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NEW ORDERING RELATIONS FOR THE HEINZ MEANS VIA HYPERBOLIC FUNCTIONS

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Abstract. In this paper, we create new types of upper and lower bounds of Heinz means using simplified relations based on hyperbolic functions. In particular, for any strictly positive operators $A,B \in B(\mathscr{H})$,we obtain the inequality

$$A\#_{2\mu-\tau}B + A\#_{1-\tau}B < 2H_\tau(A,B) < A\#_\tau B + A\#_{1-(2\mu-\tau)}B,$$

where $0 < A < B$ and $0 < \mu < \tau < 1$.

1. Introduction

If a and b are two positive numbers and $\mu \in [0,1]$, then the weighted arithmetic mean, weighted geometric mean and Heinz mean in the parameter μ are defined by

$$a\nabla_\mu b := (1-\mu)\,a + \mu b,$$

$$a\#_\mu b := a^{1-\mu}b^\mu$$

and

$$H_\mu(a,b) := \frac{1}{2}(a\#_\mu b + a\#_{1-\mu}b),$$

respectively.

It is well-known that

$$a\#_\mu b \leqslant a\nabla_\mu b. \tag{1.1}$$

we simply denote $a\nabla_{\frac{1}{2}}b$ and $a\#_{\frac{1}{2}}b$ by $a\nabla b$ and $a\#b$, receptively.

The following inequalities come directly by taking the maximum and minimum values of the Heinz mean

$$a\,\#\,b \leqslant H_\mu(a,b) \leqslant a\,\nabla\,b. \tag{1.2}$$

In [3, Therorem 2.1] and [6, Theorem 2.1], Kittaneh and Manasrah gave some refinements of inequalities (1.1) and (1.2) as follows:

$$2r_0(a\nabla b - a\#b) \leqslant a\nabla_\mu b - a\#_\mu b \leqslant 2R_0(a\nabla b - a\#b), \tag{1.3}$$

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$$2r_0(a\nabla b - a\#b) \leqslant a\nabla b - H_\mu(a,b) \leqslant 2R_0(a\nabla b - a\#b), \tag{1.4}$$

where $0 \leqslant \mu \leqslant 1$, $r_0 = \min\ \{\mu, 1-\mu\}$ and $R_0 = \max\ \{\mu, 1-\mu\}$.

Another refinement of inequality (1.2) was given in [4, Corollary2.7] and [8, Inequality2.5] as follows:

$$4\mu(1-\mu)(a\nabla b - a\#b) \leqslant a\nabla b - H_\mu(a,b) \leqslant \frac{1}{2}\mu(1-\mu)(b-a)\ln\frac{b}{a}. \tag{1.5}$$

On the other hand, Zhu [17, Theorem 1.2] obtained a lower bound for the Heinz mean as follows:

$$\frac{a^{1-\tau}b^{\tau} - a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a - \ln b)} < H_\mu(a,b), \tag{1.6}$$

where $a,b > 0$, $a \neq b$, and $(1-2\tau)^2 \leqslant (1-2\mu)^2$.

More motivating inequalities related to the Heinz means can be found in [1], [4], [9], [10] and [15].

Throughout this paper, indicate the space of bounded linear operators on a Hilbert space $\mathscr{H}$ by $B(\mathscr{H})$. For $A,B \in B(\mathscr{H})$, we write $A < B$ to mean $B-A$ is positive definite, particularly, $0 < A$ denotes that A is positive definite. By considering the definitions of means in the scalar case, such definitions can be raised up to the operator level. For $\mu \in [0,1]$, the weighted arithmetic operator mean ∇_μ and geometric operator mean $\#_\mu$ are defined as follows:

$$A\nabla_\mu B = (1-\mu)A + \mu B,$$
$$A\#_\mu B = A^{\frac{1}{2}}(A^{\frac{-1}{2}}BA^{\frac{-1}{2}})^{\mu}A^{\frac{1}{2}},$$

for positive definite operators $A,B \in B(\mathscr{H})$.

