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ON AN INEQUALITY FOR 3–CONVEX FUNCTIONS AND RELATED RESULTS

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Abstract. Majorization type theorems (such as the Karamata inequality, the Fuchs inequality) for higher convex functions are rare and the criteria given in these theorems are difficult to check (see [10, Chapter 9.]). On the other side, the Brady theorem (see [3]) gives rather simple and the straightforward criterion for such type of results. We apply Brady's theorems on inequalities originated from 3-exponential convexity of certain function and, as a by-product, we obtain improvements of AG-inequality and an interesting mean. As an equivalent version of Brady's theorem, the mean value theorems, which are usually used in the definition of Stolarsky means, are also proved.

1. Introduction and preliminaries

The initial motivation for this research was to find all functions f on $(0,\infty)$ for which the inequality

$$f(t_1)+f(t_2)+f\left(2\sqrt{t_1t_2}\right) \leqslant f(t_1+t_2)+2f\left(\sqrt{t_1t_2}\right) \tag{1}$$

holds for every $t_1,t_2>0$. Notice that for convex functions f with $f(0)=0$ it holds $f(t_1)+f(t_2)\leqslant f(t_1+t_2)$ and $f(2t)\geqslant 2f(t)$. The geometric mean in (1) has a special role, and this will become clear later, although it can be replaced with any other two variable mean, but this requires different techniques.

The importance of the inequality (1) is evident when written for $f(t)=t^p$. In this case, after rearrangement, we get

$$2^{1-p}\left(\frac{t_1^p+t_2^p}{2}-\sqrt{t_1t_2}^p\right)+\sqrt{t_1t_2}^p \leqslant \left(\frac{t_1+t_2}{2}\right)^p, \tag{2}$$

which is (when it holds) an obvious improvement of AG-inequality for $p>0$, and the reverse inequality is an obvious improvement of the inequality between the geometric mean and the power mean for $p>0$. A significant feature of (1) is that (2) becomes identity for $p=1$ and $p=2$.

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A more direct way to deduce the inequality (1) is contained in the notion of N-exponentially convex functions and their characterizations. We use the terminology from [1] (and [2]); this differs from Widder's book [11], which uses the notion of positive definite kernels. These notions in [1] and [11] are given in the infinite order case. See also [8] for more effective ways to characterize N-exponential convexity.

DEFINITION 1. ([1, p. 210]) Let $N \in \mathbb{N}$. A function $g:(a,b) \to \mathbb{R}$, $(a,b) \subseteq \mathbb{R}$, is said to be N-exponentially convex if

$$\sum_{i,j=1}^{N} \xi_i \xi_j g\left(\frac{x_i+x_j}{2}\right) \geqslant 0$$

for arbitrary $\xi_1,\ldots,\xi_N \in \mathbb{R}$, $x_1,\ldots,x_N \in (a,b)$.

Using [5, Chapter X] it follows that g is N-exponentially convex on (a,b) iff

$$\det\left(g\left(\frac{x_i+x_j}{2}\right)\right)_{i,j=1}^{k} \geqslant 0$$

for every $k=1,\ldots,N$ and for arbitrary $x_1,\ldots,x_N \in (a,b)$.

It is known that the function $g(x)=e^{x^2}$ is N-exponentially convex for all $N \in \mathbb{N}$ as a bilateral Laplace transform

$$e^{t^2} = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-\frac{x^2}{4\pi}} e^{-tx}\,dx$$

(see [11, Chapter XI]). By a straightforward calculation we get that the non-negativity of determinant $\det\left[g\left(\frac{x_i+x_j}{2}\right)\right]_{i,j=1}^{3}$ is equivalent to the inequality

$$\begin{aligned} &e^{2\left(\frac{x_1+x_2}{2}\right)^2+x_3^2}+e^{2\left(\frac{x_1+x_3}{2}\right)^2+x_2^2}+e^{2\left(\frac{x_2+x_3}{2}\right)^2+x_1^2} \\ &\leqslant e^{x_1^2+x_2^2+x_3^2}+2e^{\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{x_1+x_3}{2}\right)^2+\left(\frac{x_2+x_3}{2}\right)^2}. \end{aligned} \tag{3}$$

Setting $x_3=0$ in (3), dividing by $e^{\frac{x_1^2}{2}+\frac{x_2^2}{2}}$ and using the substitution $x_1^2=2t_1$, $x_2^2=2t_2$, the inequality (3) reduces to the inequality

$$e^{t_1}+e^{t_2}+e^{2\sqrt{t_1t_2}} \leqslant e^{t_1+t_2}+2e^{\sqrt{t_1t_2}},$$

which is exactly the inequality (1) written for $f(x)=e^x$. Our primary goal is to give a convex-like arguments for (1) and (3).

In close connection with the above mentioned problems is the identity

$$f(x)+f(y)+f(z)-f(a)-f(b)-f(c)=(xyz-abc)f'''(w) \tag{4}$$

proven in [4], where x,y,z,a,b,c satisfy (5) in the case $k=3$ and w is in the smallest interval containing x,y,z,a,b,c. Obviously, the left-hand side of (4) is closely related to inequalities (1) and (3) and the sign of it is controlled by the sign of f'''. In the final section the general form of (4) is proven.

2. On Brady's theorem

The following theorem is given in [3]. It gives a very elegant majorization type theorem for higher convexities. We add one claim to the original Corollary 4.1 in [3].

THEOREM 2. *Given real numbers* $x_1,x_2,\dots,x_k \in [a,b]$ *and* $y_1,y_2,\dots,y_k \in [a,b]$ *such that*

$$\sum_{i=1}^{k} x_i^j = \sum_{i=1}^{k} y_i^j, \quad j=1,2,\dots,k-1, \tag{5}$$

the following claims are equivalent:

(a) $\sum_{i=1}^k x_i^k \geqslant \sum_{i=1}^k y_i^k$

(b) $\max\{x_i : i=1,\dots,k\} \geqslant \max\{y_i : i=1,\dots,k\}$

(c) $(-1)^{k-1}\prod_{i=1}^k x_i \geqslant (-1)^{k-1}\prod_{i=1}^k y_i$

(d) $\sum_{i=1}^k f(x_i) \geqslant \sum_{i=1}^k f(y_i)$ *for all functions* $f:[a,b]\to\mathbb{R}$ *provided* $f^{(k)} \geqslant 0$.

