclidean space to the homogeneous group and application to $\mathcal{L}$.

REMARK 1.2. Note that, in the case $w \equiv 1$ Theorems 1.1, 1.2 and 1.3 was proved in [27]. Also in the case $w \equiv 1$ and $\varphi(x,r) \equiv |B(x,r)|^{\frac{\kappa-1}{p}}$ Corollaries 1.2, 1.4 and 1.6 was proved in [52].

Sobolev-Morrey spaces arose in the study of elliptic differential equations. Campanato considered Sobolev-Morrey spaces in [10]. More is investigated on Sobolev-Morrey spaces [13, 14, 23, 26, 31, 52, 54, 55]. The embedding relation can be found in [45, 46].

It is mentioned that since the second and higher order derivatives of vector fields are determined by Calderón-Zygmund operators rather than the fractional integral operators, we cannot use the method here to generalize estimates in Theorem 1.3 to the generalized weighted Sobolev-Morrey estimates for $\mathcal{L}$.

The plan of the paper is the following. In Section 2, we introduce some knowledge of the homogeneous group $\mathbb{G}$, the fundamental solution for $\mathcal{L}$ and the generalized weighted Morrey spaces. Section 3 is devoted to the proof of the boundedness for sublinear operators generated by Calderón-Zygmund operators T_0 in generalized weighted Morrey spaces. In addition to this, generalized weighted Morrey estimates for sublinear operators generated by fractional integral operators T_α, $0 < \alpha < Q$ are given. In Section 4 the generalized weighted Sobolev-Morrey interpolation inequalities on $\mathbb{G}$ are shown. The main results are proved in Section 5.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Preliminary

We describe first some notions on homogeneous Lie groups. For more details, we refer to the monograph [5].

Let $\circ$ be a given group law on $\mathbb{R}^N$ and assume that the map $(x,y) \to y^{-1} \circ x$ is smooth, then $\mathbb{R}^N$ together with this mapping forms a Lie group.

If there exist $0 < w_1 \leqslant w_2 \leqslant \ldots \leqslant w_N$, such that the dilations

$$\delta_\lambda : (x_1,\ldots,x_N) \mapsto (\lambda^{w_1}x_1,\ldots,\lambda^{w_N}x_N), \quad \lambda > 0$$

are group automorphisms, then the space $\mathbb{R}^N$ with this structure is called a homogeneous group, denoted by $\mathbb{G}$.

The homogeneous norm on $\mathbb{G}$ can be defined as follows. For any $x \in \mathbb{G}\setminus\{0\}$, set

$$\|x\| = \rho \Leftrightarrow |\delta_{1/\rho}x| = 1,$$

where $|\cdot|$ denotes the Euclidean norm, and set $\|0\| = 0$. For this mapping the following properties are valid.

1. $\|\delta_\lambda x\| = \lambda\|x\|$ for every $x \in \mathbb{G}$ and $\lambda > 0$;
2. there exists $c_0 \equiv c(\mathbb{G}) \geqslant 1$, such that for every $x,y \in \mathbb{G}$,

$$\|x^{-1}\| \leqslant c_0\,\|x\| \quad \textit{and} \quad \|x\circ y\| \leqslant c_0\,(\|x\|+\|y\|). \tag{2.1}$$

In view of these properties, we can define the quasidistance d by $d(x,y) = \|x\circ y^{-1}\|$ and define the d-ball by $B(x,r) \equiv B_r(x) = \{y \in G : d(x,y) < r\}$.

Let $\mathscr{B} = \{B(x,r) : x \in \mathbb{G}, r > 0\}$. By $|E|$ denote the Lebesgue measure of E. Note that $B(0,r) = \delta_r B(0,1)$, therefore

$$|B(x,r)| = r^Q|B(0,1)|, \quad x \in \mathbb{G}, r > 0, \tag{2.2}$$

where $Q = w_1 + \ldots + w_N$.

We will call that Q is the homogeneous dimension of $\mathbb{G}$ and always require $Q > 4$ in the sequel to estimate higher order derivatives of vector fields. By (2.2) the doubling condition on $\mathbb{G}$ holds, that is

$$|B(x,2r)| \leqslant c|B(x,r)|, \quad x \in \mathbb{G},\ r > 0,$$

where c is some positive constant, and so $(\mathbb{G},dx,d)$ is a space of homogeneous type.

A differential operator Y on $\mathbb{G}$ is called homogeneous of degree β $(\beta > 0)$, if for every test function φ,

$$Y(\varphi(\delta_\lambda x)) = \lambda^\beta (Y\varphi)(\delta_\lambda x), \quad \lambda > 0, x \in \mathbb{G};$$

a real function f on $\mathbb{G}$ is called homogeneous of degree α, if

$$f(\delta_\lambda x) = \lambda^\alpha f(x), \quad \lambda > 0, x \in \mathbb{G}.$$

Clearly, if Y is a homogeneous differential operator of degree β and f is a homogeneous function of degree α, then Yf is homogeneous of degree $\alpha - \beta$.

LEMMA 2.1. (See [6]) *Let $\mathscr{L}$ be a left invariant homogeneous differential operator of degree 2 on $\mathbb{G}$, then there is a unique fundamental solution $\Gamma(\cdot)$ such that for every test function u and every $x \in \mathbb{G}$,*

(a) $\Gamma(\cdot) \in C^\infty(\mathbb{G} \setminus \{0\})$;

(b) $\Gamma(\cdot)$ *is homogeneous of degree* $2 - Q$;

(c) $u(x) = (\mathscr{L}u * \Gamma)(x) = \int_{\mathbb{G}} \Gamma(x \circ y^{-1}) \mathscr{L}u(y)dy$;

(d) $X_i u(x) = \int_{\mathbb{G}} X_i \Gamma(x \circ y^{-1}) \mathscr{L}u(y)dy$.

Moreover, for $i, j = 1, \dots, m$, *there exist constants* $c_{i,j}$ *such that*

$$X_i X_j u(x) = V.P. \int_{\mathbb{G}} X_i X_j \Gamma(x \circ y^{-1}) \mathscr{L}u(y)dy + c_{ij} \mathscr{L}u(x).$$

REMARK 2.1. If we set $\Gamma_i = X_i \Gamma$, $\Gamma_{i,j} = X_i X_j \Gamma$, then it is obvious that Γ_i is homogeneous of degree $1 - Q$ and Γ_{ij} is homogeneous of degree $-Q$.

For any $f \in L_1^{\mathrm{loc}}(\mathbb{G})$, the Hardy-Littlewood maximal operator on $\mathbb{G}$ is defined by

$$Mf(x) = \sup_{r>0} \frac{1}{|B(x,r)|} \int_{B(x,r)} |f(y)|dy, \quad \text{a.e.} \quad x \in \mathbb{G}.$$

For any $f \in L_1^{\mathrm{loc}}(\mathbb{G})$, we say that T is a Calderón-Zygmund operator on $\mathbb{G}$ if

$$Tf(x) = \lim_{\varepsilon \to 0} \int_{\{y \in \mathbb{G}: \|x \circ y^{-1}\| > \varepsilon\}} K(x \circ y^{-1}) f(y)dy = V.P. \int_{\mathbb{G}} K(x \circ y^{-1}) f(y)dy,$$

where K satisfies

$$|K(x)| \leqslant \frac{c}{\|x\|^Q}, \quad |\nabla K(x)| \leqslant \frac{c}{\|x\|^{Q+1}}, \ x \in \mathbb{G} \setminus \{0\}.$$

For any $f \in L_1^{\mathrm{loc}}(\mathbb{G})$, the fractional maximal operator M_α and the fractional integral operator I_α on $\mathbb{G}$ are defined by

$$M_\alpha f(x) = \sup_{r>0} |B(x,r)|^{-1+\frac{\alpha}{Q}} \int_{B(x,r)} |f(y)|dy, \quad 0 \leqslant \alpha < Q,$$

$$I_\alpha f(x) = \int_{\mathbb{G}} \frac{f(y)}{\|x \circ y^{-1}\|^{Q-\alpha}} dy, \quad 0 < \alpha < Q,$$

respectively. If $\alpha = 0$, then $M = M_0$ is the Hardy-Littlewood maximal operator.

Suppose that T_α, $\alpha \in [0, Q)$ represents a linear or a sublinear operator, which satisfies, for any $f \in L_1(\mathbb{R}^n)$ with compact support and $x \notin supp f$, inequality

$$|T_\alpha f(x)| \leqslant C_1 \int_{\mathbb{G}} \frac{|f(y)|}{\|x \circ y^{-1}\|^{Q-\alpha}} dy, \tag{2.3}$$

where C_1 is independent of f and x.

Note that, the maximal operator M, and the Calderón-Zygmund operator T satisfy the condition (2.3) with $\alpha = 0$, and the fractional maximal operator M_α, and the fractional integral operator I_α satisfy the condition (2.3) with $0 < \alpha < Q$.

Let $0 < \alpha < Q$, $1 \leqslant p < \frac{Q}{\alpha}$ and $f \in L_p(\mathbb{G})$. Then the integral $I_\alpha f(x)$ converges absolutely for almost every $x \in \mathbb{G}$, see [18, Theorem 3.2.1]. The Hardy-Littlewood-Sobolev result states that (see [16], [17] and [18, Theorem 3.2.1]) the operator I_α is bounded from $L_p(\mathbb{G})$ to $L_q(\mathbb{G})$ if and only if $1 < p < q < \infty$ and $\alpha = Q/p - Q/q$. Also I_α is bounded from $L_1(\mathbb{G})$ to $WL_q(\mathbb{G})$ if and only if $1 < q < \infty$ and $\alpha = Q - Q/q$.

By a weight function, briefly weight, we mean a locally integrable function on $\mathbb{G}$ which takes values in $(0,\infty)$ almost everywhere. For a weight w and a measurable set E, we define $w(E) = \int_E w(x)dx$, and the characteristic function of E by χ_E. Let B be a ball on $\mathbb{G}$ and $kB(k>0)$ denote the ball with the same center as B whose radius is λ times that of B.

