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ON THE GENERALIZED POWER–TYPE TOADER MEAN

TIE-HONG ZHAO, MIAO-KUN WANG*, YE-QI DAI AND YU-MING CHU

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Abstract. This paper deals with the so-called generalized power-type Toader mean which is defined by

$$\boldsymbol{T}_n(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^n\cos^2\theta+b^n\sin^2\theta}d\theta\right)^{2/n}$$

for $a,b>0$ with non-zero integer n. In this study, we establish the following chain of inequalities

$$\begin{aligned}\boldsymbol{H}(a,b)<\boldsymbol{T}_{-1}(a,b)<\boldsymbol{G}(a,b)<\boldsymbol{T}_1(a,b)<\boldsymbol{A}(a,b)\\<\boldsymbol{T}_2(a,b)<\boldsymbol{Q}(a,b)<\boldsymbol{T}_3(a,b)<\boldsymbol{T}_4(a,b)<\boldsymbol{C}(a,b)\end{aligned}$$

for all $a,b>0$ with $a\neq b$, where $\boldsymbol{H}(a,b)=2ab/(a+b)$, $\boldsymbol{G}(a,b)=\sqrt{ab}$, $\boldsymbol{A}(a,b)=(a+b)/2$, $\boldsymbol{Q}(a,b)=[(a^2+b^2)/2]^{1/2}$ and $\boldsymbol{C}(a,b)=(a^2+b^2)/(a+b)$ are the harmonic, geometric, arithmetic, quadratic and contra-harmonic means, respectively. Further, we provide sharp bounds for $\boldsymbol{T}_{-1}(a,b)$ and $\boldsymbol{T}_4(a,b)$ in terms of bivariate means mentioned above. As applications, new bounds for complete elliptic integral of the second kind are established.

1. Introduction

Let a,b be positive real numbers and

$$r_n(\theta)=\begin{cases}(a^n\cos^2\theta+b^n\sin^2\theta)^{1/n}, & n\neq 0,\\ a^{\cos^2\theta}b^{\sin^2\theta}, & n=0.\end{cases}$$

For a strictly monotonic function $p:\mathbb{R}^+\to\mathbb{R}$, the integral quasi-arithmetic mean $\boldsymbol{M}_{p,n}(a,b)$ [1] is defined by

$$\boldsymbol{M}_{p,n}(a,b)=p^{-1}\left(\frac{1}{2\pi}\int_0^{2\pi}p(r_n(\theta))d\theta\right)=p^{-1}\left(\frac{2}{\pi}\int_0^{\pi/2}p(r_n(\theta))d\theta\right),$$

where p^{-1} is the inverse function of p.

As was shown in [2, 3], the means $\boldsymbol{M}_{p,n}$ can represent some known means for special choices of p and n. For instance,

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* Corresponding author.

Paper JMI-16-18

- The *Gaussian arithmetic-geometric* mean [4] is

$$\boldsymbol{AG}(a,b)=\frac{\pi}{2\int_0^{\pi/2}(a^2\cos^2\theta+b^2\sin^2\theta)^{-1/2}d\theta}:=\boldsymbol{M}_{1/x,2}(a,b),$$

  which is also defined by the common limit of the sequences $\{a_n\}$ and $\{b_n\}$ satisfying

$$a_0=a,\quad b_0=b,\quad a_{n+1}=\frac{a_n+b_n}{2},\quad b_{n+1}=\sqrt{a_nb_n}.$$

- The *Toader* mean [5] is

$$\boldsymbol{T}(a,b)=\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^2\cos^2\theta+b^2\sin^2\theta}d\theta:=\boldsymbol{M}_{x,2}(a,b). \tag{1.1}$$

- The *Toader-Qi* mean [6] is

$$\boldsymbol{TQ}(a,b)=\frac{\pi}{2}\int_0^{\pi/2}a^{\cos^2\theta}b^{\sin^2\theta}d\theta:=\boldsymbol{M}_{x,0}(a,b).$$

- A special *quasi-arithmetic* mean is defined by

$$\boldsymbol{E}(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a\cos^2\theta+b\sin^2\theta}d\theta\right)^2:=\boldsymbol{M}_{x^{1/2},1}(a,b), \tag{1.2}$$

  which was first treated in [7].

The eye-catching similarity between (1.1) and (1.2) allows us to raise the the following generalization

$$\boldsymbol{T}_n(a,b):=\boldsymbol{M}_{x^{n/2},n}(a,b)=\left(\frac{2}{\pi}\int_0^{\pi/2}\sqrt{a^n\cos^2\theta+b^n\sin^2\theta}d\theta\right)^{2/n} \tag{1.3}$$

for $n\in\mathbb{Z}^*=\mathbb{Z}\backslash\{0\}$. The mean of the form (1.3) will be called *generalized power-type Toader mean*. Actually, an alternative representation of $\boldsymbol{T}_n(a,b)$ can be derived from Toader mean

$$\boldsymbol{T}_n(a,b)=\left[\boldsymbol{T}\left(a^{n/2},b^{n/2}\right)\right]^{2/n},\quad (n\neq 0). \tag{1.4}$$

This is also called the $n/2$-*modification* of $\boldsymbol{T}(a,b)$ introduced in [8], where the authors study a one-parameter mean $\boldsymbol{AG}_t(a,b)=[\boldsymbol{AG}(a^t,b^t)]^{1/t}$ (that is called t-modification of $\boldsymbol{AG}(a,b)$). In particular, some well-known means are the p-modification of the classical means, such as the power mean, which is given by

$$\boldsymbol{M}_p(a,b)=\begin{cases}\left(\frac{a^p+b^p}{2}\right)^{1/p}, & p\neq 0,\\ \sqrt{ab}, & p=0.\end{cases} \tag{1.5}$$

As special cases we denote in this paper the arithmetic mean $\boldsymbol{A}=\boldsymbol{M}_1$, the harmonic mean $\boldsymbol{H}=\boldsymbol{M}_{-1}$ and the quadratic mean $\boldsymbol{Q}=\boldsymbol{M}_2$.

We recall the *Gauss identity* [9]

$$\boldsymbol{AG}(1,r')\mathscr{K}(r)=\frac{\pi}{2}$$

for $r\in(0,1)$ where and in what follows $r'=\sqrt{1-r^2}$, which shows that the Gauss arithmetic-geometric mean can be expressed by the complete elliptic integral of the first kind $\mathscr{K}$. As usual, $\mathscr{K}$ and $\mathscr{E}$ denote the complete elliptic integrals of the first and second kinds [10] given by

$$\begin{aligned}\mathscr{K}(r)&=\int_0^{\pi/2}\left(1-r^2\sin^2 d\theta\right)^{-1/2}d\theta,\\ \mathscr{E}(r)&=\int_0^{\pi/2}\left(1-r^2\sin^2 d\theta\right)^{1/2}d\theta.\end{aligned} \tag{1.6}$$

In view of (1.3) and (1.6), $\boldsymbol{T}_n(a,b)$ can be expressed by complete elliptic integrals of the second kind

$$\boldsymbol{T}_n(a,b)=\begin{cases}a\left[\frac{2\mathscr{E}\left(\sqrt{1-(b/a)^n}\right)}{\pi}\right]^{2/n}, & a^n\geqslant b^n,\\ b\left[\frac{2\mathscr{E}\left(\sqrt{1-(a/b)^n}\right)}{\pi}\right]^{2/n}, & a^n<b^n,\end{cases}\qquad (n\neq 0). \tag{1.7}$$

Legendre complete elliptic integrals have wide applications in the geometric function theory [11, 12, 13, 14, 15, 16, 17, 18, 19, 20], differential equation theory [21, 22, 23, 24, 25, 26], theory of mean values [27, 28, 29, 30] and many other fields [31, 32, 33, 34, 35]. Due to their importance, in the past few years, estimating precise bounds for the complete elliptic integrals of the first and second kinds and their generalizations have attracted the attention of many mathematicians [36, 37, 38, 39, 40, 41, 42].

Recently, one way to study complete elliptic integrals is to be applied in the theory of mean values, and present sharp bounds for the bivariate means related to complete elliptic integrals (such as $\boldsymbol{AG}(a,b)$ and $\boldsymbol{T}(a,b)$). As a consequence, several elegant double inequalities for $\mathscr{K}(r)$ and $\mathscr{E}(r)$ were established in [43, 44, 45, 46]. Specifically, the authors in [8, 47, 48] proved the double inequality

$$\boldsymbol{L}(a,b)<\boldsymbol{AG}(a,b)<\frac{\pi}{2}\boldsymbol{L}(a,b)$$

for all $a,b>0$ with $a\neq b$, where $\boldsymbol{L}(a,b)=(b-a)/(\log b-\log a)$ is the logarithmic mean of a and b.