Proof. The equivalences $(a)\Leftrightarrow(b)\Leftrightarrow(d)$ are proven in [3]. We will prove $(a)\Leftrightarrow(b)\Leftrightarrow(c)$. This will also give a more direct proof of $(a)\Leftrightarrow(b)$ than in [3]. As a by-product we get (see the proof of Corollary 4.2 in [3] and the last section of the present paper)

$$\begin{aligned}\int_a^b \sum_{i=1}^k &\left[(x_i-t)_+^{k-1} - (y_i-t)_+^{k-1}\right] dt \\ &= \frac{1}{k}\left[\sum_{i=1}^k x_i^k - \sum_{i=1}^k y_i^k\right] = (-1)^{k-1}\left[\prod_{i=1}^k x_i - \prod_{i=1}^k y_i\right],\end{aligned}$$

assuming (5). Notice that the left-hand side of the above identity is the Popoviciu spline kernel in the integral representation of $\sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i)$.

W.l.o.g. we can assume that $x_1 \geqslant x_2 \geqslant \cdots \geqslant x_k$, $y_1 \geqslant y_2 \geqslant \cdots \geqslant y_k$. Set $P_k(x) = \prod_{i=1}^k (x-x_i)$, $Q_k(x) = \prod_{i=1}^k (x-y_i)$. Obviously

$$P_k(x) = \sum_{j=0}^k (-1)^j I_j(\mathbf{x}) x^{k-j}, \quad Q_k(x) = \sum_{j=0}^k (-1)^j I_j(\mathbf{y}) x^{k-j},$$

where

$$I_j(\mathbf{x}) = I_j(x_1,\dots,x_k) = \sum_{1\leqslant i_1<i_2<\cdots i_j\leqslant k} x_{i_1}\cdots x_{i_j}, \quad j=0,1,\dots,k, \quad I_0(\mathbf{x})=1,$$

and analogously $I_j(\mathbf{y})$. Using identities [6, (9)] or Lemma 3, we conclude that $I_j(\mathbf{x})$ depends polynomially on $\mathbf{x}^{(l)} = \sum_{i=1}^k x_i^l$, $l=1,\dots,j$. Using identities (5), it follows $I_j(\mathbf{x}) = I_j(\mathbf{y})$ for $j=1,\dots,k-1$. We get

$$P_k(x) - Q_k(x) = (-1)^k (I_k(\mathbf{x}) - I_k(\mathbf{y})) = (-1)^k \left(\prod_{i=1}^k x_i - \prod_{i=1}^k y_i\right). \tag{6}$$

On the other hand, since $P_k(x_i)=0$, $i=1,\dots,k$, it follows $x_i^k=\sum_{j=1}^k(-1)^{j+1}I_j(\mathbf{x})x_i^{k-j}$, $i=1,\dots,k$, which by summation gives

$$\mathbf{x}^{(k)}=\sum_{j=1}^{k}(-1)^{j+1}I_j(\mathbf{x})\mathbf{x}^{(k-j)}=(-1)^{k+1}kI_k(\mathbf{x})+\sum_{j=1}^{k-1}(-1)^{j+1}I_j(\mathbf{x})\mathbf{x}^{(k-j)}.$$

Using analogous identity for $\mathbf{y}^{(k)}$, it follows

$$\sum_{i=1}^{k}x_i^k-\sum_{i=1}^{k}y_i^k=\mathbf{x}^{(k)}-\mathbf{y}^{(k)}=(-1)^{k+1}k\left(I_k(\mathbf{x})-I_k(\mathbf{y})\right)=(-1)^{k+1}k\left(\prod_{i=1}^{k}x_i-\prod_{i=1}^{k}y_i\right), \tag{7}$$

from which the equivalence of (a) and (c) is obvious. Using (6) and (7), we get

$$P_k(x)-Q_k(x)=-\frac{1}{k}\left[\sum_{i=1}^{k}x_i^k-\sum_{i=1}^{k}y_i^k\right]$$

from which the equivalence of (a) and (b) is obvious. □

LEMMA 3. *Let $x_1,x_2,\dots,x_n$ be real variables. Then*

$$I_k=\frac{1}{k}\sum_{l=0}^{k-1}(-1)^{k-l+1}\mathbf{x}^{(k-l)}I_l,\ \ k=1,2,\dots,n, \tag{8}$$

where

$$I_j=I_j(\mathbf{x})=I_j\left(x_1,\dots,x_n\right)=\sum_{1\leqslant i_1<i_2<\cdots i_j\leqslant n}x_{i_1}\cdots x_{i_j},\ \ j=0,1,\dots,k,\ \ I_0(\mathbf{x})=1,$$

$$\mathbf{x}^{(l)}=\sum_{i=1}^{n}x_i^l,\ \ l\in\mathbb{N}.$$

Proof. For $k=1$ identity (8) is obvious. Set

$$\left(\mathbf{x}^{(l)}|I_j\right)=\sum_{\substack{1\leqslant i_1<\cdots<i_j\leqslant n\\ i\neq i_m, m=1,\dots,j}}x_i^l x_{i_1}\cdots x_{i_j}.$$

Notice that $(\mathbf{x}|I_k)=(k+1)I_{k+1}$. The following identities are obvious:

(i) $\mathbf{x}I_k=(\sum_{i=1}^n x_i)\cdot I_k=(k+1)I_{k+1}+\left(\mathbf{x}^{(2)}|I_{k-1}\right)$

(ii) $\mathbf{x}^{(j)}I_{k-j+1}=\left(\mathbf{x}^{(j)}|I_{k-j+1}\right)+\left(\mathbf{x}^{(j+1)}|I_{k-j}\right)$, $j=1,\dots,k$.