If w is a weight function, we denote by $L_p(\mathbb{G},w)$ the weighted Lebesgue space defined by finiteness of the norm

$$\|f\|_{L_p(\mathbb{G},w)} = \Big(\int_{\mathbb{G}} |f(x)|^p w(x)dx\Big)^{\frac{1}{p}} < \infty, \quad if \quad 1 \leqslant p < \infty$$

and

$$\|f\|_{L_\infty(\mathbb{G},w)} = \operatorname*{ess\,sup}_{x\in\mathbb{G}} |f(x)|w(x), \quad if \quad p = \infty.$$

We recall a weight function w is in the Muckenhoupt's class $A_p(\mathbb{G})$, $1 < p < \infty$ [43], if

$$[w]_{A_p} := \sup_B [w]_{A_p(B)} = \sup_B \Big(\frac{1}{|B|}\int_D w(x)dx\Big)\Big(\frac{1}{|B|}\int_B w(x)^{1-p'}dx\Big)^{p-1} < \infty,$$

where the supremum is taken with respect to all balls B and $\frac{1}{p}+\frac{1}{p'}=1$. For $p=1$, $w \in A_1(\mathbb{G})$ is defined by the condition $Mw(x) \leqslant Cw(x)$ with $[w]_{A_1} = \sup_{x\in\mathbb{G}} \frac{Mw(x)}{w(x)}$, and for $p=\infty$ $A_\infty(\mathbb{G}) = \cup_{1\leqslant p<\infty} A_p(\mathbb{G})$ and $[w]_\infty = \inf_{1\leqslant p<\infty} [w]_{A_p}$.

A weight function w is in the Muckenhoupt-Wheeden class $A_{p,q}(\mathbb{G})$, $1 < p < \infty$ [44], if

$$\begin{aligned}[w]_{A_{p,q}} &:= \sup_B [w]_{A_{p,q}(B)} \\ &= \sup_B \Big(\frac{1}{|B|}\int_B w(x)^q dx\Big)^{1/q}\Big(\frac{1}{|B|}\int_B w(x)^{-p'}dx\Big)^{1/p'} < \infty,\end{aligned}$$

where the supremum is taken with respect to all balls D and $\frac{1}{p}+\frac{1}{p'}=1$. While $p=1$, $w \in A_{1,q}(\mathbb{G})$ with $1 < q < \infty$ if

$$\begin{aligned}[w]_{A_{1,q}} &:= \sup_B [w]_{A_{1,q}(B)} \\ &= \sup_B \Big(\frac{1}{|B|}\int_B w(x)^q dx\Big)^{\frac{1}{q}}\Big(\operatorname*{ess\,sup}_{x\in B} \frac{1}{w(x)}\Big) < \infty.\end{aligned}$$

Weighted norm inequalities for fractional integral operators arise naturally in harmonic analysis, and have been extensively studied by several authors. Let $0 < \alpha < Q$,

$1 \leqslant p < \frac{Q}{\alpha}$, $\alpha = Q/p - Q/q$, and $w \in A_{pq}(\mathbb{G})$, then the operator I_α is bounded from $L_p(\mathbb{G}, w^p)$ to $L_q(\mathbb{G}, w^q)$, see [33] and in the Euclidean setting see [37].

We define generalized weighed Morrey spaces as follows.

DEFINITION 2.1. Let $1 \leqslant p < \infty$, φ be a positive measurable function on $\mathbb{G} \times (0,\infty)$ and w be non-negative measurable function on $\mathbb{G}$. We denote by $M_{p,\varphi}(\mathbb{G}, w)$ the generalized weighted Morrey space, the space of all functions $f \in L_p^{\mathrm{loc}}(\mathbb{G}, w)$ with finite norm

$$\|f\|_{M_{p,\varphi}(\mathbb{G},w)} = \sup_{x\in\mathbb{G}, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)},$$

where $\|f\|_{L_p(B(x,r),w)} \equiv \|f\chi_{B(x,r)}\|_{L_p(\mathbb{G},w)}$.

Furthermore, by $WM_{p,\varphi}(\mathbb{G}, w)$ we denote the weak generalized weighted Morrey space of all functions $f \in WL_p^{\mathrm{loc}}(\mathbb{G}, w)$ for which

$$\|f\|_{WM_{p,\varphi}(\mathbb{G},w)} = \sup_{x\in\mathbb{G}, r>0} \varphi(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{WL_p(B(x,r),w)} < \infty,$$

where $WL_p(B(x,r), w)$ denotes the weak $L_{p,w}$-space of measurable functions f for which

$$\|f\|_{WL_p(B(x,r),w)} \equiv \|f\chi_{B(x,r)}\|_{WL_p(\mathbb{G},w)} = \sup_{t>0} t \Big(\int_{\{y\in B(x,r):|f(y)|>t\}} w(y)dy \Big)^{\frac{1}{p}}.$$

EXAMPLE 2.1. (1) If $w \equiv 1$, then $M_{p,\varphi}(\mathbb{G}, 1) = M_{p,\varphi}(\mathbb{G})$ is the generalized Morrey space and $WM_{p,\varphi}(\mathbb{G}, 1) = WM_{p,\varphi}(\mathbb{G})$ is the weak generalized Morrey space.

(2) If $\varphi(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\kappa}(\mathbb{G}, w)$ is the weighted Morrey space and $WM_{p,\varphi}(\mathbb{G}, w) = WL_{p,\kappa}(\mathbb{G}, w)$ is the weak weighted Morrey space.

(3) If $\varphi(x,r) \equiv v(B(x,r))^{\frac{\kappa}{p}} w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\kappa}(\mathbb{G}, v, w)$ is the two weighted Morrey space.

(4) If $w \equiv 1$ and $\varphi(x,r) = r^{\frac{\lambda-Q}{p}}$ with $0 < \lambda < Q$, then $M_{p,\varphi}(\mathbb{G}, w) = L_{p,\lambda}(\mathbb{G})$ is the classical Morrey space and $WM_{p,\varphi}(\mathbb{G}, w) = WL_{p,\lambda}(\mathbb{G})$ is the weak Morrey space.

(5) If $\varphi(x,r) \equiv w(B(x,r))^{-\frac{1}{p}}$, then $M_{p,\varphi}(\mathbb{G}, w) = L_p(\mathbb{G}, w)$ is the weighted Lebesgue space and $WM_{p,\varphi}(\mathbb{G}, w) = WL_p(\mathbb{G}, w)$ is the weak weighted Lebesgue space.

We use the following simplified notation later:

$$\|Du\|_{M_{p,\varphi}(\mathbb{G},w)} = \sum_{i=1}^{m} \|X_i u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

$$\|D^2 u\|_{M_{p,\varphi}(\mathbb{G},w)} = \sum_{i,j=1}^{m} \|X_i X_j u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|X_0 u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

and generally,

$$\|D^k u\|_{M_{p,\varphi}(\mathbb{G},w)} = \sum \|X_{ji} \ldots X_{jl} u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

where $X_{ji} \ldots X_{jl}$ is homogeneous of degree k (let us note that X_0 is homogeneous of degree two while the remaining $X_1, \ldots, X_m$ are homogeneous of degree one).

DEFINITION 2.2. For $p \in [1,\infty)$, a nonnegative integer k and a weight w, the weighted Sobolev-Morrey space $S_{k,p,\varphi}(\mathbb{G},w)$ consists of all $M_{p,\varphi}(\mathbb{G},w)$ functions such that

$$\|u\|_{S_{k,p,\varphi}(\mathbb{G},w)} = \sum_{h=0}^{k} \|D^h u\|_{M_{p,\varphi}(\mathbb{G},w)}$$

is finite.

The space $S_{k,p,\varphi}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$ consists of all functions $u \in S_{k,p}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$ with $D^h u \in M_{p,\varphi}(\mathbb{G},w)$, and is endowed by the same norm.

REMARK 2.2. For a non-negative measurable function w defined on $(0,\infty)$, we denote by $\mathscr{G}^p_w$ the set of all almost decreasing functions $\varphi : \mathbb{G} \times (0,\infty) \to (0,\infty)$ such that

$$\inf_{D\in\mathscr{B}:r_B\leqslant r_{B_0}} \varphi(B) \gtrsim \varphi(B_0) \ \text{ for all } \ B_0 \in \mathscr{B}$$

and

$$\inf_{B\in\mathscr{B}:r_B\geqslant r_{B_0}} \varphi(B)\, w^p(B)^{\frac{1}{p}} \gtrsim \varphi(B_0)\, w^p(B_0)^{\frac{1}{p}},$$

where r_B and r_{B_0} denote the radius of the d-balls B and B_0, respectively.

We will use the following statement on the boundedness of the weighted Hardy operator

$$H_w g(t) := \int_t^\infty g(s)\, w(s)\, ds, \qquad 0 < t < \infty.$$

where w is a weight. The following theorem was proved in [21].

THEOREM 2.1. [21] *Let v_1, v_2 and w be weights on $(0,\infty)$ and $v_1(t)$ be bounded outside a neighborhood of the origin. The inequality*

$$\sup_{t>0} v_2(t)\, H_w g(t) \leqslant C \sup_{t>0} v_1(t)\, g(t)$$

holds for some $C > 0$ for all non-negative and non-decreasing g on $(0,\infty)$ if and only if

$$B := \sup_{t>0} v_2(t) \int_t^\infty \frac{w(s)\, ds}{\sup_{s<\tau<\infty} v_1(\tau)} < \infty$$

3. Sublinear operators on generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{G},w)$

In this section, we shall give the boundedness of the sublinear operators T_α, $\alpha \in [0,Q)$ generated by Calderón-Zygmund operators ($\alpha = 0$) and generated by fractional integral operator ($\alpha > 0$) on generalized weighted Morrey spaces $M_{p,\varphi}(\mathbb{G},w)$.