Sándor [49, 50] proved that the double inequality

$$\sqrt{\boldsymbol{A}(a,b)\boldsymbol{G}(a,b)}<\boldsymbol{AG}(a,b)<\frac{\boldsymbol{A}(a,b)+\boldsymbol{G}(a,b)}{4}+\frac{\boldsymbol{Q}^2(a,b)\sqrt{\boldsymbol{H}(a,b)}}{2\boldsymbol{C}(a,b)\sqrt{\boldsymbol{G}(a,b)}}$$

holds true for all $a,b>0$ with $a\neq b$, where $\boldsymbol{C}(a,b)=(a^2+b^2)/(a+b)$ is contra-harmonic means of a and b.

Barnard, Pearce and Richards [36], and Alzer and Qiu [51] proved that the double inequality

$$\boldsymbol{M}_{3/2}(a,b)<\boldsymbol{T}(a,b)<\boldsymbol{M}_{\log 2/\log(\pi/2)}(a,b) \tag{1.8}$$

holds for all $a,b>0$ with $a\neq b$, where the parameters $3/2$ and $\log 2/\log(\pi/2)$ are the best possible constants satisfying the inequality (1.8).

In [52], Qian and Chu proved that $\lambda=\left(1-\sqrt{1-(2\sqrt{2}/\pi)^{4/p}}\right)/2$ and $\mu=1/2-\sqrt{p}/(4p)$ are the best possible parameters such that the double inequality

$$\begin{aligned}\boldsymbol{G}^p\big(\lambda a+(1-\lambda)b,\lambda b+(1-\lambda)a\big)\boldsymbol{A}^{1-p}(a,b)&<\boldsymbol{E}(a,b)\\ &<\boldsymbol{G}^p\big(\mu a+(1-\mu)b,\mu b+(1-\mu)a\big)\boldsymbol{A}^{1-p}(a,b)\end{aligned}$$

holds for all $p\in[1,\infty)$ and $a>b>0$.

Very recently, Zhao et al. [53] proved that the double inequalities

$$\begin{aligned}\alpha_1\left[\frac{7\boldsymbol{C}(a,b)}{16}+\frac{9\boldsymbol{H}(a,b)}{16}\right]+(1-\alpha_1)\left[\frac{3\boldsymbol{A}(a,b)}{4}+\frac{\boldsymbol{G}(a,b)}{4}\right]&<\boldsymbol{E}(a,b)\\ <\beta_1\left[\frac{7\boldsymbol{C}(a,b)}{16}+\frac{9\boldsymbol{H}(a,b)}{16}\right]&+(1-\beta_1)\left[\frac{3\boldsymbol{A}(a,b)}{4}+\frac{\boldsymbol{G}(a,b)}{4}\right],\end{aligned}$$

$$\begin{aligned}\left[\frac{7\boldsymbol{C}(a,b)}{16}+\frac{9\boldsymbol{H}(a,b)}{16}\right]^{\alpha_2}\left[\frac{3\boldsymbol{A}(a,b)}{4}+\frac{\boldsymbol{G}(a,b)}{4}\right]^{1-\alpha_2}&<\boldsymbol{E}(a,b)\\ <\left[\frac{7\boldsymbol{C}(a,b)}{16}+\frac{9\boldsymbol{H}(a,b)}{16}\right]^{\beta_2}&\left[\frac{3\boldsymbol{A}(a,b)}{4}+\frac{\boldsymbol{G}(a,b)}{4}\right]^{1-\beta_2}\end{aligned}$$

hold for all $a,b>0$ with $a\neq b$ if and only if $\alpha_1\leqslant 3/16$, $\beta_1\geqslant 64/\pi^2-6=0.4845\cdots$, $\alpha_2\leqslant 3/16$ and $\beta_2\geqslant\log[32/(3\pi^2)]/\log(7/6)=0.5038\cdots$.

The goal of this article is to establish the chain of inequalities

$$\begin{aligned}\boldsymbol{H}(a,b)<\boldsymbol{T}_{-1}(a,b)<\boldsymbol{G}(a,b)<\boldsymbol{T}_1(a,b)<\boldsymbol{A}(a,b)&\\ <\boldsymbol{T}_2(a,b)<\boldsymbol{Q}(a,b)<\boldsymbol{T}_3(a,b)&<\boldsymbol{T}_4(a,b)<\boldsymbol{C}(a,b)\end{aligned}$$

for all $a,b>0$ with $a\neq b$. Motived by this chain, it makes sense to ask what are sharp linear bounds for $\boldsymbol{T}_{-1}(a,b)$ and $\boldsymbol{T}_4(a,b)$ in terms of classical bivariate means $\boldsymbol{H},\boldsymbol{G},\boldsymbol{A}$, $\boldsymbol{Q}$ and $\boldsymbol{C}$. As applications, new bounds for complete elliptic integral of the second kind $\mathscr{E}$ are given.

2. The chain of inequalities for $\boldsymbol{T}_n(a,b)$

In order to prove our main results, we need several notations and technical lemmas which we present in this section.

The following formulas for the complete elliptic integrals $\mathscr{K}$ and $\mathscr{E}$ can be found in the literature [9],

$$\frac{d\mathscr{K}}{dr}=\frac{\mathscr{E}-r'^2\mathscr{K}}{rr'^2},\quad \frac{d\mathscr{E}}{dr}=\frac{\mathscr{E}-\mathscr{K}}{r},$$

and *Landen identities*

$$\mathscr{K}\left(\frac{2\sqrt{r}}{1+r}\right)=(1+r)\mathscr{K},\quad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2\mathscr{E}-r'^2\mathscr{K}}{1+r}. \tag{2.9}$$

A special value for $\mathscr{E}$ will be used later, which is given by

$$\mathscr{E}\left(\sqrt{2}/2\right)=\frac{4\Gamma(3/4)^2+\Gamma(1/4)^2}{8\sqrt{\pi}}=1.35064388\cdots,$$

where $\Gamma(x)=\int_0^x t^{x-1}e^{-t}dt\ \ (x>0)$ is the classical Euler gamma function [54].

LEMMA 2.1. *The function*

(i) $r\mapsto r'^c\mathscr{K}$ *is strictly decreasing from* $(0,1)$ *onto* $(0,\pi/2)$ *if* $c\in[1/2,\infty)$*;*

(ii) $r\mapsto\left(\mathscr{E}-r'^2\mathscr{K}\right)/\left(r^2\mathscr{K}\right)$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1/2)$*;*

(iii) $r\mapsto 2\mathscr{E}-r'^2\mathscr{K}$ *is strictly increasing from* $(0,1)$ *onto* $(\pi/2,2)$*;*

(iv) $r\mapsto\left[2(2\mathscr{E}-r'^2\mathscr{K})/\pi-1-r^2/4\right]/r^4$ *is strictly increasing from* $(0,1)$ *onto* $(1/64,4/\pi-5/4)$*.*

Proof. Parts (i), (ii) and (iii) can be found in the literature [9, Theorem 3.21(7), Exercises 3.43(13) and (46)] and part (iv) can be found in [55, Lemma 2.4]. □

The following lemma is derived from a consequence of (1.4) and (1.8).

LEMMA 2.2. *Let* $p\in\mathbb{R}$ *and* $n\in\mathbb{Z}^*$*. Then the following statements are true:*

(i) If $2p\log(\pi/2)/\log 2\leqslant n<0$ *or* $n\geqslant\max\{4p/3,0\}$*, then* $\boldsymbol{T}_n(a,b)>\boldsymbol{M}_p(a,b)$ *holds for all* $a,b>0$ *with* $a\neq b$*;*

(ii) If $n\leqslant\min\{4p/3,0\}$ *or* $0<n\leqslant 2p\log(\pi/2)/\log 2$*, then* $\boldsymbol{T}_n(a,b)<\boldsymbol{M}_p(a,b)$ *holds for all* $a,b>0$ *with* $a\neq b$*.*

Proof. As shown in (1.4) and (1.5), it can be easily seen that

$$\boldsymbol{T}_n(a,b) = \left[\boldsymbol{T}\left(a^{n/2}, b^{n/2}\right)\right]^{2/n}, \quad \boldsymbol{M}_{pn/2}(a,b) = \left[\boldsymbol{M}_p\left(a^{n/2}, b^{n/2}\right)\right]^{2/n}.$$

Combining this with (1.8), it follows that

$$\boldsymbol{M}_{\frac{3n}{4}}(a,b) < \boldsymbol{T}_n(a,b) < \boldsymbol{M}_{\frac{n\log 2}{2\log(\pi/2)}}(a,b) \tag{2.10}$$

if $n > 0$ and

$$\boldsymbol{M}_{\frac{n\log 2}{2\log(\pi/2)}}(a,b) < \boldsymbol{T}_n(a,b) < \boldsymbol{M}_{\frac{3n}{4}}(a,b) \tag{2.11}$$

if $n < 0$.