Using the identity (ii) from $j=2$ to $j=k$, we get:

$$\begin{aligned}\left(\mathbf{x}^{(2)}|I_{k-1}\right) &= \mathbf{x}^{(2)}I_{k-1}-\left(\mathbf{x}^{(3)}|I_{k-2}\right)=\mathbf{x}^{(2)}I_{k-1}-\mathbf{x}^{(3)}I_{k-2}+\left(\mathbf{x}^{(4)}|I_{k-3}\right)\\ &= \mathbf{x}^{(2)}I_{k-1}-\mathbf{x}^{(3)}I_{k-2}+\mathbf{x}^{(4)}I_{k-3}-\mathbf{x}^{(5)}I_{k-4}+\cdots+(-1)^k\left(\mathbf{x}^{(k)}I_1-\mathbf{x}^{(k+1)}\right)\\ &= \sum_{j=2}^{k+1}(-1)^j\mathbf{x}^{(j)}I_{k-j+1}. \end{aligned} \tag{9}$$

Using identity (i) and (9) it follows

$$\begin{aligned} I_{k+1} &= \frac{1}{k+1}\left[\mathbf{x}^{(1)}I_k-\left(\mathbf{x}^{(2)}|I_{k-1}\right)\right]\\ &= \frac{1}{k+1}\left[\mathbf{x}^{(1)}I_k+\sum_{j=2}^{k+1}(-1)^{j+1}\mathbf{x}^{(j)}I_{k-j+1}\right]=\frac{1}{k+1}\sum_{j=1}^{k+1}(-1)^{j+1}\mathbf{x}^{(j)}I_{k-j+1}\\ &= \frac{1}{k+1}\sum_{j=0}^{k}(-1)^{k-j+2}\mathbf{x}^{(k-j+1)}I_j, \end{aligned}$$

which proves (8). □

Using notations from Lemma 3 and simple induction it follows:

PROPOSITION 4. *I_k polynomially depends on $\mathbf{x}^{(j)}=\sum_{i=1}^n x_i^j$ for $j=1,\dots,k$.*

For the sake of clarity, we give some initial examples. Set: $\mathbf{x}^j=\left(\sum_{i=1}^n x_i\right)^j$.

$k=1$: $\quad I_1=(-1)^{1-0+1}\mathbf{x}^{(1)}I_0=\mathbf{x}=\sum_{i=1}^n x_i$

$k=2$: $\quad I_2=\frac{1}{2}\left[-\mathbf{x}^{(2)}+\mathbf{x}I_1\right]=\frac{1}{2}\left[\left(\sum_{i=1}^n x_i\right)^2-\sum_{i=1}^n x_i^2\right]$

$k=3$: $\quad I_3=\frac{1}{3}\left[\mathbf{x}^{(3)}-\mathbf{x}^{(2)}I_1+\mathbf{x}I_2\right]=\frac{1}{3}\left[\mathbf{x}^{(3)}-\frac{3}{2}\mathbf{x}^{(2)}\mathbf{x}+\frac{1}{2}\mathbf{x}^3\right]$

$k=4$:

$$\begin{aligned} I_4 &= \frac{1}{4}\left[-\mathbf{x}^{(4)}+\mathbf{x}^{(3)}I_1-\mathbf{x}^{(2)}I_2+\mathbf{x}I_3\right]\\ &= \frac{1}{4}\left[-\mathbf{x}^{(4)}+\frac{4}{3}\mathbf{x}^{(3)}\mathbf{x}+\frac{1}{2}\mathbf{x}^{(2)}\mathbf{x}^{(2)}-\mathbf{x}^{(2)}\mathbf{x}^2+\frac{1}{6}\mathbf{x}^4\right]\\ &= \frac{1}{4}\left[\frac{1}{6}\left(\sum_{i=1}^n x_i\right)^4-\left(\sum_{i=1}^n x_i^2\right)\left(\sum_{i=1}^n x_i\right)^2\right.\\ &\qquad \left.+\frac{1}{2}\left(\sum_{i=1}^n x_i^2\right)^2+\frac{4}{3}\left(\sum_{i=1}^n x_i^3\right)\left(\sum_{i=1}^n x_i\right)-\sum_{i=1}^n x_i^4\right] \end{aligned}$$

3. Applications of 3-convexity and related inequalities

Now we are in position to give a simple proofs of inequalities (1) and (2).

THEOREM 5. *The inequality*

$$f\left(2\left(\frac{x_1+x_2}{2}\right)^2+x_3^2\right)+f\left(2\left(\frac{x_1+x_3}{2}\right)^2+x_2^2\right)+f\left(2\left(\frac{x_2+x_3}{2}\right)^2+x_1^2\right)$$
$$\leqslant f\left(x_1^2+x_2^2+x_3^2\right)+2f\left(\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{x_1+x_3}{2}\right)^2+\left(\frac{x_2+x_3}{2}\right)^2\right) \tag{10}$$

holds if f is a 3-convex function (on an interval containing all involved expressions). Reversed inequality holds if f is a 3-concave function.

Proof. Using

$$a_1=2\left(\frac{x_1+x_2}{2}\right)^2+x_3^2,\ a_2=2\left(\frac{x_1+x_3}{2}\right)^2+x_2^2,\ a_3=2\left(\frac{x_2+x_3}{2}\right)^2+x_1^2,$$

$$b_1=x_1^2+x_2^2+x_3^2,\ b_2=b_3=\left(\frac{x_1+x_2}{2}\right)^2+\left(\frac{x_1+x_3}{2}\right)^2+\left(\frac{x_2+x_3}{2}\right)^2,$$

it is easy to verify $a_1+a_2+a_3=b_1+2b_2$, $a_1^2+a_2^2+a_3^2=b_1^2+2b_2^2$ and $b_1=\max\{a_1,a_2,a_3,b_1,b_2\}$. The claim follows from Theorem 2. □

THEOREM 6. *The inequality*

$$f(t_1)+f(t_2)+f\left(2\sqrt{t_1t_2}\right)\leqslant f(t_1+t_2)+2f\left(\sqrt{t_1t_2}\right)$$

holds if f is a 3-convex function (on an interval containing all involved expressions). Reversed inequality holds if f is a 3-concave function.

Proof. Using

$$a_1=t_1,\ a_2=t_2,\ a_3=2\sqrt{t_1t_2},$$
$$b_1=t_1+t_2,\ b_2=b_3=\sqrt{t_1t_2},$$

it is easy to verify $a_1+a_2+a_3=b_1+2b_2$, $a_1^2+a_2^2+a_3^2=b_1^2+2b_2^2$ and $b_1=\max\{a_1,a_2,a_3,b_1,b_2\}$. The claim follows from Theorem 2. □

REMARK 7. The famous inequality for 3-convex functions is the Levinson inequality (see [10, Theorem 2.42]). It is easy to verify that the basic ($n=2$) non-weighted form of this inequality can be proved using Theorem 2. It seems, that the proof of the weighted form of the Levinson inequality, even for $n=2$, is beyond this method. For the weighted versions of this method see Theorem 3.2 in [3].