The following are true for the homogeneous group space [3, 40]. Let us note that the homogeneous group is a special case of homogeneous spaces, so we can state

LEMMA 3.1. *Let $1 \leqslant p < \infty$ and $w \in A_p(\mathbb{G})$. Then the maximal operator M and Calderón-Zygmund operator T are bounded on $L_p(\mathbb{G},w)$ for $p > 1$ and from $L_1(\mathbb{G},w)$ to $WL_1(\mathbb{G},w)$.*

LEMMA 3.2. *Let $1 \leqslant p < q < \infty$, $w \in A_{pq}(\mathbb{G})$, $0 < \alpha < \frac{Q}{p}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$. Then the fractional integral operator I_α is bounded from $L_p(\mathbb{G},w^p)$ to $L_q(\mathbb{G},w^q)$ for $p > 1$ and from $L_1(\mathbb{G},w)$ to $WL_q(\mathbb{G},w^q)$.*

The following theorem is valid.

THEOREM 3.1. *Let $1 \leqslant p < \infty$, $w \in A_p(\mathbb{G})$ and T_0 be a sublinear operator satisfying condition (2.3) with $\alpha = 0$ which is bounded on $L_p(\mathbb{G},w)$ for $p > 1$, and bounded from $L_1(\mathbb{G},w)$ to $WL_1(\mathbb{G},w)$. Then, for $p > 1$ inequality*

$$\|T_0 f\|_{L_p(B,w)} \leqslant C\, w(B)^{\frac{1}{p}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x_0,t),w)}\, w(B(x_0,t))^{-\frac{1}{p}} \frac{dt}{t}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_p^{\rm loc}(\mathbb{G},w)$, where $c_0 \geqslant 1$ is the constant from the triangle inequality (2.1) and C does not depend on f, x_0 and $r > 0$.

Moreover, for $p = 1$ inequality

$$\|T_0 f\|_{WL_1(B,w)} \leqslant C\, w(B) \int_{2c_0 r}^{\infty} \|f\|_{L_1(B(x_0,t),w)}\, w(B(x_0,t))^{-1} \frac{dt}{t}, \tag{3.1}$$

holds for any ball $B = B(x_0,r)$ and for all $f \in L_1^{\rm loc}(\mathbb{G},w)$, where C does not depend on f, x_0 and $r > 0$.

Proof. Let $f \in L_p^{\rm loc}(\mathbb{G},w)$, $p \in (1,\infty)$ and $w \in A_p(\mathbb{G})$. For arbitrary $x_0 \in \mathbb{G}$, set $B = B(x_0,r)$ for the ball centered at x_0 and of radius r, $2c_0 B = B(x_0, 2c_0 r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2c_0 B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2c_0 B)}}(y), \quad r > 0,$$

and for all $x \in \mathbb{G}$ we get

$$\begin{aligned} \left|T_0 f(x)\right| &\leqslant \left|T_0 f_1(x)\right| + \left|T_0 f_2(x)\right| \\ &\lesssim \left|T_0 f_1(x)\right| + \int_{\complement_{(2c_0 B)}} \frac{|f(y)|}{\|x \circ y^{-1}\|^Q} dy. \end{aligned} \tag{3.2}$$

First we show that $T_0 f(x)$ is well-defined *a.e.* x and independent of the choice B containing x.

As T_0 is bounded on $L_p(\mathbb{G},w)$ for $p > 1$ and $f_1 \in L_p(\mathbb{G},w)$, $T_0 f_1$ is well-defined. Next, we show that the second-term of the right-hand side (3.2) converges absolutely for any $f \in L_p^{\rm loc}(\mathbb{G},w)$ and almost every $x \in \mathbb{G}$.

Observe that the inclusions $x \in B$, $y \in {}^{\complement}(2c_0B)$ imply $\|x_0 \circ y^{-1}\| \approx \|x \circ y^{-1}\|$. Then we get

$$|T_0 f_2(x)| \lesssim \int_{\complement_{(2c_0B)}} \frac{|f(y)|}{\|x \circ y^{-1}\|^Q} dy \lesssim \int_{\complement_{(2c_0B)}} \frac{|f(y)|}{\|x_0 \circ y^{-1}\|^Q} dy.$$

By Fubini's theorem we have

$$\int_{\complement_{(2c_0B)}} \frac{|f(y)|}{\|x_0 \circ y^{-1}\|^Q} dy \approx \int_{\complement_{(2c_0B)}} |f(y)| \int_{\|x_0 \circ y^{-1}\|}^{\infty} \frac{dt}{t^{Q+1}} dy$$
$$\approx \int_{2c_0r}^{\infty} \int_{B(x_0,t)\setminus B(x_0,2c_0r)} |f(y)| dy \frac{dt}{t^{Q+1}} \lesssim \int_{2c_0r}^{\infty} \int_{B(x_0,t)} |f(y)| dy \frac{dt}{t^{Q+1}}.$$

Applying Hölder's inequality, we get

$$\begin{aligned} |T_0 f_2(x)| &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_1(B(x_0,t))} t^{-Q-1} dt \\ &\lesssim \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t),w)} \|w^{-1/p}\|_{L_{p'}(B(x_0,t))} \frac{dt}{t^{Q+1}} \\ &\leqslant [w]_{A_p}^{1/p} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t),w)} w(B(x_0,t))^{-1/p} \frac{dt}{t}. \end{aligned} \tag{3.3}$$

Therefore $T_0 f_2(x)$ exists for all $x \in B$. Since $\mathbb{G} = \bigcup_{r>0} B(x_0,r)$, we obtain the existence of $T_0 f(x)$ for a.e. $x \in \mathbb{G}$.

From (3.2) we have

$$\|T_0 f\|_{L_p(B,w)} \leqslant \|T_0 f_1\|_{L_p(B,w)} + \|T_0 f_2\|_{L_p(B,w)}.$$

Since $f_1 \in L_p(\mathbb{G},w)$, $T_0 f_1 \in L_p(\mathbb{G},w)$ and from the boundedness of T_0 in $L_p(\mathbb{G},w)$ it follows that:

$$\|T_0 f_1\|_{L_p(B,w)} \leqslant \|T_0 f_1\|_{L_p(\mathbb{G},w)} \leqslant C\|f_1\|_{L_p(\mathbb{G},w)} = C\|f\|_{L_p(2c_0B,w)},$$

where constant $C > 0$ is independent of f.

Moreover, from (3.3) for all $p \in [1,\infty)$ inequality

$$\|T_0 f_2\|_{L_p(B,w)} \lesssim [w]_{A_p}^{1/p} w(B)^{\frac{1}{p}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t),w)} w(B(x_0,t))^{-1/p} \frac{dt}{t} \tag{3.4}$$

is valid. Thus

$$\|T_0 f\|_{L_p(B,w)} \lesssim \|f\|_{L_{p,w}(2c_0B)} + [w]_{A_p}^{1/p} w(B)^{\frac{1}{p}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x_0,t),w)} w(B(x_0,t))^{-1/p} \frac{dt}{t}.$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2c_0B,w)} &\approx |B|\|f\|_{L_p(2c_0B,w)}\int_{2c_0r}^{\infty}\frac{dt}{t^{Q+1}}\\ &\lesssim |B|\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x_0,t),w)}\frac{dt}{t^{Q+1}}\\ &\leqslant w(B)^{\frac{1}{p}}\|w^{-1/p}\|_{L_{p'}(B)}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x_0,t),w)}\frac{dt}{t^{Q+1}}\\ &\leqslant w(B)^{\frac{1}{p}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x_0,t),w)}\|w^{-1/p}\|_{L_{p'}(B(x_0,t))}\frac{dt}{t^{Q+1}}\\ &\leqslant [w]_{A_p}^{1/p}w(B)^{\frac{1}{p}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x_0,t),w)}w(B(x_0,t))^{-1/p}\frac{dt}{t}.\end{aligned} \tag{3.5}$$

Thus

$$\|T_0f\|_{L_p(B,w)} \lesssim [w]_{A_p}^{1/p}w(B)^{\frac{1}{p}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x_0,t),w)}w(B(x_0,t))^{-1/p}\frac{dt}{t}.$$

Let $f \in L_1^{\mathrm{loc}}(\mathbb{G},w)$ and $w \in A_1(\mathbb{G})$. As T_0 is bounded from $L_1(\mathbb{G},w)$ to $WL_1(\mathbb{G},w)$ and $f_1 \in L_1(\mathbb{G},w)$, T_0f_1 is well-defined. Next, we show that the second-term of the right-hand side (3.2) converges absolutely for any $f \in L_1^{\mathrm{loc}}(\mathbb{G},w)$ and almost every $x \in \mathbb{G}$.

Indeed,

$$\begin{aligned}|T_0f_2(x)| &\lesssim \int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t))}t^{-Q-1}dt\\ &\lesssim \int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t),w)}\|w^{-1}\|_{L_\infty(B(x_0,t))}\frac{dt}{t^{Q+1}}\\ &\leqslant [w]_{A_1}\int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t),w)}w(B(x_0,t))^{-1}\frac{dt}{t}.\end{aligned} \tag{3.6}$$

Therefore $T_0f_2(x)$ exists for all $x \in B$ and for $f \in L_1^{\mathrm{loc}}(\mathbb{G},w)$ and $w \in A_1(\mathbb{G})$ we get the existence of $T_0f(x)$ for a.e. $x \in \mathbb{G}$.

From the weak $(1,1)$ boundedness of T_0 and (3.5) it follows that:

$$\begin{aligned}&\|T_0f_1\|_{WL_1(B,w)} \leqslant \|T_0f_1\|_{WL_1(\mathbb{G},w)} \lesssim \|f_1\|_{L_1(\mathbb{G},w)} = \|f\|_{L_1(2c_0B,w)}\\ &\approx |B|\|f\|_{L_1(2c_0B,w)}\int_{2c_0r}^{\infty}\frac{dt}{t^{Q+1}} \lesssim |B|\int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t),w)}\frac{dt}{t^{Q+1}}\\ &\leqslant w(B)\|w^{-1}\|_{L_\infty(B)}\int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t),w)}\frac{dt}{t^{Q+1}}\\ &\leqslant w(B)\int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t),w)}\|w^{-1}\|_{L_\infty(B(x_0,t))}\frac{dt}{t^{Q+1}}\\ &\leqslant [w]_{A_1}w(B)\int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t),w)}w(B(x_0,t))^{-1}\frac{dt}{t}.\end{aligned} \tag{3.7}$$

Then by (3.4) and (3.7) we get inequality (3.1). □

THEOREM 3.2. *Let* $1 \leqslant p < \infty$, $w \in A_p(\mathbb{G})$ *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s)\, w(B(x,s))^{\frac{1}{p}}}{w(B(x,t))^{\frac{1}{p}}} \frac{dt}{t} \leqslant C \varphi_2(x,r), \tag{3.8}$$

where C *does not depend on* x *and* r.