If $\mu = \frac{1}{2}$, we write $A\nabla B$ and $A\#B$ to denote the arithmetic operator mean and geometric operator mean, respectively. By recognizing the above definitions, the Heinz operator mean is given by

$$H_\mu(A,B) = \frac{A\#_\mu B + A\#_{1-\mu}B}{2}.$$

An operator version of (1.1) and (1.2) due to Furuta [2] and Kittaneh et al. [5] are the following inequalities

$$A\#_\mu B \leqslant A\nabla_\mu B, \tag{1.7}$$

$$A\#B \leqslant H_\mu(A,B) \leqslant A\nabla B. \tag{1.8}$$

Recently, such operator mean inequalities are under active investigations. The authors in [7, Theorem 4] and [6, Corollary 3.1] established the following refinements of the inequalities in (1.7) and (1.8) as follows:

$$2r_0(A\nabla B - A\#B) \leqslant A\nabla_\mu B - A\#_\mu B \leqslant 2R_0(A\nabla B - A\#B), \tag{1.9}$$

$$2r_0(A\nabla B - A\#B) \leqslant A\nabla B - H_\mu(A,B) \leqslant 2R_0[A\nabla B - A\#B], \tag{1.10}$$

where $0 \leqslant \mu \leqslant 1$, $r_0 = \min\{\mu, 1-\mu\}$ and $R_0 = \max\{\mu, 1-\mu\}$.

The generalized hyperbolic cosine and hyperbolic sine functions which play a vital role in our research are introduced in [13] as:

$$\cos h_a(z) = \frac{a^z + a^{-z}}{2} \quad \text{and} \quad \sin h_a(z) = \frac{a^z - a^{-z}}{2},$$

where $a, z \in \mathbb{R}$ and $a > 1$.

In this paper, we are interested in finding new ordering relations for the Heinz operator means by adapting generalizations of hyperbolic functions and using the monotonicity principle for bounded self-adjoint operators on the Hilbert space $\mathscr{H}$ [15]: Let $T \in B(\mathscr{H})$ be self-adjoint with a spectrum $Sp(T)$ and let f and g be continuous real functions such that $f(t) \geqslant g(t)$ for all $t \in Sp(T)$. Then $f(T) \geqslant g(T)$.

2. Preliminaries

We use the simplified relations

$$\cos hax = \frac{a^{1-\tau}b^{\tau} + a^{\tau}b^{1-\tau}}{2\sqrt{ab}} = \frac{H_{\tau}(a,b)}{\sqrt{ab}}, \tag{2.1}$$

$$\frac{\sin hax}{ax} = \frac{a^{1-\tau}b^{\tau} - a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a - \ln b)} \frac{1}{\sqrt{ab}}, \tag{2.2}$$

where $0 < a < b$, $0 < \tau < 1$, $\tau \neq \frac{1}{2}$, $a = 1 - 2\tau$ and $x = \frac{1}{2}(\ln a - \ln b)$.

If $x, y \in \mathbb{R}$ and $x < y$, then we have [12]

$$e^{-(y-x)\ln(a)} < \frac{\cos h_a(y)}{\cos h_a(x)} < e^{(y-x)\ln(a)} \tag{2.3}$$

In addition, for $z \in (0,1)$, the inequality

$$e^{\alpha z^2} < \cos h_a(z) < e^{\beta z^2}, \tag{2.4}$$

where $\alpha = \ln(a + \frac{1}{a}) - \ln 2$ and $\beta = \frac{\ln^2 a}{2}$ holds [12].

For $z \neq 0$, we have [11,12]

$$\frac{\ln(a)}{\cos h_a(z)} < \frac{\sin h_a(z)}{z} < \ln(a)\cos h_a(z), \tag{2.5}$$

and

$$\frac{\sin h_a(z)}{z} < \frac{(2 + \cos h_a(z))\ln(a)}{3} \tag{2.6}$$

Moreover,

$$\cos h_a(z) < \left(\frac{\sin h_a(z)}{z}\right)^q, \tag{2.7}$$

for every $q \geqslant 3$.

Furthermore, it is known that [12] if $z \in (0,\infty)$, then

$$\ln(a)\cos h_a\left(\frac{z}{2}\right) < \frac{\sin h_a(z)}{z} < \ln(a)\cos h^2{}_a\left(\frac{z}{2}\right). \tag{2.8}$$

Note that, when $a = e$, the ineuality (2.6) is reduced to the hyperbolic Cusa-Huygens inequality [14].

3. Scalar results

We start our work by establishing some new scalar inequalities related to the Heinz means by the help of hyperbolic functions.

THEOREM 3.1. *Let* $0 < a < b$ *and* $0 < \mu < \tau < 1$. *Then*

$$\left(\frac{a}{b}\right)^{\tau-\mu} H_\mu(a,b) < H_\tau(a,b) < \left(\frac{a}{b}\right)^{\mu-\tau} H_\mu(a,b). \tag{3.1}$$

Proof. Let $\alpha = 1-2\tau$ and $\beta = 1-2\mu$. Then for $x = \frac{1}{2}(\ln a - \ln b)$, we have $\beta x < \alpha x$.