REMARK 8. It is tempting to consider the higher analogues of the inequality (10). It is straightforward to see what the form is, in the simplest case, of the inequality is to use $\det\left(g\left(\frac{x_i+x_j}{2}\right)\right)_{i,j=1}^{4}$, where $g(x)=e^{x^2}$ (see (3)). Now we have $4!=24$ terms and analogously as in the proof Theorem 5, we can define $a_1,\ldots,a_{12}$, $b_1,\ldots,b_{12}$ (counting with multiplicities) and try to check the conditions in Theorem 2. Calculations (by Wolfram Mathematica) show that $\sum_{i=1}^{12} a_i^j = \sum_{i=1}^{12} b_i^j$ holds for $j=1,\ldots,5$ and $\sum_{i=1}^{12} a_i^6 < \sum_{i=1}^{12} b_i^6$ for $x_i \neq x_j$, $i \neq j$. Obviously, Theorem 2 cannot be applied in this case. But there is still chance that the analogous inequality to (10) holds for 6-convex functions (see for example Lemma 2.2 in [3]). By constructing an example (take $(x_1,x_2,x_3,x_4)=(10,7,5,1)$), it can be shown that the spline kernel $\sum_{i=1}^{12}(b_i-t)_+^5-(a_i-t)_+^5$ is not non-negative for all t (see also Lemma 2.2 in [3]).

The inequality (1) for $f(t)=t^p$, after rearranging, can be written as

$$2^{1-p}\left(\frac{t_1^p+t_2^p}{2}-\sqrt{t_1t_2}^{\,p}\right)+\sqrt{t_1t_2}^{\,p}\leqslant\left(\frac{t_1+t_2}{2}\right)^p,\ t_1,t_2>0. \tag{11}$$

Using Theorem 6, it follows that (11) holds for $p\in[0,1]\cup[2,\infty)$ and the reverse inequality holds for $p\in(-\infty,0]\cup[1,2]$. In the first case it can be regarded as an improvement of AG-inequality and for $p\in[1,2]$, the reverse inequality in (11), can be regarded as an improvement of the inequality between the p-power mean and the geometric mean (of t_1 and t_2).

Another way of writing (11) for $p\geqslant 2$ is:

$$\sqrt{t_1t_2}\leqslant\left(\frac{(t_1+t_2)^p-t_1^p-t_2^p}{2^p-2}\right)^{\frac{1}{p}} \tag{12}$$

and the reverse inequality holds for $1\leqslant p\leqslant 2$. Notice that the reverse inequality in (12) holds for every $p\leqslant 1$ (using $2^p-2<0$ or $p<0$ and the reverse form of (11)). For the critical cases $p=1$ and $p=0$ see below.

The right-hand side of (12) has some interesting features. Set:

$$H_p(t_1,t_2):=\left(\frac{(t_1+t_2)^p-t_1^p-t_2^p}{2^p-2}\right)^{\frac{1}{p}} \tag{13}$$

We list the following basic properties of $H_p(t_1,t_2)$. The proofs are elementary.

PROPOSITION 9. *Let* $t_1,t_2>0$.

1. $\lim\limits_{p\to 0} H_p(t_1,t_2)=\dfrac{2t_1t_2}{t_1+t_2}$
2. $\lim\limits_{p\to 1} H_p(t_1,t_2)=\dfrac{1}{2\log 2}\log\dfrac{(t_1+t_2)^{t_1+t_2}}{t_1^{t_1}t_2^{t_2}}$
3. $t_1,t_2\mapsto H_p(t_1,t_2)$ *is an increasing function in both variables for any* $p\in\mathbb{R}$.

4. $\lim_{p\to\infty} H_p(t_1,t_2) = \frac{t_1+t_2}{2}$

5. $\lim_{p\to-\infty} H_p(t_1,t_2) = \min\{t_1,t_2\}$

6. $H_p(\lambda t_1,\lambda t_2) = \lambda H_p(t_1,t_2)$ *for any* $\lambda > 0$.

7. $H_p(t_1,t_2) = H_p(t_2,t_1)$, $H_p(t_1,t_1) = t_1$

8. $H_2(t_1,t_2) = \sqrt{t_1 t_2}$

This proposition shows that $H_p(t_1,t_2)$ is a mean of positive reals t_1,t_2 for any $p\in\mathbb{R}$ (with the obvious extension for $p=0$ and $p=1$ imposing continuity in the variable p).

The inequality (13) can be also proved by using differential calculus. For $p\in\mathbb{N}$ an instructive proof can be given by using binomial theorem and AG-inequality. In the following lemma much more is proved, namely, the mean $H_p(t_1,t_2)$ is increasing in the variable p.

LEMMA 10. *The function* $f:\mathbb{R}\to\mathbb{R}$ *defined by*

$$f(t) = \frac{1}{t}\log\frac{(1+x)^t - x^t - 1}{2^t - 2},$$

assuming $f(0) = \lim_{t\to 0} f(t)$ *and* $f(1) = \lim_{t\to 1} f(t)$ *(see Proposition 9), is an increasing function for any* $x > 1$.

Proof. The first derivative gives

$$f'(t) = -\frac{1}{t^2}\left(\log\frac{(x+1)^t - x^t - 1}{2^t-2} + t\frac{2^t\log 2}{2^t-2} - t\frac{x^t\log x - (1+x)^t\log(1+x)}{1+x^t-(1+x)^t}\right)$$

It is enough to prove that

$$g(x) = t\frac{x^t\log x - (1+x)^t\log(1+x)}{1+x^t-(1+x)^t} - \log\frac{(x+1)^t - x^t - 1}{2^t-2} - t\frac{2^t\log 2}{2^t-2} \tag{14}$$

is non-negative for $x \geqslant 1$ and $t\in\mathbb{R}$. Obviously $g(1) = 0$. It follows

$$\begin{aligned} g'(x) &= -\frac{t^2}{x(x+1)\left(x^t-(x+1)^t+1\right)^2} \\ &\quad\cdot\left(x^t\left((x+1)^t - x - 1\right)\log x - (x+1)^t\left(x^t - x\right)\log(x+1)\right). \end{aligned}$$

It is enough to prove $g'(x) > 0$ for $x > 1$.