Let T_0 *be a sublinear operator satisfying condition* (2.3) *with* $\alpha = 0$ *which is bounded on* $L_p(\mathbb{G},w)$ *for* $p > 1$, *and bounded from* $L_1(\mathbb{G},w)$ *to* $WL_1(\mathbb{G},w)$. *Then the operator* T_0 *is bounded from* $M_{p,\varphi_1}(\mathbb{G},w)$ *to* $M_{p,\varphi_2}(\mathbb{G},w)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(\mathbb{G},w)$ *to* $WM_{1,\varphi_2}(\mathbb{G},w)$.

Proof. By condition (3.8), Theorems 2.1 and 3.1 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} w(B(x,t))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_p(B(x,r),w)}$ and $\omega(r) = w(B(x,r))^{-\frac{1}{p}} r$ we have for $p > 1$

$$\begin{aligned} \|T_0 f\|_{M_{p,\varphi_2}(\mathbb{G},w)} &\lesssim \sup_{x\in\mathbb{G},\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x,t),w)}\, w(B(x,t))^{-\frac{1}{p}} \frac{dt}{t} \\ &= \sup_{x\in\mathbb{R}^n,\, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w)} = \|f\|_{M_{p,\varphi_1}(\mathbb{G},w)} \end{aligned}$$

and for $p = 1$

$$\begin{aligned} \|T_0 f\|_{WM_{1,\varphi_2}(\mathbb{G},w)} &\lesssim \sup_{x\in\mathbb{G},\, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_1(B(x,t),w)}\, w(B(x,t))^{-1} \frac{dt}{t} \\ &= \sup_{x\in\mathbb{G},\, r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-1} \|f\|_{L_1(B(x,r),w)} = \|f\|_{M_{1,\varphi_1}(\mathbb{G},w)}. \qquad \square \end{aligned}$$

COROLLARY 3.1. *Let* $1 \leqslant p < \infty$, $w \in A_p(\mathbb{G})$ *and* (φ_1, φ_2) *satisfy the condition* (3.8). *Then the operators* M, T *are bounded from* $M_{p,\varphi_1}(\mathbb{G},w)$ *to* $M_{p,\varphi_2}(\mathbb{G},w)$ *for* $p > 1$ *and from* $M_{1,\varphi_1}(\mathbb{G},w)$ *to* $WM_{1,\varphi_2}(\mathbb{G},w)$.

COROLLARY 3.2. *Let* $1 \leqslant p < \infty$, $w \in A_p(\mathbb{G})$ *and* $\varphi \in \mathcal{G}_w^p$ *satisfy the condition* (1.5). *Let* T_0 *be a sublinear operator satisfying condition* (2.3) *with* $\alpha = 0$ *which is bounded on* $L_p(\mathbb{G},w)$ *for* $p > 1$, *and bounded from* $L_1(\mathbb{G},w)$ *to* $WL_1(\mathbb{G},w)$. *Then the operator* T_0 *is bounded on* $M_{p,\varphi}(\mathbb{G},w)$ *for* $p > 1$ *and from* $M_{1,\varphi}(\mathbb{G},w)$ *to* $WM_{1,\varphi}(\mathbb{G},w)$.

COROLLARY 3.3. *Let* $1 \leqslant p < \infty$, $w \in A_p(\mathbb{G})$ *and* $\varphi \in \mathcal{G}_w^p$ *satisfy the condition* (1.5). *Then the operators* M, T *are bounded on* $M_{p,\varphi}(\mathbb{G},w)$ *for* $p > 1$ *and from* $M_{1,\varphi}(\mathbb{G},w)$ *to* $WM_{1,\varphi}(\mathbb{G},w)$.

Note that for $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, from Theorem 3.2 we get the following new result.

COROLLARY 3.4. *Let $w \in A_p$, $1 \leqslant p < Q$ and $0 < \kappa < 1$. Let T_0 be a sublinear operator satisfying condition (2.3) with $\alpha = 0$ which is bounded on $L_p(\mathbb{G}, w)$ for $p > 1$, and bounded from $L_1(\mathbb{G}, w)$ to $WL_1(\mathbb{G}, w)$. Then the operator T_0 is bounded on $L_{p,\kappa}(\mathbb{G}, w)$ for $p > 1$ and from $L_{1,\kappa}(\mathbb{G}, w)$ to $WL_{1,\kappa}(\mathbb{G}, w)$.*

The following corollary for the operators M and T was proved in [31].

COROLLARY 3.5. *Let $w \in A_p$, $1 \leqslant p < Q$ and $0 < \kappa < 1$. Then for $p > 1$, the operators M, T are bounded on $L_{p,\kappa}(\mathbb{G}, w)$ and for $p = 1$, the operators M, T are bounded from $L_{1,\kappa}(\mathbb{G}, w)$ to $WL_{1,\kappa}(\mathbb{G}, w)$.*

REMARK 3.1. Note that, in the Euclidean setting Theorems 3.1 and 3.2 were proved in [20].

Next we state one of our main results. First we present some estimates which are the main tools for proving our theorems, on the boundedness of the operators T_α with $\alpha \in (0, Q)$ on the generalized weighted Morrey spaces.

THEOREM 3.3. *Let $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $w \in A_{p,q}$. Let also T_α be a sublinear operator satisfying condition (2.3), bounded from $L_p(\mathbb{G}, w^p)$ to $L_q(\mathbb{G}, w^q)$ for $p > 1$, and bounded from $L_1(\mathbb{G}, w)$ to $WL_q(\mathbb{G}, w^q)$ for $p = 1$.*

Then, for $1 < p < \frac{Q}{\alpha}$ inequality

$$\|T_\alpha f\|_{L_q(B(x,r),w^q)} \leqslant C w^q(B(x,r))^{\frac{1}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_p(B(x,t),w^p)} w^q(B(x,t))^{-\frac{1}{q}} \frac{dt}{t}$$

holds for any ball $B(x,r)$ and for all $f \in L_p^{\mathrm{loc}}(\mathbb{G}, w)$, where C does not depend on f, x and $r > 0$.

Moreover, for $p = 1$ inequality

$$\|T_\alpha f\|_{WL_q(B(x,r),w^q)} \leqslant C w^q(B(x,r))^{\frac{1}{q}} \int_{2c_0 r}^{\infty} \|f\|_{L_1(B(x,t),w)} w^q(B(x,t))^{-\frac{1}{q}} \frac{dt}{t}, \qquad (3.9)$$

holds for any ball $B(x,r)$ and for all $f \in L_1^{\mathrm{loc}}(\mathbb{G}, w)$, where C does not depend on f, x and $r > 0$.

Proof. Let $1 < p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, and $w \in A_{p,q}(\mathbb{G})$. For arbitrary $x \in \mathbb{G}$, set $B = B(x,r)$, $2c_0 B \equiv B(x, 2c_0 r)$. We represent f as

$$f = f_1 + f_2, \quad f_1(y) = f(y)\chi_{2c_0 B}(y), \quad f_2(y) = f(y)\chi_{\complement_{(2c_0 B)}}(y), \quad r > 0,$$

and have

$$\|T_\alpha f\|_{L_q(B,w^q)} \leqslant \|T_\alpha f_1\|_{L_q(B,w^q)} + \|T_\alpha f_2\|_{L_q(B,w^q)}.$$

Since $f_1 \in L_p(\mathbb{G}, w^p)$, $T_\alpha f_1 \in L_q(\mathbb{G}, w^q)$ and from the boundedness of T_α from $L_p(\mathbb{G}, w^p)$ to $L_q(\mathbb{G}, w^q)$ it follows that:

$$\|T_\alpha f_1\|_{L_q(B,w^q)} \leqslant \|T_\alpha f_1\|_{L_q(\mathbb{G},w^q)} \leqslant C\|f_1\|_{L_p(\mathbb{G},w^p)} = C\|f\|_{L_p(2c_0 B,w^p)},$$

where constant $C>0$ is independent of f.

It is clear that $z\in B$, $y\in {}^{\complement}(2c_0B)$ implies $\|x\circ y^{-1}\|\approx\|z\circ y^{-1}\|$. We get

$$|T_\alpha f_2(z)|\lesssim\int_{\complement_{(2c_0B)}}\frac{|f(y)|}{\|x\circ y^{-1}\|^{Q-\alpha}}dy.$$

By Fubini's theorem we have

$$\begin{aligned}\int_{\complement_{(2c_0B)}}\frac{|f(y)|}{\|x\circ y^{-1}\|^{Q-\alpha}}dy&\approx\int_{\complement_{(2c_0B)}}|f(y)|\Big(\int_{\|x\circ y^{-1}\|}^{\infty}\frac{dt}{t^{Q+1-\alpha}}\Big)\,dy\\&\approx\int_{2c_0r}^{\infty}\Big(\int_{2c_0r\leqslant\|x\circ y^{-1}\|<t}|f(y)|dy\Big)\,\frac{dt}{t^{Q+1-\alpha}}\\&\leqslant\int_{2c_0r}^{\infty}\Big(\int_{B(x,t)}|f(y)|dy\Big)\frac{dt}{t^{Q+1-\alpha}}.\end{aligned}$$

By applying Hölder's inequality, we get

$$\begin{aligned}|T_\alpha f_2(x)|&\lesssim\int_{2c_0r}^{\infty}\|f\|_{L_1(B(x_0,t))}\,t^{\alpha-Q-1}dt\\&\lesssim\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\,\|w^{-1}\|_{L_{p'}(B(x,t))}\,\frac{dt}{t^{Q+1-\alpha}}\\&\lesssim\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\,w^q(B(x,t))^{-\frac{1}{q}}\,\frac{dt}{t}.\end{aligned}\tag{3.10}$$