It is well-known that $p \mapsto \boldsymbol{M}_p(a,b)$ is strictly increasing for fixed $a,b > 0$ with $a \neq b$. Thus, if $2p\log(\pi/2)/\log 2 \leqslant n < 0$, that is $p \leqslant n\log 2/[2\log(\pi/2)]$ and $n < 0$, then from (2.11) we obtain

$$\boldsymbol{M}_p(a,b) \leqslant \boldsymbol{M}_{\frac{n\log 2}{2\log(\pi/2)}}(a,b) < \boldsymbol{T}_n(a,b).$$

If $n \geqslant \max\{4p/3, 0\}$, namely, $n > 0$ and $p \leqslant 3n/4$, then it follows from (2.10) that

$$\boldsymbol{M}_p(a,b) \leqslant \boldsymbol{M}_{\frac{3n}{4}}(a,b) < \boldsymbol{T}_n(a,b).$$

This completes the proof of part (i) and similar for the proof of part (ii). □

As special cases of $\boldsymbol{M}_p(a,b)$, Lemma 2.2 can derive the following Proposition 2.3.

PROPOSITION 2.3. *The inequality*

(1) $\boldsymbol{T}_n(a,b) < \boldsymbol{H}(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $n \leqslant -2$*;*

(2) $\boldsymbol{T}_n(a,b) < \boldsymbol{G}(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $n \leqslant -1$*;*

(3) $\boldsymbol{T}_n(a,b) < \boldsymbol{A}(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $n \leqslant 1$*;*

(4) $\boldsymbol{T}_n(a,b) < \boldsymbol{Q}(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $n \leqslant 2$*;*

(5) $\boldsymbol{T}_n(a,b) < \boldsymbol{M}_3(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if and only if* $n \leqslant 3$.

PROPOSITION 2.4. *The inequality* $\boldsymbol{T}_n(a,b) < \boldsymbol{C}(a,b)$ *holds for all* $a,b > 0$ *with* $a \neq b$ *if* $n \leqslant 4$. *Moreover, neither* $\boldsymbol{T}_n(a,b) > \boldsymbol{C}(a,b)$ *nor* $\boldsymbol{T}_n(a,b) < \boldsymbol{C}(a,b)$ *for all* $a,b > 0$ *with* $a \neq b$ *if* $n \geqslant 5$.

Proof. Without loss of generality, we may assume $a > b > 0$. A simple calculation

$$\frac{a^3+b^3}{2} - \left(\frac{a^2+b^2}{a+b}\right)^3 = -\frac{(a-b)^4\left(a^2+ab+b^2\right)}{2(a+b)^3} < 0$$

gives $\boldsymbol{M}_3(a,b) < \boldsymbol{C}(a,b)$. According to this and Proposition 2.3(5), the first half of Proposition 2.4 will be proved if we can show $\boldsymbol{T}_4(a,b) < \boldsymbol{C}(a,b)$ for all $a,b>0$ with $a \neq b$.

Let us denote $r_n = \sqrt{1-(b/a)^n} \in (0,1)$ for $n>0$ in what follows. Then from (1.7) we obtain

$$\frac{\boldsymbol{T}_4(a,b)}{\boldsymbol{C}(a,b)} = \left[\frac{2}{\pi}\mathscr{E}(r_4)\right]^{1/2} \frac{1+\sqrt{r_4'}}{1+r_4'}. \tag{2.12}$$

Due to $4/\pi - 5/4 = 0.023239\cdots < 1/40$, Lemma 2.1 (iv) enables us to know

$$\frac{2}{\pi}\left(2\mathscr{E} - r'^2\mathscr{K}\right) < 1 + \frac{r^2}{4} + \frac{r^4}{40} \tag{2.13}$$

for $r \in (0,1)$.

Let $r_4 = 2\sqrt{r}/(1+r)$. Then (2.12) and (2.13) together with Landen identity (2.9) lead to the conclusion that

$$\begin{aligned}
\frac{\boldsymbol{T}_4(a,b)}{\boldsymbol{C}(a,b)} &= \left[\frac{2}{\pi}\left(2\mathscr{E} - r'^2\mathscr{K}\right)\left(\frac{\sqrt{1+r}+\sqrt{1-r}}{2}\right)^2\right]^{1/2} \\
&< \left[\left(1 + \frac{1-r'^2}{4} + \frac{\left(1-r'^2\right)^2}{40}\right) \cdot \frac{1+r'}{2}\right]^{1/2} \\
&= \left\{1 - \frac{(1-r')^2}{80}\left[4(r'+7) + (1-r')(1+4r'+r'^2)\right]\right\}^{1/2} < 1
\end{aligned}$$

for $r \in (0,1)$, which completes the proof of $\boldsymbol{T}_n(a,b) < \boldsymbol{C}(a,b)$ for $n \leqslant 4$.

For the second half of Proposition 2.4, we first prove that there exist $a>b>0$ such that $\boldsymbol{T}_n(a,b) < \boldsymbol{C}(a,b)$ for each $n \geqslant 5$. For simplicity we denote $\sigma = b/a \in (0,1)$. As we know, the function $r \to \mathscr{E}(r)$ is strictly decreasing from $(0,1)$ onto $(1,\pi/2)$. This in conjunction with (1.7) yields

$$\frac{\boldsymbol{T}_n(a,b)}{\boldsymbol{C}(a,b)} = \left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-\sigma^n}\right)\right]^{2/n} \frac{1+\sigma}{1+\sigma^2} \leqslant \left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-\sigma^5}\right)\right]^{2/n} \frac{1+\sigma}{1+\sigma^2} \tag{2.14}$$

for $n \geqslant 5$.

Note that

$$\lim_{\sigma \to 0^+}\left\{\left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{1-\sigma^5}\right)\right]^{2/n} \frac{1+\sigma}{1+\sigma^2}\right\} = \left(\frac{2}{\pi}\right)^{2/n} < 1 \tag{2.15}$$

for each $n \geqslant 5$.

Equation (2.14) and inequality (2.15) imply that there exists small enough $\varepsilon_n \in (0,1)$ such that $\boldsymbol{T}_n(a,b) < \boldsymbol{C}(a,b)$ for $b/a \in (0,\varepsilon_n)$ with each $n \geqslant 5$.

Next, we prove that there exist $a>b>0$ such that $\boldsymbol{T}_n(a,b)>\boldsymbol{C}(a,b)$ for each $n\geqslant 5$. As in (2.12), for $n\geqslant 5$ we obtain

$$\frac{\boldsymbol{T}_n(a,b)}{\boldsymbol{C}(a,b)}=\left[\frac{2}{\pi}\mathscr{E}(r_n)\right]^{2/n}\frac{1+{r'_n}^{2/n}}{1+{r'_n}^{4/n}}. \tag{2.16}$$

Let $b=a/\sqrt[n]{2}$ for $n\geqslant 5$. Then it is easy to see that $r_n=\sqrt{2}/2$ and so $r'_n=\sqrt{2}/2$. Numerical experiment shows that $\log\left[\frac{2}{\pi}\mathscr{E}\left(\sqrt{2}/2\right)\right]/\log(\sqrt{2}/2)=0.435\cdots<11/25$, which gives

$$\frac{2}{\pi}\mathscr{E}\left(\sqrt{2}/2\right)>\left(\sqrt{2}/2\right)^{11/25}={r'}_n^{11/25}. \tag{2.17}$$

It follows from (2.16) and (2.17) that

$$\frac{\boldsymbol{T}_n(a,b)}{\boldsymbol{C}(a,b)}>\left({r'}_n^{11/25}\right)^{2/n}\frac{1+{r'_n}^{2/n}}{1+{r'_n}^{4/n}}:=\rho({r'}_n^{2/n}), \tag{2.18}$$

where

$$\rho(x)=x^{11/25}\cdot\frac{1+x}{1+x^2}.$$

Differentiating $\rho(x)$ gives

$$\rho'(x)=\frac{\hat{\rho}(x)}{25x^{14/25}(1+x^2)^2}, \tag{2.19}$$

where $\hat{\rho}(x)=11+36x-39x^2-14x^3$.