Let $x > 1$. Using (14), $g'(x) > 0$ is equivalent to

$$x^t\left((x+1)^t - x - 1\right)\log x < (x+1)^t\left(x^t - x\right)\log(x+1). \tag{15}$$

By rearranging, (15) is equivalent to

$$\frac{(1+x)^{1-t}-1}{x^{1-t}-1} < \frac{\log(1+x)}{\log x}, \text{ for } t > 1,$$

and to

$$\frac{(1+x)^{1-t}-1}{x^{1-t}-1} > \frac{\log(1+x)}{\log x}, \text{ for } t < 1.$$

Since obviously $\lim_{t\to 1} \frac{(1+x)^{1-t}-1}{x^{1-t}-1} = \frac{\log(x+1)}{\log x}$, it is enough to prove that $t \mapsto \frac{(1+x)^{1-t}-1}{x^{1-t}-1}$ is a decreasing function, which, by substitution $u = 1-t$, is reduced to the claim that

$$\phi(u) = \frac{(1+x)^u - 1}{x^u - 1},\ \phi(0) = \lim_{u\to 0} \phi(u) = \frac{\log(x+1)}{\log x}$$

is an increasing function on $\mathbb{R}$. It is easy to check that $\phi'(u) > 0$ is equivalent to (15), so, it seems, the argument by the derivative doesn't work here. Due to continuity of ϕ at $u = 0$ it is enough to prove $\phi(u_1) < \phi(u_2)$ assuming $u_1 < u_2 < 0$ or $0 < u_1 < u_2$. The idea is to prove $\phi(\lambda u) > \phi(u)$ for $\lambda > 1$ and $u > 0$ and $\phi(\lambda u) < \phi(u)$ in the case $\lambda > 1$ and $u < 0$. Assuming this is proven, we have in the case $0 < u_1 < u_2$:

$$\phi(u_2) = \phi\left(\frac{u_2}{u_1}u_1\right) > \phi(u_1), \text{ using } u_1 > 0,\ u_2/u_1 > 1,$$

and in the case $u_1 < u_2 < 0$:

$$\phi(u_1) = \phi\left(\frac{u_1}{u_2}u_2\right) < \phi(u_2), \text{ using } u_2 < 0,\ u_1/u_2 > 1.$$

It remains to show $\phi(\lambda u) > \phi(u)$ for $\lambda > 1$ and $u > 0$, and the opposite inequality for $\lambda > 1$ and $u < 0$. For $n \in \mathbb{N}$, we have:

$$\phi(nu) = \frac{(1+x)^{nu}-1}{x^{nu}-1} = \phi(u)\frac{\sum_{k=0}^{n-1}(1+x)^{ku}}{\sum_{k=0}^{n-1} x^{ku}},$$

which obviously implies $\phi(nu) > \phi(u)$ for $u > 0$ and the opposite inequality for $u < 0$. Let $q = (1+x)^{u/n}$ and $q_1 = x^{u/n}$. Set $S_n(q) = \sum_{k=0}^{n} q^k$. In this terms inequality $\phi\left(\frac{n+1}{n}u\right) > \phi(u)$ can be written as

$$\frac{S_n(q)}{S_n(q_1)} > \frac{S_{n-1}(q)}{S_{n-1}(q_1)} \Leftrightarrow \frac{S_n(q)}{S_{n-1}(q)} > \frac{S_n(q_1)}{S_{n-1}(q_1)} \Leftrightarrow \frac{q^n}{S_{n-1}(q)} > \frac{q_1^n}{S_{n-1}(q_1)},$$

which obviously holds for $u > 0$ (since in this case $q > q_1$ holds). The same argument gives $\phi\left(\frac{n+1}{n}u\right) < \phi(u)$ for $u < 0$ (which implies $q < q_1$). By iteration of this property, we get $\phi\left(\frac{m}{n}u\right) > \phi(u)$ for $m > n$, $m,n \in \mathbb{N}$ and $u > 0$ and the opposite inequality for $u < 0$. The standard continuity argument gives the claim. This finishes the proof. □

REMARK 11. Log-convexity of the function $g(t)=\frac{(1+x)^t-x^t-1}{2^t-2}$ would imply that $f(t)=\frac{1}{t}\log g(t)$ from the previous lemma is an increasing function (since $g(0)=1$), and g seems to be easier to handle. We will show that g is not a log-convex function for any $x>1$. Otherwise $g^2(t_1+t_2)\leqslant g(2t_1)g(2t_2)$ holds for any $t_1,t_2\in\mathbb{R}$. Let $t_1=2$, $t_2=t$ and $x>1$. By extracting the dominant terms for $t\approx\infty$, we get

$$\begin{aligned}
&g(4)g(2t)-g^2(2+t)\\
&=\frac{(x+1)^4-x^4-1}{14}\frac{(x+1)^{2t}-x^{2t}-1}{2^{2t}-2}-\left(\frac{(1+x)^{2+t}-x^{2+t}-1}{2^{2+t}-2}\right)^2\\
&=\frac{(1+x)^{2t}}{2^{2t}}\left(\frac{(x+1)^4-x^4-1}{14}\varepsilon_1(x,t)-\frac{(x+1)^4}{16}\varepsilon_2(x,t)\right)\\
&=\frac{(1+x)^{2t}}{2^{2t}}\left(\frac{(x+1)^4-x^4-1}{14}-\frac{(x+1)^4}{16}+\varepsilon(x,t)\right)\\
&=\frac{(1+x)^{2t}}{2^{2t}}\left(-\frac{1}{112}(x-1)^2\left(7x^2+10x+7\right)+\varepsilon(x,t)\right),
\end{aligned}\tag{16}$$

where obviously $\lim_{t\to\infty}\varepsilon_1(x,t)=\lim_{t\to\infty}\varepsilon_2(x,t)=1$ and $\lim_{t\to\infty}\varepsilon(x,t)=0$. It follows from (16) that g is not a log-convex function for t big enough.

That the role of the geometric mean in the inequalities

$$2A\left(f(t_1),f(t_2)\right)-2f\left(M(t_1,t_2)\right)\leqslant f\left(2A(t_1,t_2)\right)-f\left(2M(t_1,t_2)\right),$$

where M is any mean, is special is exactly due to possibility to apply Theorem 2. To illustrate this we examine the case of the harmonic mean:

$$f(t_1)+f(t_2)-2f\left(\frac{2t_1t_2}{t_1+t_2}\right)\leqslant f(t_1+t_2)-f\left(2\frac{2t_1t_2}{t_1+t_2}\right).\tag{17}$$

Theorem 2 cannot be applied since the identity (5) holds only for $j=1$ and for $j=2$ equality holds iff $t_1=t_2$. This also implies that (17) cannot hold for all 3-convex functions. The Popoviciu criterion (using the kernel $\sum_{j=1}^{3}(b_j-t)_+-(a_j-t)_+$) can easily give that inequality (17) is not valid for all convex functions. As above it remains to consider the case $f(x)=x^p$. In this case (17) can be written as

$$2^{1-p}\left[\frac{t_1^p+t_2^p}{2}-\left(\frac{2t_1t_2}{t_1+t_2}\right)^p\right]+\left(\frac{2t_1t_2}{t_1+t_2}\right)^p\leqslant\left(\frac{t_1+t_2}{2}\right)^p.\tag{18}$$

It follows from Proposition 9 and Lemma 10

$$H(t_1,t_2)=\frac{2t_1t_2}{t_1+t_2}\leqslant H_p(t_1,t_2)\tag{19}$$

for every $p>0$ and the reverse inequality for $p<0$, where $H_p(t_1,t_2)$ is defined in (13). It is easy to see that (19) is equivalent to (18) for $p>1$. Also, (19) gives the reverse inequality in (18) for $p<1$. We conclude that (18) is an improvement of

AH-inequality for $p>1$ and the reverse inequality in (18) is an improvement of the inequality between p-power mean and the harmonic mean for $0<p<1$. Compare this to the discussion below (11).