Moreover, for all $p\in[1,\infty)$ inequality

$$\|T_\alpha f_2\|_{L_q(B,w^q)}\lesssim w^q(B)^{\frac{1}{q}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\,w^q(B(x,t)^{-\frac{1}{q}}\,\frac{dt}{t}\tag{3.11}$$

is valid. Thus

$$\|T_\alpha f\|_{L_q(B,w^q)}\lesssim\|f\|_{L_{p,w^p}(2c_0B)}+w^q(B)^{\frac{1}{q}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\,\|w^{-1}\|_{L_{p'}(B(x,t))}\,\frac{dt}{t^{Q+1-\alpha}}.$$

On the other hand,

$$\begin{aligned}\|f\|_{L_p(2c_0B,w^p)}&\approx|B|^{1-\frac{\alpha}{Q}}\|f\|_{L_p(2c_0B,w^p)}\int_{2c_0r}^{\infty}\frac{dt}{t^{Q+1-\alpha}}\\&\leqslant|B|^{1-\frac{\alpha}{Q}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\frac{dt}{t^{Q+1-\alpha}}\\&\lesssim w^q(B)^{\frac{1}{q}}\|w^{-1}\|_{L_{p'}(B)}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\frac{dt}{t^{Q+1-\alpha}}\\&\lesssim w^q(B)^{\frac{1}{q}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\,\|w^{-1}\|_{L_{p'}(B(x,t))}\,\frac{dt}{t^{Q+1-\alpha}}\\&\lesssim[w]_{A_{p,q}}\,w^q(B)^{\frac{1}{q}}\int_{2c_0r}^{\infty}\|f\|_{L_p(B(x,t),w^p)}\,w^q(B(x,t))^{-\frac{1}{q}}\,\frac{dt}{t}.\end{aligned}\tag{3.12}$$

Thus

$$\|T_\alpha f\|_{L_q(B,w^q)} \lesssim w^q(B)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_p(B(x,t),w^p)}\, w^q(B(x,t))^{-\frac{1}{q}} \frac{dt}{t}.$$

Let $p=1$. From the weak $(1,q)$ boundedness of T_α and (3.12) it follows that:

$$\begin{aligned}
\|T_\alpha f_1\|_{WL_q(B,w^q)} &\leqslant \|T f_1\|_{WL_q(\mathbb{G},w^q)} \\
&\lesssim \|f_1\|_{L_1(\mathbb{G},w)} = \|f\|_{L_1(2c_0B,w)} \\
&\approx |B|^{1-\frac{\alpha}{Q}} \|f\|_{L_1(2c_0B,w)} \int_{2c_0r}^{\infty} \frac{dt}{t^{Q+1-\alpha}} \\
&\leqslant |B|^{1-\frac{\alpha}{Q}} \int_{2c_0r}^{\infty} \|f\|_{L_1(B(x,t),w)} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(B)^{\frac{1}{q}} \|w^{-1}\|_{L_{p'}(B)} \int_{2c_0r}^{\infty} \|f\|_{L_1(B(x,t),w)} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(B)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_1(B(x,t),w)} \|w^{-1}\|_{L_\infty(B(x,t))} \frac{dt}{t^{Q+1-\alpha}} \\
&\lesssim w^q(B)^{\frac{1}{q}} \int_{2c_0r}^{\infty} \|f\|_{L_1(B(x,t),w)}\, w^q(B(x,t))^{-\frac{1}{q}} \frac{dt}{t}.
\end{aligned} \tag{3.13}$$

By (3.11) and (3.13) we get inequality (3.9). □

THEOREM 3.4. *Let* $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$, *and* (φ_1, φ_2) *satisfy the condition*

$$\int_r^{\infty} \frac{\operatorname*{ess\,inf}_{t<s<\infty} \varphi_1(x,s) w^p(B(x,s))^{\frac{1}{p}}}{w^q(B(x,t))^{\frac{1}{q}}} \frac{dt}{t} \leqslant C \varphi_2(x,r), \tag{3.14}$$

where C *does not depend on* x *and* r. *Let* T_α *be a sublinear operator satisfying condition* (2.3) *with* $\alpha \in (0,Q)$, *bounded from* $L_{p,w^p}(\mathbb{G})$ *to* $L_{q,w^q}(\mathbb{G})$ *for* $p>1$, *and bounded from* $L_{1,w}(\mathbb{G})$ *to* $WL_{q,w^q}(\mathbb{G})$ *for* $p=1$. *Then the operator* T_α *is bounded from* $M_{p,\varphi_1}(\mathbb{G},w^p)$ *to* $M_{q,\varphi_2}(\mathbb{G},w^q)$ *for* $p>1$ *and from* $M_{1,\varphi_1}(\mathbb{G},w)$ *to* $WM_{q,\varphi_2}(\mathbb{G},w^q)$ *for* $p=1$. *Moreover, for* $p>1$

$$\|T_\alpha f\|_{M_{q,\varphi_2}(\mathbb{G},w^q)} \lesssim \|f\|_{M_{p,\varphi_1}(\mathbb{G},w^p)},$$

and for $p=1$

$$\|T_\alpha f\|_{WM_{q,\varphi_2}(\mathbb{G},w^q)} \lesssim \|f\|_{M_{1,\varphi_1}(\mathbb{G},w)}.$$

Proof. By condition (3.8), Theorems 2.1 and 3.3 with $v_2(r) = \varphi_2(x,r)^{-1}$, $v_1(r) = \varphi_1(x,r)^{-1} w^p(B(x,t))^{-\frac{1}{p}}$, $g(r) = \|f\|_{L_p(B(x,r),w)}$ and $\omega(r) = w^q(B(x,r))^{-\frac{1}{q}} r^{-1}$ we have for $p>1$

$$\begin{aligned}
\|T_\alpha f\|_{M_{q,\varphi_2}(\mathbb{G},w^q)} &\lesssim \sup_{x\in\mathbb{G}, r>0} \varphi_2(x,r)^{-1} \int_r^{\infty} \|f\|_{L_p(B(x,t),w^p)}\, w^q(B(x,t))^{-\frac{1}{q}} \frac{dt}{t} \\
&\lesssim \sup_{x\in\mathbb{G}, r>0} \varphi_1(x,r)^{-1} w^p(B(x,r))^{-\frac{1}{p}} \|f\|_{L_p(B(x,r),w^p)} = \|f\|_{M_{p,\varphi_1}(\mathbb{G},w^p)}
\end{aligned}$$

and for $p=1$

$$\|T_\alpha f\|_{WM_{q,\varphi_2}(\mathbb{G},w^q)} \lesssim \sup_{x\in\mathbb{G},r>0} \varphi_2(x,r)^{-1} \int_r^\infty \|f\|_{L_p(B(x,t),w)}\, w^q(B(x,t))^{-\frac{1}{q}} \frac{dt}{t}$$
$$= \sup_{x\in\mathbb{G},r>0} \varphi_1(x,r)^{-1} w(B(x,r))^{-1} \|f\|_{L_1(B(x,r),w)} = \|f\|_{M_{1,\varphi_1}(\mathbb{G},w)}. \qquad \square$$

COROLLARY 3.6. *Let $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$ and (φ_1, φ_2) satisfy condition (3.14). Then the fractional maximal operator M_α and the fractional integral operator I_α are bounded from $M_{p,\varphi_1}(\mathbb{G},w^p)$ to $M_{q,\varphi_2}(\mathbb{G},w^q)$ for $p>1$ and from $M_{1,\varphi_1}(\mathbb{G},w)$ to $WM_{q,\varphi_2}(\mathbb{G},w^q)$ for $p=1$.*

In Theorem 3.4 if take $\varphi_1 \equiv \varphi \in \mathcal{G}_w^p$, $\varphi_2(r) = r^\alpha \varphi(r)$, then we get the following new corollary.

COROLLARY 3.7. *Let $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$ and $\varphi \in \mathcal{G}_p$ satisfy the condition*

$$\int_r^\infty t^{\alpha-1} \varphi(t)\, dt \leqslant C r^\alpha \varphi(r), \tag{3.15}$$

where C does not depend on r. Let T_α be a sublinear operator satisfying condition (2.3) with $\alpha \in (0,Q)$, bounded from $L_{p,w^p}(\mathbb{G})$ to $L_{q,w^q}(\mathbb{G})$ for $p>1$, and bounded from $L_{1,w}(\mathbb{G})$ to $WL_{q,w^q}(\mathbb{G})$ for $p=1$. Then the operator T_α is bounded from $M_{p,\varphi}(\mathbb{G},w)$ to $M_{q,r^\alpha\varphi(r)}(\mathbb{G},w)$ for $p>1$ and from $M_{1,\varphi}(\mathbb{G},w)$ to $WM_{q,r^\alpha\varphi(r)}(\mathbb{G},w)$ for $p=1$.

COROLLARY 3.8. *Let $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $w \in A_{p,q}(\mathbb{G})$ and $\varphi \in \mathcal{G}_w^p$ satisfy the condition (3.15). Then the operators M_α and I_α are bounded from $M_{p,\varphi}(\mathbb{G},w)$ to $M_{q,r^\alpha\varphi(r)}(\mathbb{G},w)$ for $p>1$ and from $M_{1,\varphi}(\mathbb{G},w)$ to $WM_{q,r^\alpha\varphi(r)}(\mathbb{G},w)$ for $p=1$.*

For $\varphi_1(x,r) = \varphi_2(x,r) \equiv w(B(x,r))^{\frac{\kappa-1}{p}}$, from Theorem 3.4 we get the following new result.