Simple calculations lead to

$$\hat{\rho}(1/2)=35/2,\quad \hat{\rho}(1)=-6, \tag{2.20}$$

$$\hat{\rho}'(x)=-\frac{3}{2}\left[9+(2x-1)(33+14x)\right]<0. \tag{2.21}$$

Equations (2.19)–(2.21) lead to the conclusion that there exists $x_0\in(1/2,1)$ such that $\rho(x)$ is strictly increasing on $(1/2,x_0)$ and strictly decreasing on $(x_0,1)$. According to this and $1/\sqrt[5]{2}=0.87\cdots\in(1/2,1)$, it follows that

$$\rho(x)\geqslant\min\{\rho(1/\sqrt[5]{2}),\rho(1)\}=1 \tag{2.22}$$

for $x\in[1/\sqrt[5]{2},1)$.

Therefore, $\boldsymbol{T}_n(a,b)>\boldsymbol{C}(a,b)$ for each $n\geqslant 5$ and $a,b>0$ with $b=a/\sqrt[n]{2}$ follows from (2.18) and (2.22) together with ${r'}_n^{2/n}\in[1/\sqrt[5]{2},1)$. □

We are now in a position to state our first Theorem.

THEOREM 2.5. *The chain of inequalities*

$$\begin{aligned}\boldsymbol{H}(a,b)&<\boldsymbol{T}_{-1}(a,b)<\boldsymbol{G}(a,b)<\boldsymbol{T}_1(a,b)<\boldsymbol{A}(a,b)\\&<\boldsymbol{T}_2(a,b)<\boldsymbol{Q}(a,b)<\boldsymbol{T}_3(a,b)<\boldsymbol{T}_4(a,b)<\boldsymbol{C}(a,b)\end{aligned} \tag{2.23}$$

holds for all $a,b>0$ with $a\neq b$. Moreover, the sequence $\{\boldsymbol{T}_n(a,b)\}$ is strictly increasing for $n\in[-43,43]\cap\mathbb{Z}^*$ and all $a,b>0$ with $a\neq b$.

Proof. The chain (2.23) can be obtained from a consequence of Proposition 2.3 and Proposition 2.4. This makes reasonable to study the monotonicity of the sequence $\{\boldsymbol{T}_n(a,b)\}$.

Let us first consider the sequence $\{\boldsymbol{T}_n(a,b)\}$ for $n \geqslant 1$. Then the monotonicity is valid if we can find a suitable number $p = p(n)$ such that

$$\boldsymbol{T}_n(a,b) < \boldsymbol{M}_p(a,b) < \boldsymbol{T}_{n+1}(a,b) \tag{2.24}$$

for all fixed $a,b > 0$ with $a \neq b$.

Lemma 2.2 enables us to know that the inequality (2.24) holds for all $a,b > 0$ with $a \neq b$ if

$$n \leqslant \frac{2p\log(\pi/2)}{\log 2}, \quad n+1 \geqslant \frac{4p}{3},$$

which is equivalent to

$$\frac{n\log 2}{2\log(\pi/2)} \leqslant p \leqslant \frac{3(n+1)}{4}. \tag{2.25}$$

The inequality (2.25) can hold only if

$$\frac{n\log 2}{2\log(\pi/2)} \leqslant \frac{3(n+1)}{4} \quad \Leftrightarrow \quad n \leqslant \frac{3(\log\pi - \log 2)}{5\log 2 - 3\log\pi} = 42.9448\cdots.$$

For $1 \leqslant n \leqslant 42$, we can take

$$p := p(n) = \frac{1}{2}\left[\frac{3(n+1)}{4} + \frac{n\log 2}{2\log(\pi/2)}\right]$$

to make inequality (2.25) hold and so is (2.24).

Similarly, for $-42 \leqslant n \leqslant -1$, numerical calculations show that

$$n \geqslant -\frac{3(\log\pi - \log 2)}{5\log 2 - 3\log\pi} = -42.9448\cdots \quad \Leftrightarrow \quad \frac{n\log 2}{2\log(\pi/2)} \geqslant \frac{3(n-1)}{4}.$$

This allows us to take

$$p^* := p^*(n) = \frac{1}{2}\left[\frac{n\log 2}{2\log(\pi/2)} + \frac{3(n-1)}{4}\right]$$

such that

$$\frac{3(n-1)}{4} \leqslant p* \leqslant \frac{n\log 2}{2\log(\pi/2)} \quad \Leftrightarrow \quad n-1 \leqslant \frac{4p^*}{3} \text{ and } \frac{2\log(\pi/2)p^*}{\log 2} \leqslant n < 0.$$

According to this and Lemma 2.2, we obtain

$$\boldsymbol{T}_{n-1}(a,b) < \boldsymbol{M}_p(a,b) < \boldsymbol{T}_n(a,b) \tag{2.26}$$

for all $a,b > 0$ with $a \neq b$. This gives the monotonicity of $\boldsymbol{T}_n(a,b)$ for $-42 \leqslant n \leqslant -1$.

Therefore, it can be easily seen from (2.24) and (2.26) together with $\boldsymbol{T}_{-1}(a,b) < \boldsymbol{T}_1(a,b)$ that the sequence $\{\boldsymbol{T}_n(a,b)\}$ is strictly increasing for $n \in [-43,43] \cap \mathbb{Z}^*$. $\square$

As $\boldsymbol{AG}_t(a,b)$ and $\boldsymbol{M}_p(a,b)$ mentioned in the introduction, their monotonicity corresponding to the parameter and Theorem 2.5 allow us to pose the following conjecture.

CONJECTURE 2.6. *The generalized power-type Toader mean sequence* $\{\boldsymbol{T}_n(a,b)\}$ *is strictly increasing for* $n \in \mathbb{Z}^*$ *and all fixed* $a,b>0$ *with* $a \neq b$.

3. Sharp bounds for $\boldsymbol{T}_{-1}(a,b)$ and $\boldsymbol{T}_4(a,b)$

In this section, we will present several sharp bounds for $\boldsymbol{T}_{-1}(a,b)$ and $\boldsymbol{T}_4(a,b)$ by the convex combination of $\boldsymbol{G},\boldsymbol{H},\boldsymbol{A},\boldsymbol{Q},\boldsymbol{C}$ and a homotopy between $\boldsymbol{A}$ and $\boldsymbol{C}$. The proofs rely on the monotonicity of functions related to complete elliptic integrals.

THEOREM 3.1. *The double inequality*

$$\alpha_1\boldsymbol{G}(a,b)+(1-\alpha_1)\boldsymbol{H}(a,b)<\boldsymbol{T}_{-1}(a,b)<\beta_1\boldsymbol{G}(a,b)+(1-\beta_1)\boldsymbol{H}(a,b) \tag{3.27}$$

holds for $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_1\leqslant 0$ *and* $\beta_1\geqslant 1/4$.

Proof. Since $\boldsymbol{H}(a,b)$, $\boldsymbol{G}(a,b)$ and $\boldsymbol{T}_{-1}(a,b)$ are symmetric and homogeneous of degree one, without loss of generality, we assume that $a>b>0$. Let $r=\sqrt{1-b/a}\in(0,1)$. Then it can be easily obtained from (1.7) that

$$\boldsymbol{T}_{-1}(a,b)=b\left[\frac{\pi}{2\mathscr{E}(r)}\right]^2, \quad \boldsymbol{G}(a,b)=\frac{b}{r'}, \quad \boldsymbol{H}(a,b)=\frac{2b}{1+r'^2}. \tag{3.28}$$

Further, substituting $x=\sqrt{1-(b/a)^2}$ into (1.7) and (1.8), we obtain

$$\frac{2}{\pi}\mathscr{E}(x)>\left(\frac{1+x'^{3/2}}{2}\right)^{2/3} \tag{3.29}$$

for $x\in(0,1)$.