Notice that (19) is a consequence of Lemma 10, which is technically demanding. It is easy to see that for $p>2$ (18) is a trivial consequence of Theorem 6 and HG-inequality. We give an alternative proof of (18) for $1<p<2$ based on different ideas.

THEOREM 12. *Let $p\geqslant 1$ and $x>1$. Then*

$$1+x^p-2\left(\frac{2x}{1+x}\right)^p \leqslant (1+x)^p-\left(2\frac{2x}{1+x}\right)^p,\ x\geqslant 1. \tag{20}$$

Proof. For $p\geqslant 2$ inequality (20) is a trivial consequence of Theorem 6 and HG-inequality (see also discussion below (11)). Let $1<p<2$. Using substitution $u=\frac{x-1}{x+1}$ and rearranging, (20) is equivalent to inequality

$$(1-u)^p+(1+u)^p+(2^p-2)\left(1-u^2\right)^p \leqslant 2^p,\ u\in[0,1].$$

Using binomial expansions we get

$$\begin{aligned}
F(u) &= (1-u)^p+(1+u)^p+(2^p-2)\left(1-u^2\right)^p\\
&= 2\sum_{k=0}^{\infty}\binom{p}{2k}u^{2k}+(2^p-2)\sum_{k=0}^{\infty}(-1)^k\binom{p}{k}u^{2k}\\
&= 2^p+p\,(p+1-2^p)\,u^2+\sum_{k=2}^{\infty}\left[2\binom{p}{2k}+(-1)^k(2^p-2)\binom{p}{k}\right]u^{2k}.
\end{aligned}$$

Set $f(u)=p\,(p+1-2^p)\,u^2+\sum_{k=2}^{\infty}\left[2\binom{p}{2k}+(-1)^k(2^p-2)\binom{p}{k}\right]u^{2k}$. Since $2^p=F(1)=2^p+f(1)$, obviously $f(1)=0$. Also, for $1<p<2$ it holds: $p+1-2^p<0$, $\binom{p}{2l}>0$ and $\binom{p}{2l+1}<0$ for any $l\in\mathbb{N}$. This gives

$$2\binom{p}{2k}+(-1)^k(2^p-2)\binom{p}{k}>0,\ k\geqslant 2.$$

It follows:

$$\begin{aligned}
F(u) &= 2^p+p\,(p+1-2^p)\,u^2+\sum_{k=2}^{\infty}\left[2\binom{p}{2k}+(-1)^k(2^p-2)\binom{p}{k}\right]u^{2k}\\
&= 2^p+u^2\left(p\,(p+1-2^p)+\sum_{k=2}^{\infty}\left[2\binom{p}{2k}+(-1)^k(2^p-2)\binom{p}{k}\right]u^{2k-2}\right)\\
&\leqslant 2^p+u^2\left(p\,(p+1-2^p)+\sum_{k=2}^{\infty}\left[2\binom{p}{2k}+(-1)^k(2^p-2)\binom{p}{k}\right]\right)\\
&= 2^p+u^2f(1)=2^p. \quad \square
\end{aligned}$$

REMARK 13. It is not obvious what multi-variable analogues of inequality (1) could be. See Remark 8 for one possibility. A natural generalization of inequality (1) is given by

$$\sum_{i=1}^{n} f(t_i) - nf\left(\prod_{i=1}^{n} t_i^{1/n}\right) \leqslant f\left(\sum_{i=1}^{n} t_i\right) - f\left(n\prod_{i=1}^{n} t_i^{1/n}\right). \tag{21}$$

It can be written as

$$nA\left(f(t_1),\ldots,f(t_n)\right) - nf\left(G(t_1,\ldots,t_n)\right) \leqslant f\left(nA(t_1,\ldots,t_n)\right) - f\left(nG(t_1,\ldots,t_n)\right),$$

where A and G are the arithmetic and the geometric mean, respectively. It seems that the case $n=2$ is in some sense an exceptional case. Theorem 2 cannot be applied for $n \geqslant 3$. Identities (5) don't hold and in this case are inequalities (after simple rearranging)

$$\begin{aligned}\left(\prod_{i=1}^{n} t_i\right)^{\frac{k}{n}} &\leqslant \frac{1}{n^k-n}\left[\left(\sum_{i=1}^{n} t_i\right)^{k} - \sum_{i=1}^{n} t_i^k\right] \\ &= \frac{1}{n^k-n}\sum_{j_1+j_2+\cdots+j_n=k,0\leqslant j_i<k}\binom{k}{j_1,j_2,\ldots,j_n}\prod_{i=1}^{n} t_i^{j_i},\end{aligned} \tag{22}$$

which is, since obviously

$$\sum_{j_1+j_2+\cdots+j_n=k,0\leqslant j_i<k}\binom{k}{j_1,j_2,\ldots,j_n} = n^k - n,$$

a consequence of the AG-inequality (we skip the combinatorial details in the proof). Equality holds (except in the case $n=k=2$) iff $t_1=t_2=\cdots=t_n$.

Using Popoviciu criterion for linear (in f) inequalities in the convex case, it is easy to see that (21) cannot generally hold for convex functions.