COROLLARY 3.9. *Let $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $0 < \kappa < \frac{p}{q}$ and $w \in A_{p,q}(\mathbb{G})$. Let also T_α be a sublinear operator satisfying condition (2.3) with $\alpha \in (0,Q)$ bounded from $L_p(\mathbb{G},w^p)$ to $L_q(\mathbb{G},w^q)$ for $p>1$, and bounded from $L_1(\mathbb{G},w)$ to $WL_q(\mathbb{G},w^q)$. Then the operator T_α is bounded from $L_{p,\kappa}(\mathbb{G},w^p,w^q)$ to $L_{q,\kappa q/p}(\mathbb{G},w^q)$ for $p>1$ and from $L_{1,\kappa}(\mathbb{G},w,w^q)$ to $WL_{q,\kappa q}(\mathbb{G},w^q)$ for $p=1$.*

The following corollary for the operator I_α was proved in [31].

COROLLARY 3.10. [31] *Let $1 \leqslant p < q < \infty$, $0 < \alpha < \frac{Q}{p}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{Q}$, $0 < \kappa < \frac{p}{q}$ and $w \in A_{p,q}(\mathbb{G})$. Then the operators M_α and I_α are bounded from $L_{p,\kappa}(\mathbb{G},w^p,w^q)$ to $L_{q,\kappa q/p}(\mathbb{G},w^q)$ for $p>1$ and from $L_{1,\kappa}(\mathbb{G},w,w^q)$ to $WL_{q,\kappa q}(\mathbb{G},w^q)$ for $p=1$.*

REMARK 3.2. Note that, in the Euclidean setting of Theorems 3.3 and 3.4 were proved in [20].

4. Weighted Morrey inequalities

In this section, we will derive some interpolation inequalities for generalized weighted Sobolev-Morrey norms. We recall a statement in [52]:

LEMMA 4.1. *Let* $K \in C(\mathbb{G} \setminus \{0\})$ *be homogeneous of degree* α $(\alpha \in \mathbb{R})$ *with respect to the dilations* $(\delta_\lambda)_{\lambda>0}$*, then there exists a constant* $c > 0$ *such that*

$$|K(z)| \leqslant c\,\|z\|^{\alpha},$$

where $c = \sup\limits_{\Sigma_N} |K(z)|$, Σ_N *denotes the unit sphere of* $\mathbb{G}$.

Observe that if the integral kernel $K(\cdot)$ is homogeneous of degree $-Q$, then

$$Tf(x) = V.P. \int_{\mathbb{G}} K(x \circ y^{-1}) f(y) dy$$

is obviously a Calderón-Zygmund operator.

Given two balls B_{r_1}, B_{r_2} and a function $\phi \in C_0^\infty(\mathbb{G})$, let us write $B_{r_1} \prec \phi \prec B_{r_2}$ to mean that $0 \leqslant \phi(x) \leqslant 1$, $\phi(x) \equiv 1$ on B_{r_1} and $\operatorname{supp}\phi \subseteq B_{r_2}$. Now we show several interpolation inequalities in generalized weighted Sobolev-Morrey spaces on $\mathbb{G}$.

LEMMA 4.2. *Let* $1 < p < \infty$, $w \in A_p(\mathbb{G})$ *and* φ *satisfy the condition* (1.3). *Then there exists a constant* $c > 0$ *such that for any* $\varepsilon > 0$ *and any test function* u*, the following inequality holds*

$$\|Du\|_{M_{p,\varphi}(\mathbb{G},w)} \leqslant \varepsilon \|D^2 u\|_{M_{p,\varphi}(\mathbb{G},w)} + \frac{c}{\varepsilon} \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

Proof. From Lemma 2.1, we have

$$X_i u(x) = \int_{\mathbb{G}} X_i \Gamma(x \circ y^{-1}) \mathscr{L} u(y) dy = \int_{\mathbb{G}} \Gamma_i(x \circ y^{-1}) \mathscr{L} u(y) dy.$$

Let ϕ be a cutoff function with $B_{1/2}(0) \prec \phi \prec B_1(0)$, and split Γ_i as

$$\Gamma_i = \phi \Gamma_i + (1-\phi)\Gamma_i = K_0 + K_\infty,$$

where K_0 and K_∞ are all homogeneous of degree $1-Q$, then

$$X_i u(x) = \int_{\{y \in \mathbb{G}: \|x \circ y^{-1}\| < 1\}} K_0(x \circ y^{-1}) \mathscr{L} u(y) dy + \int_{\{y \in \mathbb{G}: \|x \circ y^{-1}\| \geqslant 1/2\}} K_\infty(x \circ y^{-1}) \mathscr{L} u(y) dy := I + II. \tag{4.1}$$

In terms of Lemma 4.1 (see, [31, pp. 1332]),

$$|I| \leqslant \int_{\{y \in \mathbb{G}: \|x \circ y^{-1}\| < 1\}} |K_0(x \circ y^{-1})| |\mathscr{L} u(y)| dy \leqslant CM\mathscr{L} u(x), \tag{4.2}$$

where C does not depend on x.

Using Corollary 3.1, we infer that

$$\|I\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|M\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{4.3}$$

In terms of Lemma 4.1 (see, [31, pp. 1332]),

$$|II| = \left| \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\geqslant 1/2\}} \widetilde{K}_\infty(x^{-1}\circ y)\mathscr{L}u(y)dy \right| \lesssim Mu(x). \tag{4.4}$$

It follows by Lemma 3.5 that

$$\|II\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|Mu\|_{M_{p,\varphi}(\mathbb{G},w)} \leqslant C\|u\|_{M_{p,\varphi}(\mathbb{G},w)}, \tag{4.5}$$

where C does not depend on x.

Summing (4.3) and (4.5), we obtain

$$\|Du\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

A dilation argument leads to

$$\varepsilon\|Du\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \varepsilon^2\|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

and the proof of the lemma is concluded. □

In the case of Euclidean space, the interpolation result on higher order derivatives can be deduced by the induction. But in our context the interpolation lemma on higher order derivatives of vector fields cannot be deduced simply from that on lower order derivative by the induction. Now we need to use the representation formula of the higher order derivative on homogeneous groups to arrive at our aim.

LEMMA 4.3. (See [6]) *Let $Q > 4$, for every integer $k \geqslant 2$ and any couple of left invariant differential monomials P^{2k-1} and P^{2k-2}, homogeneous of degree $2k-1$ and $2k-2$, respectively, we can determine two kernels $K^{(1)}$, $K^{(2)} \in C^\infty(\mathbb{G}\setminus\{0\})$ which are homogeneous of degree $1-Q$ and $2-Q$, respectively, such that for any test function u,*

$$P^{2k-1}u(x) = \left((\mathscr{L}^k u) * K^{(1)}\right)(x), \qquad P^{2k-2}u(x) = \left((\mathscr{L}^k u) * K^{(2)}\right)(x),$$

where $\mathscr{L}^k = \underbrace{\mathscr{L}\mathscr{L}\ldots\mathscr{L}}_{k \text{ times}}$.

LEMMA 4.4. *Let $1 < p < \infty$, $w \in A_p(\mathbb{G})$ and φ satisfy the condition (1.3). If $k \geqslant 2$ is an integer, then there exists a constant $c = c(Q,k) > 0$ such that for every $\varepsilon > 0$ and any test function u,*

$$\|D^{2k-1}u\|_{M_{p,\varphi}(\mathbb{G},w)} \leqslant \varepsilon\,\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \frac{c}{\varepsilon^{2k-1}}\,\|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

Proof. Suppose that ϕ is a cutoff function with $B_{1/2}(0) \prec \phi \prec B_1(0)$. By Lemma 4.3, we have

$$P^{2k-1}u(x) = \Big((\mathscr{L}^k u) * K^{(1)}\Big)(x).$$

Now let us split $K^{(1)}$ in the following way

$$K^{(1)} = \phi K^{(1)} + (1-\phi)K^{(1)} = K_0^{(1)} + K_\infty^{(1)},$$

where $K_0^{(1)}$ and $K_\infty^{(1)}$ are homogeneous of degree $1-Q$. Thus

$$\begin{aligned} P^{2k-1}u(x) &= \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} K_0^{(1)}(x\circ y^{-1})\,\mathscr{L}^k u(y)dy \\ &\quad + \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\geqslant 1/2\}} K_\infty^{(1)}(x\circ y^{-1})\,\mathscr{L}^k u(y)dy \\ &= I_1(x) + I_2(x). \end{aligned} \tag{4.6}$$

It is easy to see with (4.2) that

$$|I_1(x)| \leqslant \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} |K_0^{(1)}(x\circ y^{-1})|\,|\mathscr{L}^k u(y)|dy \lesssim M\mathscr{L}^k u(x).$$

From Corollary 3.1

$$\|I_1(\cdot)\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|M\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{4.7}$$

We have by using Lemma 4.1 and the way in (4.4) (see, [31, pp. 1333]),

$$|I_2(x)| \lesssim \sum_{i=0}^{\infty} \int_{\{y\in\mathbb{G}:2^{i-1}\leqslant\|x\circ y^{-1}\|<2^i\}} \frac{|u(y)|dy}{\|x\circ y^{-1}\|^{Q+2k-1}} \lesssim Mu(x).$$

Applying Lemma 3.5

$$\|I_2(\cdot)\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|Mu\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{4.8}$$

Combining (4.7) and (4.8), we have from (4.6) that

$$\|D^{2k-1}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

A dilation argument shows

$$\varepsilon^{2k-1}\|D^{2k-1}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \varepsilon^{2k}\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

and this ends the proof. □

LEMMA 4.5. *Let $1<p<\infty$, $w\in A_p(\mathbb{G})$ and φ satisfy the condition (1.3). If $k\geqslant 2$ is an integer, then there exists a constant $c=c(Q,k)>0$ such that for every $\varepsilon>0$ and any test function u,*

$$\|D^{2k-2}u\|_{M_{p,\varphi}(\mathbb{G},w)} \leqslant \varepsilon^2\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \frac{c}{\varepsilon^{2k-2}}\|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

Proof. Let ϕ be a cutoff function with $B_{1/2}(0) \prec \phi \prec B_1(0)$. By Lemma 4.3, we see

$$P^{2k-2}u(x) = \Big((\mathscr{L}^k u) * K^{(2)}\Big)(x).$$

Split $K^{(2)}$ as

$$K^{(2)} = \phi K^{(1)} + (1-\phi)K^{(2)} = K_0^{(2)} + K_\infty^{(2)},$$

where $K_0^{(2)}$ and $K_\infty^{(2)}$ are homogeneous of degree $2-Q$, then

$$\begin{aligned} P^{2k-2}u(x) &= \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|<1\}} K_0^{(2)}(x\circ y^{-1})\mathscr{L}^k u(y)dy \\ &\quad + \int_{\{y\in\mathbb{G}:\|x\circ y^{-1}\|\geqslant 1/2\}} K_\infty^{(2)}(x\circ y^{-1})\mathscr{L}^k u(y)dy \\ &= J_1(x) + J_2(x). \end{aligned}$$

Analogously to the proof of Lemma 4.4, it yields

$$\|J_1(\cdot)\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|M\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

$$\|J_2(\cdot)\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|Mu\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

Therefore

$$\|D^{2k-2}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

A dilation argument deduces

$$\varepsilon^{2k-2}\|D^{2k-2}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \varepsilon^{2k}\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.$$

This completes the proof. □

5. Proof of the main theorems

The following result is known, see [6, 15].