It is obvious that the left-side of inequality (3.27) for $\alpha_1=0$ can be derived from Theorem 2.5. Thus, Theorem 3.1 will be proved if we can show the right-side of inequality (3.27) and the optimal constants satisfying the inequality (3.27) are $a_1=0,\beta_1=1/4$.

Due to $\boldsymbol{H}(a,b)<\boldsymbol{G}(a,b)$, it follows that $p\mapsto p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)$ is strictly increasing with respect to $p\in\mathbb{R}$. For $p\geqslant 1/4$, then (3.28) and (3.29) lead to

$$\begin{aligned}
&\boldsymbol{T}_{-1}(a,b)-[p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)]\leqslant \boldsymbol{T}_{-1}(a,b)-\left[\frac{1}{4}\boldsymbol{G}(a,b)+\frac{3}{4}\boldsymbol{H}(a,b)\right]\\
&=b\left[\left(\frac{\pi}{2\mathscr{E}(r)}\right)^2-\frac{1}{4r'}-\frac{3}{2\left(1+r'^2\right)}\right]<b\left[\left(\frac{2}{1+r'^{3/2}}\right)^{4/3}-\frac{1+6r'+r'^2}{4r'\left(1+r'^2\right)}\right]\\
&:=bf(\sqrt{r'}),
\end{aligned} \tag{3.30}$$

where

$$f(u)=\left(\frac{2}{1+u^3}\right)^{4/3}-\frac{1+6u^2+u^4}{4u^2\left(1+u^4\right)}.$$

In order to determine the sign of $f(u)$, it suffices to verify that

$$\begin{aligned}
&2^4\left[4u^2\left(1+u^4\right)\right]^3-\left(1+u^3\right)^4\left(1+6u^2+u^4\right)^3\\
&=-(1-u)^4\left(1+4u+28u^2+96u^3+342u^4+972u^5+1426u^6+1588u^7\right.\\
&\quad+1561u^8+2460u^9+3012u^{10}+2460u^{11}+1561u^{12}+1588u^{13}+1426u^{14}\\
&\quad\left.+972u^{15}+342u^{16}+96u^{17}+28u^{18}+4u^{19}+u^{20}\right)<0.
\end{aligned}$$

Therefore, $\boldsymbol{T}_{-1}(a,b)<p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)$ for $p\geqslant 1/4$ and all $a,b>0$ with $a\neq b$ follows from (3.30) and $f(u)<0$.

It remains to prove $a_1=0$ and $\beta_1=1/4$ are the best possible constants.

Let $0<p<1/4$. Then from (3.28) we clearly see that

$$\boldsymbol{T}_{-1}(a,b)-[p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)]=b\left[\left(\frac{\pi}{2\mathscr{E}(r)}\right)^2-\frac{p}{r'}-\frac{2(1-p)}{1+r'^2}\right].\quad(3.31)$$

As $r\to 0^+$, by using of Taylor expansion and (3.31), we obtain

$$\boldsymbol{T}_{-1}(a,b)-[p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)]=\frac{1}{8}\left(\frac{1}{4}-p\right)r^4+o\left(r^4\right),$$

which in conjunction with $p<1/4$ implies that there exists small enough $\delta_1\in(0,1)$ such that $\boldsymbol{T}_{-1}(a,b)<p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)$ for all $a,b>0$ with $(1-\delta_1)a<b<a$.

On the other hand, if $p>0$, then it is easy to see that

$$\lim_{r\to 1^-}\left[\left(\frac{\pi}{2\mathscr{E}(r)}\right)^2-\frac{p}{r'}-\frac{2(1-p)}{1+r'^2}\right]=-\infty.$$

Combining this with (3.31) leads to the conclusion that there exists small enough $\delta_2\in(0,1)$ such that $\boldsymbol{T}_{-1}(a,b)>p\boldsymbol{G}(a,b)+(1-p)\boldsymbol{H}(a,b)$ for all $a,b>0$ with $0<b<\delta_2 a$. $\square$

THEOREM 3.2. *The double inequality*

$$\alpha_2\boldsymbol{C}(a,b)+(1-\alpha_2)\boldsymbol{A}(a,b)<\boldsymbol{T}_4(a,b)<\beta_2\boldsymbol{C}(a,b)+(1-\beta_2)\boldsymbol{A}(a,b)$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\alpha_2\leqslant\sqrt{8/\pi}-1=0.5957\cdots$ *and* $\beta_2\geqslant 1$.

Proof. Without loss of generality, we may assume that $a=1$ and $b=\sqrt{\frac{1-r}{1+r}}$ with $r\in(0,1)$. Then it can be easily seen from (1.7) and Landen identity (2.9) that

$$\boldsymbol{T}_4(a,b)=\left[\frac{2}{\pi}\mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)\right]^{1/2}=\sqrt{\frac{2}{\pi}}\left(\frac{2\mathscr{E}-r'^2\mathscr{K}}{1+r}\right)^{1/2}\quad(3.32)$$

and

$$\boldsymbol{A}(a,b)=\frac{1+r+r'}{2(1+r)},\qquad \boldsymbol{C}(a,b)=\frac{2}{1+r+r'}.\quad(3.33)$$

Let $p \in \mathbb{R}$. Then from (3.32) and (3.33) we obtain

$$\begin{aligned}
&\frac{\boldsymbol{T}_4^2(a,b)-[p\boldsymbol{C}(a,b)+(1-p)\boldsymbol{A}(a,b)]^2}{\boldsymbol{A}^2(a,b)}\\
&=\left[\frac{\boldsymbol{T}_4(a,b)}{\boldsymbol{A}(a,b)}\right]^2-\left[p\frac{\boldsymbol{C}(a,b)}{\boldsymbol{A}(a,b)}+(1-p)\right]^2\\
&=\frac{(4/\pi)(2\mathscr{E}-r'^2\mathscr{K})}{1+r'}-\left(\frac{2p}{1+r'}+1-p\right)^2:=\frac{g_p(r)}{(1+r')^2},
\end{aligned} \tag{3.34}$$

where

$$g_p(r)=\frac{4}{\pi}\left(1+r'\right)\left(2\mathscr{E}-r'^2\mathscr{K}\right)-(1-p)^2r'^2-2(1-p^2)r'-(1+p)^2.$$

Elaborated computations lead to

$$g_p(0^+)=0, \qquad g_p(1^-)=\frac{8}{\pi}-(1+p)^2, \tag{3.35}$$

$$g_p'(r)=\frac{4r}{\pi r'}\hat{g}_p(r), \tag{3.36}$$

where

$$\hat{g}_p(r)=-\left(2\mathscr{E}-r'^2\mathscr{K}\right)+\frac{r'\left(1+r'\right)\left(\mathscr{E}-r'^2\mathscr{K}\right)}{r^2}+\frac{\pi}{2}(1-p)^2(1+r')+\pi p(1-p).$$

Moreover,

$$\hat{g}_p(0^+)=\pi(1-p), \quad \hat{g}_p(1^-)=\frac{\pi}{2}\left(1-p^2\right)-2. \tag{3.37}$$

We divide the proof into three cases.

Case 2.1 $p=\sqrt{8/\pi}-1$. Then from (3.35) and (3.37) we obtain

$$g_p(0^+)=g_p(1^-)=0, \tag{3.38}$$

$$\hat{g}_p(0^+)=2(\pi-\sqrt{2\pi})=1.269\cdots \quad \hat{g}_p(1^-)=2(\sqrt{2\pi}-3)=-0.986\cdots. \tag{3.39}$$

Lemma 2.1 (i), (ii) and (iii) enable us to know that the functions $r\mapsto-(2\mathscr{E}-r'^2\mathscr{K})$ and $r\mapsto\left[r'(1+r')(\mathscr{E}-r'^2\mathscr{K})\right]/r^2$ are strictly decreasing on $(0,1)$, which yields $\hat{g}_p(r)$ is strictly decreasing on $(0,1)$. According to this with (3.39) and (3.36), it follows that there exists $r_1\in(0,1)$ such that $g_p(r)$ is strictly increasing on $(0,r_1)$ and strictly decreasing on $(r_1,1)$. We conclude that $g_p(r)>0$ for $r\in(0,1)$ from (3.38) and the piecewise monotonicity of $g_p(r)$.

Therefore, the inequality $\boldsymbol{T}_4(a,b)>\left(\sqrt{8/\pi}-1\right)\boldsymbol{C}(a,b)+\left(2-\sqrt{8/\pi}\right)\boldsymbol{A}(a,b)$ for all $a,b>0$ with $a\neq b$ follows easily from (3.34) and $g_p(r)>0$.