It remains to consider the case $f(x)=x^p$. Rearranging (21), we get for $p>1$ or $p<0$:

$$\prod_{i=1}^{n} t_i^{\frac{1}{n}} \leqslant \left(\frac{(\sum_{i=1}^{n} t_i)^p - \sum_{i=1}^{n} t_i^p}{n^p-n}\right)^{\frac{1}{p}} = H_p\left(t_1,\ldots,t_n\right) \tag{23}$$

and the reverse inequality for $0<p<1$. The right-hand side of the inequality (23) has the same properties as stated in Proposition 9, except that the geometric mean and the harmonic means cannot be obtained for any $p\in\mathbb{R}$ (including limit cases) for $n\geqslant 3$. We don't go into details of this investigation since the method is beyond the method used in this paper. Our conjecture is that the inequality (23) depends on the inequality $n^{p-1}\geqslant p$ (using multi-variable calculus and the Sylvester criteria for local extrema), which for $n\geqslant 3$ holds for $p\geqslant 1$ and $p<p_0$, where $p_0<1$ is the non-trivial solution of the equation $n^{p-1}=p$. Notice that for $n=2$ this inequality covers discussion below (11). For $p\in\mathbb{N}$ the inequality (23) reduces to (22).

4. Mean value theorems

It was proved in [4] that

$$f(x)+f(y)+f(z)-f(a)-f(b)-f(c)=\frac{1}{2}(xyz-abc)f'''(w) \tag{24}$$

holds, using conditions given in (5) for $k=3$, for some w in the smallest interval containing x,y,z,a,b,c and f''' continuous. Notice that Theorem 2 (in the case $k=3$) easily follows from (24). The purpose of this section is to present the generalizations of (24). The proofs are based on the ideas given in [7] and similar papers (see for example [9]).

THEOREM 14. *Let $x_1,\ldots,x_k,\ y_1,\ldots,y_k\in[a,b]$ be non-identical k-tuples such that (5) holds and such that (a) or (b) or (c) in Theorem 2 holds. If $f\in C^k([a,b])$, then there exists $\xi\in[a,b]$ such that*

$$\begin{aligned}\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i) &= \frac{f^{(k)}(\xi)}{k!}\left(\sum_{i=1}^{k} x_i^k-\sum_{i=1}^{k} y_i^k\right)\\ &= (-1)^{k+1}\frac{f^{(k)}(\xi)}{(k-1)!}\left[\prod_{i=1}^{k} x_i-\prod_{i=1}^{k} y_i\right].\end{aligned} \tag{25}$$

Proof. Set $m=\min_{x\in[a,b]} f^{(k)}(x)$, $M=\max_{x\in[a,b]} f^{(k)}(x)$. Obviously

$$f_1(x)=\frac{M}{k!}x^k-f(x),\ \ f_2(x)=f(x)-\frac{m}{k!}x^k$$

are k-convex functions. Applying Theorem 2 on f_1 and f_2 and rearranging, we get

$$\frac{m}{k!}\left(\sum_{i=1}^{k} x_i^k-\sum_{i=1}^{k} y_i^k\right)\leqslant\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i)\leqslant\frac{M}{k!}\left(\sum_{i=1}^{k} x_i^k-\sum_{i=1}^{k} y_i^k\right),$$

from which (25) obviously follows. The second identity in (25) is given in (7). Notice that $\sum_{i=1}^{k} x_i^k>\sum_{i=1}^{k} y_i^k$ since the involved k-tuples are not identical (otherwise $P_k(x)\equiv Q_k(x)$; see the proof of Theorem 2). $\square$

Mean value theorems of a type given in Theorem 14 are useful in defining suitable means. If $f^{(k)}$ is an invertible function, then

$$\xi=\left(f^{(k)}\right)^{-1}\left(k!\frac{\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i)}{\sum_{i=1}^{k} x_i^k-\sum_{i=1}^{k} y_i^k}\right)\in[a,b], \tag{26}$$

assuming that conditions (5) hold. Notice that $\max_i\{x_i\}=\max_i\{y_i\}$ implies that k-tuples $\mathbf{x}$ and $\mathbf{y}$ are identical (assuming decreasing order). At this instance assumption $\max_i\{x_i\}>\max_i\{y_i\}$ is not necessary since the reversed inequality implies obviously the reversed inequality in the part (d) in Theorem 2. The expression (26) is a natural generalization of the classical Stolarsky mean, which is given by (26) for $k=1$.

The most interesting case is $f(x)=x^r$, which gives

$$\xi = M(\mathbf{x},\mathbf{y};k,r) = \left(\frac{k!}{\prod_{j=0}^{k-1}(r-j)}\frac{\sum_{i=1}^k x_i^r - \sum_{i=1}^k y_i^r}{\sum_{i=1}^k x_i^k - \sum_{i=1}^k y_i^k}\right)^{\frac{1}{r-k}},$$

assuming continuous extensions for $r=0,1,\ldots,k$. In this way we get Stolarsky type means under the assumptions $\sum_{i=1}^k x_i^j = \sum_{i=1}^k y_i^j$, $j=1,\ldots,k-1$ and $\max_i x_i \neq \max_i y_i$. It can give a useful information on mutual position of r-power means (or logarithmic type means for $r=0,1,\ldots,k$) for given k-tuples, assuming their jth power means, $j=1,\ldots,k-1$, are equal.

EXAMPLE 15. For $k=2$, we get

$$M(\mathbf{x},\mathbf{y};2,r) = \left(\frac{2}{r(r-1)}\frac{x_1^r+x_2^r-y_1^r-y_2^r}{x_1^2+x_2^2-y_1^2-y_2^2}\right)^{\frac{1}{r-2}}, \ r\neq 0,1,2,\ x_1+x_2=y_1+y_2, x_1\neq y_1.$$

Notice that this is a special case of Stolarsky type means for four points given in [7]. We easily get

$$\lim_{x_1\to y_1} M(\mathbf{x},\mathbf{y};2,r) = \left(\frac{1}{r-1}\frac{y_1^{r-1}-y_2^{r-1}}{y_1-y_2}\right)^{\frac{1}{r-2}}, r\neq 1,2,\ y_1\neq y_2,$$

which is classical two parameter Stolarsky mean. For $r=2$ we get

$$M(\mathbf{x},\mathbf{y};2,2) = \lim_{r\to 2} M(\mathbf{x},\mathbf{y};2,r) = e^{-\frac{3}{2}}\left(\frac{x_1^{x_1^2}x_2^{x_2^2}}{y_1^{y_1^2}y_2^{y_2^2}}\right)^{\frac{1}{x_1^2+x_2^2-y_1^2-y_2^2}}.$$

The inequality

$$M(\mathbf{x},\mathbf{y};2,2) \leqslant \frac{x_1+x_2}{2} = \frac{y_1+y_2}{2} = M(\mathbf{x},\mathbf{y};2,3)$$

can be proved using results given at the end of this section.

The Cauchy type mean value theorem is as follows.