LEMMA 5.1. *For some integer h with $0<\gamma<Q$, assume that $K_\gamma \in C^\infty(\mathbb{G}\setminus\{0\})$ is homogeneous of degree $\gamma-Q$, f is an integrable function and T_γ is defined by*

$$T_\gamma f = f * K_\gamma,$$

P^γ is a left invariant homogeneous differential operator of degree γ, then

$$P^\gamma T_\gamma f = V.P.(f * P^\gamma K_\gamma) + cf$$

for some constant c depending on K_γ and P^γ.

Proof of Theorem 1.1. Let $u \in S_{2,p,\varphi}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$. It holds from Lemma 2.1

$$X_iX_ju(x) = V.P.\int_{\mathbb{G}} \Gamma_{ij}(x\circ y^{-1})\mathscr{L}u(y)dy + c_{ij}\mathscr{L}u(x),$$

and using Corollary 3.1,

$$\|X_iX_ju\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{5.1}$$

Due to $a_0X_0u = \mathscr{L}u - \sum\limits_{i,j=1}^{m} a_{ij}X_iX_j$, it follows that

$$\|X_0u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{5.2}$$

Then by (5.1) and (5.2)

$$\|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{5.3}$$

From Lemma 4.2, we have

$$\begin{aligned}\|Du\|_{M_{p,\varphi}(\mathbb{G},w)} &\lesssim \varepsilon\,\|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)} + \frac{1}{\varepsilon}\,\|u\|_{M_{p,\varphi}(\mathbb{G},w)}\\ &\lesssim \varepsilon\,\|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \frac{1}{\varepsilon}\,\|u\|_{M_{p,\varphi}(\mathbb{G},w)}.\end{aligned} \tag{5.4}$$

Combining (5.3) and (5.4), the proof is ended.

$$\begin{aligned}\|u\|_{S_{2,p,\varphi}(\mathbb{G},w)} &= \|u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|Du\|_{M_{p,\varphi}(\mathbb{G},w)} + \|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)}\\ &\lesssim \|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}. \quad \square\end{aligned}$$

Proof of Theorem 1.2. Let $u \in S_{2k,p,\varphi}(\mathbb{G},w) \cap S^0_{1,p}(\mathbb{G},w)$, where k is a positive integer. In order to prove the conclusion, we need to establish the following inequality: If k is a positive integer, there exists a constant $C>0$ such that for every test function u,

$$\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} \leqslant C\,\|D^{2k-2}\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{5.5}$$

When $k=1$, by (5.3),

$$\|D^2u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|\mathscr{L}u\|_{M_{p,\varphi}(\mathbb{G},w)}. \tag{5.6}$$

When $k \geqslant 2$, since X_0 cannot be expressed as the composition of two vector fields with homogeneity of degree 1, it follows that D^k cannot be obtained from $D(D^{k-1})$ directly. But P^k can be written as X_0P^{2k-2} or X_iP^{2k-1} $(i=1,\dots,m)$, denoted by P^2P^{2k-2} and PP^{2k-1}, respectively. Furthermore, it holds from Lemma 4.3 that

$$P^{2k-1}u(x) = \Big((\mathscr{L}^ku)*K^{(1)}\Big)(x), \quad P^{2k-2}u(x) = \Big((\mathscr{L}^ku)*K^{(2)}\Big)(x),$$

where $K^{(1)}$, $K^{(2)}$ are homogeneous of degree $1-Q$ and $2-Q$, respectively.

In the case of $P^{2k} = P^2 P^{2k-2}$, we have from Lemma 5.1,

$$P^{2k}u(x) = V.P. \int_{\mathbb{G}} P^2 K^{(2)}(x \circ y^{-1})(\mathscr{L}^k u)(y)dy + c_{ij}(\mathscr{L}^k u)(x),$$

where $P^2 K^{(2)}$ is homogeneous of degree $-Q$. Applying Corollary 3.1,

$$\begin{aligned}\|P^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} &\lesssim \Big\| \int_{\mathbb{G}} P^2 K^{(2)}(\cdot \circ y^{-1})(\mathscr{L}^k u)(y)dy \Big\|_{M_{p,\varphi}(\mathbb{G},w)} + \|\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \\ &\lesssim \|\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k-2}\mathscr{L} u\|_{M_{p,\varphi}(\mathbb{G},w)}.\end{aligned}$$

In the case of $P^{2k} = PP^{2k-1}$, we obtain by using Lemma 5.1,

$$P^{2k}u(x) = V.P. \int_{\mathbb{G}} PK^{(1)}(x \circ y^{-1})(\mathscr{L}^k u)(y)dy + c_{ij}(\mathscr{L}^k u)(x),$$

where $PK^{(1)}$ is homogeneous of degree $-Q$. By virtue of Corollary 3.1,

$$\begin{aligned}\|P^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} &\lesssim \Big\| \int_{\mathbb{G}} PK^{(1)}(\cdot \circ y^{-1})(\mathscr{L}^k u)(y)dy \Big\|_{M_{p,\varphi}(\mathbb{G},w)} + \|\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \\ &\lesssim \|\mathscr{L}^k u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k-2}\mathscr{L} u\|_{M_{p,\varphi}(\mathbb{G},w)}.\end{aligned}$$

As a consequence

$$\|D^{2k}u\|_{M_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k-2}\mathscr{L} u\|_{M_{p,\varphi}(\mathbb{G},w)},$$

and (5.5) is proved.

Then

$$\|D^{2k+2}u\|_{L_{p,\varphi}(\mathbb{G},w)} \lesssim \|D^{2k}\mathscr{L} u\|_{L_{p,\varphi}(\mathbb{G},w)}. \tag{5.7}$$

Lemma 4.4 implies that

$$\begin{aligned}\|D^{2k+1}u\|_{L_{p,\varphi}(\mathbb{G},w)} &\lesssim \varepsilon \|D^{2k+2}u\|_{L_{p,\varphi}(\mathbb{G},w)} + \frac{1}{\varepsilon^{2k+1}} \|u\|_{L_{p,\varphi}(\mathbb{G},w)} \\ &\lesssim \varepsilon \|D^{2k}\mathscr{L} u\|_{L_{p,\varphi}(\mathbb{G},w)} + \frac{1}{\varepsilon^{2k+1}} \|u\|_{L_{p,\varphi}(\mathbb{G},w)}. \end{aligned} \tag{5.8}$$

Combining (5.7) and (5.8), we have

$$\begin{aligned}&\|u\|_{S_{2k+2,p,\varphi}(\mathbb{G},w)} = \|u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|D^{2k+1}u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|D^{2k+2}u\|_{M_{p,\varphi}(\mathbb{G},w)} \\ &\lesssim \|D^{2k}\mathscr{L} u\|_{M_{p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)} = \|\mathscr{L} u\|_{S_{2k,p,\varphi}(\mathbb{G},w)} + \|u\|_{M_{p,\varphi}(\mathbb{G},w)}.\end{aligned}$$

Theorem 1.2 is proved. □

Proof of Theorem 1.3. From Lemma 2.1, we get

$$X_i u(x) = \int_{\mathbb{G}} \Gamma_i(x \circ y^{-1})\mathscr{L} u(y)dy.$$

Since the function $\Gamma_i(\cdot)$ is homogeneous of degree $1-Q$, it follows by Lemma 4.1 that

$$|X_i u(x)| \lesssim \int_{\mathbb{G}} \frac{|\mathscr{L}u(y)|}{\|x\circ y^{-1}\|^{Q-1}}\,dy,$$

and we finish the proof by applying Corollary 3.6 with $\alpha = 1$.

$$\|X_i u\|_{M_{q,\varphi_2}(\mathbb{G},w)} \lesssim \Big\| \int_{\mathbb{G}} \frac{|\mathscr{L}u(y)|}{\|\cdot\circ y^{-1}\|^{Q-1}}\,dy \Big\|_{M_{q,\varphi_2}(\mathbb{G},w)}$$
$$\lesssim \|\mathscr{L}u\|_{M_{p,\varphi_1}(\mathbb{G},w)}, \quad i = 1,2,\ldots,m. \quad \square$$

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REFERENCES

[1] D. R. ADAMS, *A note on Riesz potentials*, Duke Math. J. **42**, 765–778 (1975).

[2] D. R. ADAMS, J. XIAO, *Morrey spaces in harmonic analysis*, Ark. Mat. **50**, 201–230 (2012).

[3] A. BERNARDIS, S. HARTZSTEIN, G. PRADOLINI, *Weighted inequalities for commutators of fractional integrals on spaces of homogeneous type*, J. Math. Anal. Appl. **322**, 825–846 (2006).

[4] A. BONFIGLIOLI, E. LANCONELLI, *Lie groups related to Hörmander operators and Kolmogorov-Fokker-Planck equations*, Commun. Pure Appl. Anal. **11** (5), 1587–1614 (2012).

[5] A. BONFIGLIOLI, E. LANCONELLI, F. UGUZZONI, *Stratified Lie Groups and Potential Theory for Their Sub-Laplacians*, Springer Monographs in Mathematics, Springer, Berlin, 2007.