Case 2.2 $p=1$. Then the inequality $\boldsymbol{T}_4(a.b)<\boldsymbol{C}(a,b)$ for all $a,b>0$ with $a\neq b$ follows directly from Proposition 2.4.

Case 2.3 $\sqrt{8/\pi}-1<p<1$. Then from (3.35) and (3.37) we clearly see that

$$g_p(1^-)<0 \tag{3.40}$$

and

$$\hat{g}_p(0^+) > 0. \tag{3.41}$$

Equations (3.34)–(3.36) and inequalities (3.40) and (3.41) lead to the conclusion that there exist small enough $\delta_3, \delta_4 \in (0,1)$ such that $\boldsymbol{T}_4(a,b) < p\boldsymbol{C}(a,b) + (1-p)\boldsymbol{A}(a,b)$ for all $a,b > 0$ with $0 < b < \delta_3 a$, and $\boldsymbol{T}_4(a,b) > p\boldsymbol{C}(a,b) + (1-p)\boldsymbol{A}(a,b)$ for all $a,b > 0$ with $(1-\delta_4)a < b < a$.

Therefore, Theorem 3.2 follows from Cases 2.1–2.3 and the monotonicity of $p \mapsto p\boldsymbol{C}(a,b) + (1-p)\boldsymbol{A}(a,b)$. □

THEOREM 3.3. *The double inequality*

$$\alpha_3\boldsymbol{C}(a,b) + (1-\alpha_3)\boldsymbol{Q}(a,b) < \boldsymbol{T}_4(a,b) < \beta_3\boldsymbol{C}(a,b) + (1-\beta_3)\boldsymbol{Q}(a,b)$$

holds for all $a,b > 0$ *with* $a \neq b$ *if and only if* $\alpha_3 \leqslant (\sqrt{2}+1)(2/\sqrt{\pi}-1) = 0.3099\cdots$ *and* $\beta_2 \geqslant 1$.

Proof. Without loss of generality, we may assume that $a = 1$ and $b = \sqrt{\frac{1-r}{1+r}}$ with $r \in (0,1)$. Let $p \in \mathbb{R}$. Then from (3.32) and (3.33) we obtain

$$\left[\frac{\boldsymbol{T}_4(a,b)}{\boldsymbol{A}(a,b)}\right]^2 - \left[p\frac{\boldsymbol{C}(a,b)}{\boldsymbol{A}(a,b)} + (1-p)\frac{\boldsymbol{Q}(a,b)}{\boldsymbol{A}(a,b)}\right]^2$$

$$= \frac{(4/\pi)(2\mathscr{E} - r'^2\mathscr{K})}{1+r'} - \left[\frac{2p}{1+r'} + (1-p)\sqrt{\frac{2}{1+r'}}\right]^2 := \frac{h_p(r)}{1+r'}, \tag{3.42}$$

where

$$h_p(r) = \frac{4}{\pi}(2\mathscr{E} - r'^2\mathscr{K}) - \frac{4p^2}{1+r'} - \frac{4\sqrt{2}p(1-p)}{\sqrt{1+r'}} - 2(1-p)^2.$$

Simple calculations lead to

$$h_p(0^+) = 0, \quad h_p(1^-) = \frac{8}{\pi} - 2\left[\left(\sqrt{2}-1\right)p+1\right]^2, \tag{3.43}$$

$$h_p'(r) = \frac{r}{r'}\hat{h}_p(r), \tag{3.44}$$

where

$$\hat{h}_p(r) = \frac{4}{\pi}\frac{r'(\mathscr{E} - r'^2\mathscr{K})}{r^2} - \frac{4p^2}{(1+r')^2} - \frac{2\sqrt{2}p(1-p)}{(1+r')^{3/2}}.$$

Moreover,

$$\hat{h}_p(0^+) = 1-p, \quad \hat{h}_p(1^-) = -2\sqrt{2}p[(\sqrt{2}-1)p+1]. \tag{3.45}$$

We divide the proof into three cases.

Case 3.1 $p = (\sqrt{2}+1)(2/\sqrt{\pi}-1)$. Then it can be easily seen from (3.43) and (3.45) that

$$h_p(0^+) = h_p(1^-) = 0, \tag{3.46}$$

$$\hat{h}_p(0^+) = 0.69006\cdots > 0, \quad \hat{h}_p(1^-) = -0.9891\cdots < 0. \tag{3.47}$$

Lemma 2.1 (i) and (ii) show that the function $r \mapsto \left[r'(\mathscr{E} - r'^2\mathscr{K})\right]/r^2$ is strictly decreasing on $(0,1)$, which yields $\hat{h}(r)$ is strictly decreasing on $(0,1)$. Combining this with (3.44) and (3.47), we clearly see that there exists $r_2 \in (0,1)$ such that $h_p(r)$ is strictly increasing on $(0,r_2)$ and strictly decreasing on $(r_2,1)$.

Therefore, the inequality

$$\boldsymbol{T}_4(a,b) > \left(\sqrt{2}+1\right)(2/\sqrt{\pi}-1)\boldsymbol{C}(a,b) + \left[1-(\sqrt{2}+1)(2/\sqrt{\pi}-1)\right]\boldsymbol{Q}(a,b)$$

for all $a,b>0$ with $a \neq b$ follows easily from (3.42) and (3.46) together with the piecewise monotonicity of $h_p(r)$.

Case 3.2 $p=1$. In this case, Theorem 3.3 is valid from Proposition 2.4.

Case 3.3 $(\sqrt{2}+1)(2/\sqrt{\pi}-1) < p < 1$. Then from (3.43) and (3.45) we obtain

$$h_p(1^-) < 0, \quad \hat{h}_p(0^+) > 0. \tag{3.48}$$

Equation (3.42)–(3.44) and inequality (3.48) imply that there exist small enough $\delta_5, \delta_6 \in (0,1)$ such that $\boldsymbol{T}_4(a,b) < p\boldsymbol{C}(a,b) + (1-p)\boldsymbol{Q}(a,b)$ for all $a,b>0$ with $0<b<\delta_5 a$, and $\boldsymbol{T}_4(a,b) > p\boldsymbol{C}(a,b)+(1-p)\boldsymbol{Q}(a,b)$ for all $a,b>0$ with $(1-\delta_6)a < b < a$.

Therefore, Theorem 3.3 follows from Cases 3.1–3.3 and the monotonicity of $p \mapsto p\boldsymbol{C}(a,b)+(1-p)\boldsymbol{Q}(a,b)$. □

Let $p \in [1/2,1]$ and

$$\boldsymbol{C}_p(a,b) = \boldsymbol{C}[pa+(1-p)b, pb+(1-p)a] \tag{3.49}$$

be the one-parameter contra-harmonic mean of a and b, which is also a homotopy between $\boldsymbol{A}(a,b)$ and $\boldsymbol{C}(a,b)$. It can be easily verified that $\boldsymbol{C}_p(a,b)$ is strictly increasing with respect to $p \in [1/2,1]$ for $a,b>0$ with $a \neq b$ and so

$$\boldsymbol{C}_{1/2}(a,b) = \boldsymbol{A}(a,b) < \boldsymbol{T}_4(a,b) < \boldsymbol{C}(a,b) = \boldsymbol{C}_1(a,b). \tag{3.50}$$

Inspired by (3.50), it makes sense to ask what are the optimal parameters $\alpha_4, \beta_4 \in [1/2,1]$ such that $\boldsymbol{C}_{\alpha_4}(a,b) < \boldsymbol{T}_4(a,b) < \boldsymbol{C}_{\beta_4}(a,b)$ for all $a,b>0$ with $a \neq b$. This question will be answered in Theorem 3.4.

THEOREM 3.4. *Let* $\alpha_4, \beta_4 \in [1/2,1]$. *Then the double inequality*

$$\boldsymbol{C}_{\alpha_4}(a,b) < \boldsymbol{T}_4(a,b) < \boldsymbol{C}_{\beta_4}(a,b)$$

holds for all $a,b>0$ *with* $a \neq b$ *if and only if* $\alpha_4 \leqslant \left(1+\sqrt{\sqrt{8/\pi}-1}\right)/2 = 0.8859\cdots$ *and* $\beta_4 = 1$.