THEOREM 16. *Let $x_1,\ldots,x_k$, $y_1,\ldots,y_k \in [a,b]$ be non-identical k-tuples such that (5) holds and such that (a) or (b) or (c) in Theorem 2 holds. If $f,g\in C^k([a,b])$, then there exists $\xi\in[a,b]$ such that*

$$\frac{\sum_{i=1}^k f(x_i) - \sum_{i=1}^k f(y_i)}{\sum_{i=1}^k g(x_i) - \sum_{i=1}^k g(y_i)} = \frac{f^{(k)}(\xi)}{g^{(k)}(\xi)},$$

provided the denominators are non-zero.

Proof. Set

$$h(x)=\left(\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i)\right)g(x)-\left(\sum_{i=1}^{k} g(x_i)-\sum_{i=1}^{k} g(y_i)\right)f(x).$$

Obviously $\sum_{i=1}^{k} h(x_i)-\sum_{i=1}^{k} h(y_i)=0$ and the claim follows from Theorem 14 and $\sum_{i=1}^{k} x_i^k-\sum_{i=1}^{k} y_i^k>0$. $\square$

Again, if $f^{(k)}/g^{(k)}$ is invertible, then

$$\xi=\left(\frac{f^{(k)}}{g^{(k)}}\right)^{-1}\left(\frac{\sum_{i=1}^{k} f(x_i)-\sum_{i=1}^{k} f(y_i)}{\sum_{i=1}^{k} g(x_i)-\sum_{i=1}^{k} g(y_i)}\right)\in\left[\min_i\{x_i,y_i\},\max_i\{x_i,y_i\}\right]$$

is a mean. The most important case is $f(x)=x^r$, $g(x)=x^s$, which gives Stolarsky type means

$$\xi=M(\mathbf{x},\mathbf{y},k;r,s)=\left(\frac{\prod_{j=0}^{k-1}(s-j)}{\prod_{j=0}^{k-1}(r-j)}\frac{\sum_{i=1}^{k} x_i^r-\sum_{i=1}^{k} y_i^r}{\sum_{i=1}^{k} x_i^s-\sum_{i=1}^{k} y_i^s}\right)^{\frac{1}{r-s}},$$

under obvious restrictions $r\neq s$, $r,s\neq 0,1,\ldots,k-1$, in which cases can be continuously extended (using $\sum_{i=1}^{k} x_i^j=\sum_{i=1}^{k} y_i^j$, $j=0,1,\ldots,k-1$). The important property of monotonicity in variables r and s can be proved using the notion of the exponential convexity. We give just a sketch of the proof.

The fundamental claim is that the function

$$\Phi(r)=\frac{1}{\prod_{j=0}^{k-1}(r-j)}\left(\sum_{i=1}^{k} x_i^r-\sum_{i=1}^{k} y_i^r\right),$$

assuming $\max\{x_i: i=1,\ldots,k\}>\max\{y_i: i=1,\ldots,k\}$, is an exponentially convex function on $\mathbb{R}$ (with obvious extensions for $r=0,1,\ldots,k-1$), which by definition means

$$\sum_{i,j=1}^{n}\xi_i\xi_j\Phi\left(\frac{r_i+r_j}{2}\right)\geqslant 0,$$

for every $n\in\mathbb{N}$, $\xi_i,\xi_j\in\mathbb{R}$, $i,j=1,\ldots,n$. See Definition 1. Actually, we want to prove that $\Phi(r)$ is a log-convex function, which is equivalent to 2-exponential convexity (see the determinant criterion below Definition 1 for $k=2$). It is instructive to compare the method given below with Theorem 10 and Remark 11.

The basic tool is Theorem 2 and the function

$$F_r(x)=\begin{cases}\frac{1}{\prod_{j=0}^{k-1}(r-j)}\left(x^r-\sum_{j=0}^{k-1}\binom{r}{j}(x-1)^j\right), & r\neq 0,1,\ldots,k-1\\ \frac{1}{\frac{d}{dr}\prod_{j=0}^{k-1}(r-j)\Big|_{r=l}}\left(x^l\log x-\sum_{j=0}^{k-1}\frac{d}{dr}\binom{r}{j}\Big|_{r=l}(x-1)^j\right), & r=l=0,1,\ldots,k-1\end{cases}.$$

It is easy to see that $\lim_{r\to l}F_r(x)=F_l(x)$ for every $l=0,1,\ldots,k-1$ and every $x>0$, and $\frac{d^kF_r}{dx^k}=x^{r-k}$ for every $x>0$. In this way, we get two crucial (and obvious) properties of $F_r(x)$:

1. $x \mapsto F_r(x)$ is a k-convex function on $(0,\infty)$,

2. $r \mapsto \frac{d^k F_r(x)}{dx^k} = x^{r-k}$ is an exponentially convex function on $\mathbb{R}$.

Note that $\sum_{i=1}^{k}\left(F_r\left(x_i\right)-F_r\left(y_i\right)\right)=\Phi(r)$, assuming that k-tuples $x_1,\ldots,x_k$, $y_1,\ldots,y_k$ satisfy (5). Set

$$F(x)=\sum_{i,j=1}^{n}\xi_i\xi_j F_{\frac{r_i+r_j}{2}}(x).$$

Using the property from Definition 1 it is obvious that F is a k-convex function, which by Theorem 2 gives

$$\begin{aligned}
0 &\leqslant \sum_{l=1}^{k}F\left(x_l\right)-\sum_{l=1}^{k}F\left(y_l\right)\\
&=\sum_{i,j=1}^{n}\xi_i\xi_j\sum_{l=1}^{k}\left(F_{\frac{r_i+r_j}{2}}\left(x_l\right)-F_{\frac{r_i+r_j}{2}}\left(y_l\right)\right)\\
&=\sum_{i,j=1}^{n}\xi_i\xi_j\Phi\left(\frac{r_i+r_j}{2}\right).
\end{aligned}$$

This gives exponential convexity of the function $r \mapsto \Phi(r)$, and particularly log-convexity, which implies (see [10, p. 2]):

$$\frac{\log\Phi(r_1)-\log\Phi(r_2)}{r_1-r_2}\leqslant\frac{\log\Phi(s_1)-\log\Phi(s_2)}{s_1-s_2},$$

or equivalently

$$M(\mathbf{x},\mathbf{y},k;r_1,r_2)\leqslant M(\mathbf{x},\mathbf{y},k;s_1,s_2),$$

for $r_1 \leqslant s_1$, $r_2 \leqslant s_2$.

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