[6] M. BRAMANTI, L. BRANDOLINI, *L^p estimates for uniformly hypoelliptic operators with discontinuous coefficients on homogeneous groups*, Rend. Semin. Mat. Univ. Politec. Torino **58**, 389–433 (2000).

[7] M. BRAMANTI, L. BRANDOLINI, *L^p estimates for nonvariational hypoelliptic operators with VMO coefficients*, Trans. Amer. Math. Soc. **352** (2), 781–822 (2000).

[8] M. BRAMANTI, L. BRANDOLINI, *Estimates of BMO type for singular integrals on spaces of homogeneous type and applications to hypoelliptic PDES*, Rev. Mat. Iberoam. **21** (2), 511–556 (2005).

[9] M. BRAMANTI, M. C. CERUTTI, *L^p estimates for some ultraparabolic operators with discontinuous coefficients*, J. Math. Anal. Appl. **200**, 332–354 (1996).

[10] S. CAMPANATO, *Proprietá di inclusione per spazi di Morrey*, Ricerche Mat. **12**, 67–86 (1963).

[11] L. CAPOGNA, D. DANIELLI, S. D. PAULS, J. T. TYSON, *An Introduction to the Heisenberg Group and the Sub-Riemannian Isoperimetric Problem*, Progress in Mathematics, vol. 259, Birkhauser-Verlag, Basel, 2007.

[12] F. CHIARENZA, M. FRASCA, *Morrey spaces and Hardy-Littlewood maximal function*, Rend. Mat. Appl. **7**, 273–279 (1987).

[13] G. DI FAZIO, D. I. HAKIM, Y. SAWANO, *Elliptic equations with discontinuous coefficients in generalized Morrey spaces*, Eur. J. Math. **3** (3), 728–762 (2017).

[14] G. DI FAZIO, M. A. RAGUSA, *Interior estimates in Morrey spaces for strong solutions to nondivergence form equations with discontinuous coefficients*, J. Funct. Anal. **112** (2), 241–256 (1993).

[15] G. B. FOLLAND, *Subelliptic estimates and function spaces on nilpotent Lie groups*, Ark. Mat. **13**, 161–207 (1975).

[16] G. B. FOLLAND, E. M. STEIN, *Hardy Spaces on Homogeneous Groups*, Mathematical Notes Vol. 28, Princeton Univ. Press, Princeton, 1982.

[17] V. S. GULIYEV, *Integral operators on function spaces on the homogeneous groups and on domains in* $\mathbb{R}^n$, [Russian Doctor's degree dissertation], Moscow: Mat. Inst. Steklov 1994, p. 329.

[18] V. S. GULIYEV, *Function spaces, integral operators and two weighted inequalities on homogeneous groups, some applications*, (Russian) Baku: Casioglu 1999, p. 332.

[19] V. S. GULIYEV, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl. Art. ID 503948, 20 pp. (2009).

[20] V. S. GULIYEV, *Generalized weighted Morrey spaces and higher order commutators of sublinear operators*, Eurasian Math. J. **3** (3), 33–61 (2012).

[21] V. S. GULIYEV, *Generalized local Morrey spaces and fractional integral operators with rough kernel*, J. Math. Sci. (N. Y.) **193** (2), 211–227 (2013).

[22] V. S. GULIYEV, R. V. GULIYEV, M. N. OMAROVA, *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math. **17** (1), 56–71 (2018).

[23] V. S. GULIYEV, T. S. GADJIEV, A. SERBETCI, *The Dirichlet problem for the uniformly higher-order elliptic equations in generalized weighted Sobolev-Morrey spaces*, Nonlinear Stud. **26** (4), 831–842 (2019).

[24] V. S. GULIYEV, *Characterizations for the fractional maximal operator and its commutators in generalized weighted Morrey spaces on Carnot groups*, Anal. Math. Physics **10** (2), paper no. 13, 1–20 (2020).

[25] V. S. GULIYEV, *Calderoó-Zygmund operators with kernels of Dini's type on generalized weighted variable exponent Morrey spaces*, Positivity **25** (5), 1771–1788 (2021).

[26] V. S. GULIYEV, M. N. OMAROVA, M. A. RAGUSA, A. SCAPELLATO, *Regularity of solutions of elliptic equations in divergence form in modified local generalized Morrey spaces*, Anal. Math. Physics **11** (1), paper no. 13, 1–20 (2021).

[27] V. S. GULIYEV, *Generalized Sobolev-Morrey estimates for hypoelliptic operators on homogeneous groups*, Rev. R. Acad. Cienc. Exactas Fis. Nat. Ser. A Mat. RACSAM **115** (2), paper no. 69, 1–23 (2021).

[28] C. E. GUTIERREZ, E. LANCONELLI, *Schauder estimates for sub-elliptic equations*, J. Evol. Equ. **9**, 707–726 (2009).

[29] V. H. HAMZAYEV, *Sublinear operators with rough kernel generated by Calderon-Zygmund operators and their commutators on generalized weighted Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci. Mathematics **38** (1), 79–94 (2018).

[30] L. HÖRMANDER, *Hypoelliptic second order differential equations*, Acta Math. **119**, 147–161 (1967).

[31] Y. HOU, P. NIU, *Weighted Sobolev-Morrey estimates for hypoelliptic operators with drift on homogeneous groups*, J. Math. Anal. Appl. **428**, 1319–1338 (2015).

[32] Y. HOU, X. W. CUI, X. J. FENG, *Global Hölder estimates for hypoelliptic operators with drift on homogeneous groups*, Miskolc Math. Notes **13**, 392–401 (2012).

[33] A. KAIREMA, *Sharp weighted bounds for fractional integral operators in a space of homogeneous type*, Math. Scand. **114** (2), 226–253 (2014).

[34] A. E. KOGOJ, E. LANCONELLI, *An invariant Harnack inequality for a class of hypoelliptic ultraparabolic equations*, Mediterr. J. Math. **1**, 51–80 (2004).

[35] V. KOKILASHVILI, A. MESKHI, M. A. RAGUSA, *Weighted extrapolation in grand Morrey spaces and applications to partial differential equations*, Atti Accad. Naz. Lincei Rend. Lincei Mat. Appl. **30** (1), 67–92 (2019).

[36] Y. KOMORI, S. SHIRAI, *Weighted Morrey spaces and a singular integral operator*, Math. Nachr. **282** (2), 219–231 (2009).

[37] M. T. LACEY, K. MOEN, C. PEREZ, R. H. TORRES, *Sharp weighted bounds for fractional integral operators*, J. Funct. Anal. **259**, 1073–1097 (2010).

[38] E. LANCONELLI, A. PASCUCCI, S. POLIDORO, *Linear and nonlinear ultraparabolic equations of Kolmogorov type arising in diffusion theory and in finance*, in: Nonlinear Problems in Mathematical Physics and Related Topics II, Kluwer/Plenum, New York, 243–265 (2002).

[39] E. LANCONELLI, S. POLIDORO, *On a class of hypoelliptic evolution operators*, Rend. Semin. Mat. Univ. Politec. Torino **52** (1), 29–63 (1994).

[40] G. Z. LU, *Weighted Poincare and Sobolev inequalities for vector fields satisfying Hörmander's condition and applications*, Rev. Mat. Iberoam. **8** (3), 367–439 (1992).

[41] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc. **43**, 126–166 (1938).

[42] T. MIZUHARA, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis (S. Igari, Editor), ICM 90 Satellite Proceedings, Springer-Verlag, Tokyo 183–189 (1991).

[43] B. MUCKENHOUPT, *Weighted norm inequalities for the Hardy maximal function*, Trans. Amer. Math. Soc. **165**, 207–226 (1972).

[44] B. MUCKENHOUPT, R. L. WHEEDEN, *Weighted norm inequalities for fractional integral*, Trans. Amer. Math. Soc. **192**, 261–274 (1974).

[45] A. M. NAJAFOV, *On some properties of the functions from Sobolev-Morrey type spaces*, Cent. Eur. J. Math. **3** (3), 496–507 (2005).

[46] A. M. NAJAFOV, *Embedding theorems in the Sobolev-Morrey type spaces $S^l_{p,a,\kappa,r}W(G)$ with dominant mixed derivatives*, Sib. Math. J. **47** (3), 613–625 (2006).

[47] E. NAKAI, *Hardy-Littlewood maximal operator, singular integral operators and Riesz potentials on generalized Morrey spaces*, Math. Nachr. **166**, 95–103 (1994).

[48] S. NAKAMURA, *Generalized weighted Morrey spaces and classical operators*, Math. Nachr. **289** (17–18), 2235–2262 (2016).

[49] S. NAKAMURA, Y. SAWANO, *The singular integral operator and its commutator on weighted Morrey spaces*, Collect. Math. **68** (2), 145–174 (2017).

[50] S. NAKAMURA, Y. SAWANO, H. TANAKA, *The fractional operators on weighted Morrey spaces*, J. Geom. Anal. **28** (2), 1502–1524 (2018).

[51] S. NAKAMURA, Y. SAWANO, H. TANAKA, *Weighted local Morrey spaces*, Ann. Acad. Sci. Fenn. Math. **45** (1), 67–93 (2020).

[52] P. C. NIU, X. J. FENG, *Global Sobolev-Morrey estimates for hypoelliptic operators with drift on homogeneous group*, (in Chinese) Sci. Sin. Math. **42** (9), 905–920 (2012).

[53] C. PEREZ, R. WHEEDEN, *Uncertainty principle estimates for vector fields*, J. Funct. Anal. **181**, 146–188 (2001).

[54] S. POLIDORO, M. A. RAGUSA, *Sobolev-Morrey spaces related to an ultraparabolic equation*, Manuscripta Math. **96**, 371–392 (1998).

[55] M. A. RAGUSA, *On weak solutions of ultraparabolic equations*, Nonlinear Anal. **47** (1), 503–511 (2001).

[56] Y. SAWANO, *A thought on generalized Morrey spaces*, J. Indonesian Math. Soc. **25** (3), 210–281 (2019).

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