Proof. Let $p \in [1/2,1]$, $r \in (0,1)$, $a=1$ and $b=\sqrt{(1-r)/(1+r)} \in (0,1)$. Then from (1.7) and (3.49) we obtain

$$\begin{aligned}
&\left[\frac{T_4(a,b)}{A(a,b)}\right]^2 - \left[\frac{C_p(a,b)}{A(a,b)}\right]^2 \\
&= \frac{(4/\pi)(2\mathscr{E} - r'^2\mathscr{K})}{1+r'} - \left[\frac{2\left(1-2p+2p^2+2p(1-p)r'\right)}{1+r'}\right]^2 \\
&:= \frac{g_{(2p-1)^2}(r)}{(1+r')^2},
\end{aligned}$$

where $g_p(r)$ is defined as in Theorem 3.2.

Therefore, Theorem 3.4 derives immediately from Theorem 3.2. □

The following corollary can be derived from Theorems 3.1–3.3.

COROLLARY 3.5. *Let* $\tau=(\sqrt{2}+1)(2/\sqrt{\pi}-1)=0.3099\cdots$ *and* $r'=\sqrt{1-r^2}$. *Then the inequality*

$$\max_{r\in(0,1)}\left\{\sqrt{\frac{r'(1+r'^2)}{1+6r'+r'^2}},\ \left[\tau\frac{1+r'}{1+\sqrt{r'}}+(1-\tau)\sqrt{\frac{1+r'}{2}}\right]^2\right\}<\frac{2}{\pi}\mathscr{E}(r)$$
$$<\min_{r\in(0,1)}\left\{\sqrt{\frac{1+r'^2}{2}},\ \left(\frac{1+r'}{1+\sqrt{r'}}\right)^2\right\}$$

holds for all $r\in(0,1)$.

Proof. It follows from Theorems 3.2 and 3.3 that

$$\frac{2}{\pi}\mathscr{E}(r)>\left[\frac{2\sqrt{r'}+\sqrt{2/\pi}(1-\sqrt{r'})^2}{1+\sqrt{r'}}\right]^2,\quad \frac{2}{\pi}\mathscr{E}(r)>\left[\tau\frac{1+r'}{1+\sqrt{r'}}+(1-\tau)\sqrt{\frac{1+r'}{2}}\right]^2.$$

Thus, Corollary 3.5 will be proved if we can show

$$\tau\frac{1+u^2}{1+u}+(1-\tau)\sqrt{\frac{1+u^2}{2}}\geqslant\frac{2u+\sqrt{2/\pi}(1-u)^2}{1+u}\tag{3.51}$$

for $u\in(0,1)$, where we denote $u=\sqrt{r'}$.

In order to prove (3.51), it suffices to verify

$$\tau\frac{1+u^2}{1+u}+(1-\tau)\sqrt{\frac{1+u^2}{2}}-\frac{2u+\sqrt{2/\pi}(1-u)^2}{1+u}$$
$$=\frac{\sqrt{2}(1-\tau)(1+u)\sqrt{1+u^2}-2\left[(\sqrt{2/\pi}-\tau)(1-u)^2+2(1-\tau)u\right]}{2(1+u)}\geqslant 0,$$

which is equivalent to

$$\left[\sqrt{2}(1-\tau)(1+u)\sqrt{1+u^2}\right]^2-4\left[(\sqrt{2/\pi}-\tau)(1-u)^2+2(1-\tau)u\right]^2$$
$$=\frac{8(\pi-2)^2u(1-u)^2}{\pi(2+\pi+2\sqrt{2\pi})}>0$$

for $u\in(0,1)$. □

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REFERENCES

[1] G. TOADER, *Some mean values related to the arithmetic-geometric mean*, J. Math. Anal. Appl., 1998, **218** (2), 358–368.

[2] H. HARUKI, *New characterizations of the arithmetic-geometric mean of Gauss and other well-known mean values*, Publ. Math. Debrecen., 1991, **38**, 323–332.

[3] H. HARUKI AND T. M. RASSIAS, *New characterizations of some mean-values*, J. Math. Anal. Appl., 1996, **202** (1), 333–348.

[4] T. ONO, *A generalization of Gauss' theorem on arithmetic-geometric means*, Proc. Japan Acad. Ser. A Math. Sci., 1983, **59** (4), 154–157.

[5] J.-L. WANG, W.-M. QIAN, Z.-Y. HE AND Y.-M. CHU, *On approximating the Toader mean by other bivariate means*, J. Funct. Spaces, 2019, **2019**, Article ID 6082413, 7 pages.

[6] W.-M. QIAN, X.-H. ZHANG AND Y.-M. CHU, *Sharp bounds for the Toader-Qi mean in terms of harmonic and geometric means*, J. Math. Inequal., 2017, **11** (1), 121–127.

[7] M.-L. MENG, *Inequalities for a class of new arithmetic means*, Thesis (B. S.), Huzhou University, 2017 (in Chinese).

[8] M. K. VAMANAMURTHY AND M. VUORINEN, *Inequalities for means*, J. Math. Anal. Appl., 1994, **183** (1), 155–166.

[9] G. D. ANDERSON, M. K. VAMANAMURTHY AND M. VUORINEN, *Conformal Invariants, Inequalities, and Quasiconformal Maps*, John Wiley & Sons, New York, 1997.

[10] M. ABRAMOWITZ AND I. A. STEGUN, *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables*, U.S. Government Printing Office, Washington, 1964.

[11] T.-H. ZHAO, Z.-Y. HE AND Y.-M. CHU, *On some refinements for inequalities involving zero-balanced hypergeometric function*, AIMS Math., 2020, **5** (6), 6479–6495.

[12] T.-H. ZHAO, M.-K. WANG, AND Y.-M. CHU, *A sharp double inequality involving generalized complete elliptic integral of the first kind*, AIMS Math., 2020, **5**(5), 4512–4528.

[13] T.-H. ZHAO, L. SHI AND Y.-M. CHU, *Convexity and concavity of the modified Bessel functions of the first kind with respect to Hölder means*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2020, **114** (2), Article 96, 14 pages.

[14] T.-H. ZHAO, M.-K. WANG, W. ZHANG AND Y.-M. CHU, *Quadratic transformation inequalities for Gaussian hypergeometric function*, J. Inequal. Appl., 2018, **2018**, Article 251, 15 pages.

[15] Y.-M. CHU AND T.-H. ZHAO, *Concavity of the error function with respect to Hölder means*, Math. Inequal. Appl., 2016, **19** (2), 589–595.

[16] S.-B. CHEN, S. RASHID, M. A. NOOR, R. ASHRAF AND Y.-M. CHU, *A new approach on fractional calculus and probability density function*, AIMS Math., 2020, **5** (6), 7041–7054.

[17] S.-S. ZHOU, S. RASHID, M. A. NOOR, K. I. NOOR, F. SAFDAR AND Y.-M. CHU, *New Hermite-Hadamard type inequalities for exponentially convex functions and applications*, AIMS Math., 2020, **5** (6), 6874–6901.

[18] S.-B. CHEN, H. JAHANSHAHI, Q. ALHADJI ABBA, J. E. SOLÍS-PÉREZ, S. BEKIROS, J. F. GÓMEZ-AGUILAR, A. YOUSEFPOUR AND Y.-M. CHU, *The effect of market confidence on a financial system from the perspective of fractional calculus: numerical investigation and circuit realization*, Chaos Solitons Fractals, 2020, **140**, Article ID 110223, 15 pages.

[19] R. SAIMA, S. SOBIA, K. YELIZ, K. AASMA AND Y.-M. CHU, *Some further extensions considering discrete proportional fractional operators*, Fractals, 2022, **30** (1), Article ID 2240026, 12 pages, `https://dx.doi.org/10.1142/S0218348X22400266`.

[20] M.-K. WANG, H.-H. CHU AND Y.-M. CHU, *On the approximation of some special functions in Ramanujan's generalized modular equation with signature* 3, Ramanujan J., 2021, **56**, 1–22, `https://doi.org/10.1007/s11139-021-00437-4`.

[21] S.-B. CHEN, S. RASHID, Z. HAMMOUCH, M. A. NOOR, R. ASHRAF AND Y.-M. CHU, *Integral inequalities via Raina's fractional integrals operator with respect to a monotone function*, Adv. Difference Equ., 2020, **2020**, Article 647, 20 pages.

[22] S.-B. CHEN, S. RASHID, M. A. NOOR, Z. HAMMOUCH AND Y.-M. CHU, *New fractional approaches for n-polynomial P-convexity with applications in special function theory*, Adv. Difference Equ., 2020, **2020**, Article 543, 31 pages.

[23] M. A. ALI, N. ALP, H. BUDAK, Y.-M. CHU AND Z.-Y. ZHANG, *On some new quantum midpoint-type inequalities for twice quantum differentiable convex functions,* Open Math., 2021, **19** (1), 427–439.

[24] Y.-M. CHU, S. RASHID, T. ABDELJAWAD, A. KHALID, H. KALSOOM, *On new generalized unified bounds via generalized exponentially harmonically s-convex functions on fractal sets*, Adv. Difference Equ., 2021, **2021**, Article 218, 33 pages.

[25] R. HAMEED, G. MUSTAFA, D. BALEANU AND Y.-M. CHU, *A divided differences based medium to analyze smoothness of the binary bivariate refinement schemes*, Adv. Difference Equ., 2021, **2021**, Article 180, 31 page.

[26] Y.-M. CHU, N. A. SHAH, P. AGARWAL AND D. JAE, *Analysis of fractional multi-dimensional Navier-Stokes equation*, Adv. Difference Equ., 2021, **2021**, Article 91, 18 pages.

[27] W.-M. QIAN, M.-K. WANG, H.-Z. XU AND Y.-M. CHU, *Approximations for the complete elliptic integral of the second Kind,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 2021, **115** (2), Article 88, 11 pages.

[28] W.-M. QIAN, H.-Z. XU, Z.-Y. HE AND Y.-M. CHU, *Bounding the Sándor-Yang means for the combinations of contraharmonic and arithmetic means,* J. Math. Inequal, 2021, **15** (2), 655–666.

[29] M.-Y. HONG, M.-K. WANG AND Y.-M. CHU, *A necessary and sufficient condition for the convexity of the one-parameter generalized inverse trigonometric sine function according to power mean,* J. Math. Inequal., 2021, **15** (2), 559–573.

[30] H.-H. CHU, T.-H. ZHAO, Y.-M. CHU, *Sharp bounds for the Toader mean of order* 3 *in terms of arithmetic, quadratic and contraharmonic means*, Math. Slovaca, 2020, **70** (5), 1097–1112.

[31] M. ADIL KHAN, S. KHAN, I. ULLAH, K. ALI KHAN AND Y.-M. CHU, *A novel approach to the Jensen gap through Taylor's theorem,* Math. Methods Appl. Sci., 2021, **44** (5), 3324–3333.

[32] J.-F. LI, H. JAHANSHAHI, S. LACAR, Y.-M. CHU, J. F. GÓMEZ-AGUILAR, N. D. ALOTAIBI AND K. H. ALHARBI, *On the variable-order fractional memristor oscillator: data security applications and synchronization using a type-2 fuzzy disturbance observer-based robust control,* Chaos Solitons Fractals, 2021, **145**, Article ID 110681, 13 pages.

[33] Y.-X. LI, T. MUHAMMAD, M. BILAL, M. ALTAF KHAN, A. AHMANIDAN AND B. A. PANSERA, *Fractional simulation for Darcy-Forchheimer hybrid nanoliquid flow with partial slip over a spinning disk,* Alex. Eng. J., **60** (2021), 4787–4796.

[34] Y.-X. LI, A. RAUF, M. NAEEM, M. A. BINYAMIN AND A. ASLAM, *Valency-based topological properties of linear hexagonal chain and hammer-like benzenoid,* Complexity, **2021** (2021), Article ID 9939469, 16 pages.

[35] Y.-M. CHU, U. NAZIR, M. SOHAIL, M. M. SELIM AND J.-R. LEE, *Enhancement in thermal energy and solute particles using hybrid nanoparticles by engaging activation energy and chemical reaction over a parabolic surface via finite element approach*, Fractal Fract., 2021, **5** (3), Article 119, 17 pages.

[36] R. W. BARNARD, K. PEARCE, K. C. RICHARDS, *An inequality involving the generalized hypergeometric function and the arc length of an ellipse,* SIAM J. Math. Anal., 2000, **31** (3), 693–699.

[37] Y.-M. CHU, S.-L. QIU, M.-K. WANG, *Sharp inequalities involving the power mean and complete elliptic integral of the first kind,* Rocky Mountain J. Math., 2013, **43** (5), 1489–1496.

[38] T.-H. ZHAO, Z.-Y. HE, Y.-M. CHU, *Sharp bounds for the weighted Hölder mean of the zero-balanced generalized complete elliptic integrals,* Comput. Methods Funct. Theory, 2021, **21** (3), 413–426.

[39] T.-H. ZHAO, B. A. BHAYO, Y.-M. CHU, *Inequalities for generalized Grötzsch ring function,* Comput. Methods Funct. Theory, 2021, `https://doi.org/10.1007/s40315-021-00415-3`.

[40] T.-H. ZHAO, M.-K. WANG, Y.-M. CHU, *Concavity and bounds involving generalized elliptic integral of the first kind*, J. Math. Inequal., 2021, **15** (2), 701–724.

[41] T.-H. ZHAO, M.-K. WANG, Y.-M. CHU, *Monotonicity and convexity involving generalized elliptic integral of the first kind,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 2021, **115** (2): 46.

[42] T.-H. ZHAO, Z.-H. SHEN, Y.-M. CHU, *Sharp power mean bounds for the lemniscate type means,* Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat., 2021, **115** (4): 175.

[43] Y.-M. CHU, M.-K. WANG, *Optimal Lehmer mean bounds for the Toader mean,* Results Math., 2012, **61** (3–4), 223–229.

[44] Y.-M. CHU, M.-K. WANG, S.-L. QIU, *Optimal combinations bounds of root-square and arithmetic means for Toader mean,* Proc. Indian Acad. Sci. Math. Sci., 2012, **122** (1), 41–51.

[45] T.-H. ZHAO, W.-M. QIAN, Y.-M. CHU, *On approximating the arc lemniscate functions*, Indian J. Pure Appl. Math., 2021, https://doi.org/10.1007/s13226-021-00016-9.
[46] M.-K. WANG, Y.-M. CHU, Y.-P. JIANG, S.-L. QIU, *Bounds of the perimeter of an ellipse using arithmetic, geometric and harmonic means*, Math. Inequal. Appl., 2014, **17** (1), 101–111.
[47] B. C. CARLSON AND VUORINEN, *Inequalities of the AGM and the logarithmic mean*, SIAM Review, 1991, **33**, 655–655.
[48] P. BRACKEN, *An arithmetic-geometric mean inequality*, Expo. Math., 2001, **19** (3), 273–279.
[49] J. SÁNDOR, *On certain inequalities for means*, J. Math. Anal. Appl., 1995, **189** (2), 602–606.
[50] J. SÁNDOR, *On certain inequalities for means II*, J. Math. Anal. Appl., 1996, **199** (2), 629–635.
[51] H. ALZER AND S.-L. QIU, *Monotonicity theorems and inequalities for the complete elliptic integrals*, J. Comput. Appl. Math., 2004, **172** (2), 289–312.
[52] W.-M. QIAN, Y.-M. CHU, *Sharp bounds for a special quasi-arithmetic mean in terms of arithmetic and geometric means with two-parameters*, J. Inequal. Appl., 2017, **2017**, Article 274, 10 pages.
[53] T.-H. ZHAO, B.-C. ZHOU, M.-K. WANG AND Y.-M. CHU, *On approximating the quasi-arithmetic mean*, J. Inequal. Appl., 2019, **2019**, Article 42, 12 pages.
[54] Z.-H. YANG, W.-M. QIAN, Y.-M. CHU AND W. ZHANG, *On rational bounds for the gamma function*, J. Inequal. Appl., 2017, **2017**, Article 210, 17 pages.
[55] Q. DING AND T.-H. ZHAO, *Optimal bounds for arithmetic-geometric and Toader means in terms of generalized logarithmic mean*, J. Inequal. Appl., 2017, **2017**, Article 102, 12 pages.

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Tie-Hong Zhao
School of Mathematics
Hangzhou Normal University
Hangzhou 311121, China
e-mail: tiehong.zhao@hznu.edu.cn

Miao-Kun Wang
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: wangmiaokun@zjhu.edu.cn

Ye-Qi Dai
Department of Mathematics
Huzhou University
Huzhou 313000, P. R. China
e-mail: daiyeqi2020@163.com

Yu-Ming Chu
Department of Mathematics
Huzhou University
Huzhou 313000, China
e-mail: chuyuming2005@126.com