So, inequality (2.3) implies

$$e^{-(\alpha x-\beta x)} < \frac{\cos h\alpha x}{\cos h\beta x} < e^{(\alpha x-\beta x)}.$$

Hence,

$$e^{-(\alpha x-\beta x)} < \frac{H_\tau(a,b)}{H_\mu(a,b)} < e^{(\alpha x-\beta x)}.$$

By using the equation $\alpha - \beta = 2(\mu - \tau)$, we have

$$e^{-2(\mu-\tau)x} H_\mu(a,b) < H_\tau(a,b) < e^{2(\mu-\tau)x} H_\mu(a,b).$$

By taking $x = \frac{1}{2}(\ln a - \ln b)$, we complete the proof. □

A nice result is given in the following corollary by letting $\mu = \frac{1}{2}$ in theorem3.1.

COROLLARY 3.1. *Let* $0 < a < b$ *and* $\frac{1}{2} < \tau < 1$. *Then*

$$a^\tau b^{1-\tau} < H_\tau(a,b) < a^{1-\tau} b^\tau \tag{3.2}$$

The following theorem provides a new upper and lower bounds for the Heinz means.

THEOREM 3.2. *Let* $0 < a < b$ *and* $0 < \tau < \frac{1}{2}$. *Then*

$$\left(\frac{a^2+b^2}{2ab}\right)^{\tau^2} < \frac{H_{\frac{1-2\tau}{2}}(a,b)}{\sqrt{ab}} < \left(\frac{b}{a}\right)^{\frac{\tau^2 \ln(\frac{b}{a})}{2}} \tag{3.3}$$

Proof. Inequality (2.4) implies

$$e^{\alpha\tau^2} < \frac{\left(\frac{b}{a}\right)^\tau + \left(\frac{b}{a}\right)^{-\tau}}{2} < e^{\beta\tau^2},$$

where $\alpha = \ln\left(\frac{b}{a} + \frac{a}{b}\right) - \ln 2$ and $\beta = \frac{1}{2}\ln^2\left(\frac{b}{a}\right)$.

Thus,

$$\left(\frac{a^2+b^2}{2ab}\right)^{\tau^2} < \frac{\left(\frac{b}{a}\right)^{\tau}+\left(\frac{b}{a}\right)^{-\tau}}{2} < \left(\frac{b}{a}\right)^{\frac{\tau^2 \ln\left(\frac{b}{a}\right)}{2}}$$

So, we complete the proof. □

In the following theorem we construct other ordering relations for the Heinz means using inequalities (2.5), (2.6), (2.7) and the relations (2.1) and (2.2).

THEOREM 3.3. *Let* $0<a<b$ *and* $0<\tau<1$ *with* $\tau \neq \frac{1}{2}$. *Then the following inequalities hold*

i.
$$\frac{ab}{H_\tau(a,b)} < \frac{a^{1-\tau}b^{\tau}-a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a-\ln b)} < H_\tau(a,b). \tag{3.4}$$

ii.
$$\frac{a^{1-\tau}b^{\tau}-a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a-\ln b)} < \frac{2}{3}\sqrt{ab}+\frac{1}{3}H_\tau(a,b). \tag{3.5}$$

iii.
$$H_\tau(a,b) < (a\#b)^{1-q}\left(\frac{a^{1-\tau}b^{\tau}-a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a-\ln b)}\right)^{q},\ q \geqslant 3. \tag{3.6}$$

Consider the function $f_\mu : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ defined by

$$f_\mu(a,b) = \frac{a^{1-\mu}b^{\mu}-a^{\mu}b^{1-\mu}}{(1-2\mu)(\ln a-\ln b))},\ \ \mu\in[0,1],\ \ \mu\neq\frac{1}{2}\ \text{ and }\ a\neq b.$$

It is clear that f_μ is decreasing on $0 \leqslant \mu < \frac{1}{2}$ and increasing on $\frac{1}{2} < \mu \leqslant 1$. So, Theorem 3.3 (i) yields

$$\frac{a^{1-\tau}b^{\tau}-a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a-\ln b)} < \frac{a^{1-\mu}b^{\mu}-a^{\mu}b^{1-\mu}}{(1-2\mu)(\ln a-\ln b)} < H_\mu(a,b), \tag{3.7}$$

for $0<a<b$ and $0<\mu<\tau<\frac{1}{2}$. Thus, inequality (3.7) is more precise than inequality (1.6).

The following series of inequalities presents some new comparisons of Heinz operator means with the function $f_\mu(a,b)$ with different parameters.

THEOREM 3.4. *Let* $0<a<b$ *and* $\frac{1}{2}<\mu<\tau<1$. *Then*

$$H_{\frac{1+2\mu}{4}}(a,b) < \frac{a^{1-\tau}b^{\tau}-a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a-\ln b)} < \frac{1}{\sqrt{ab}}H^2_{\frac{1+2\mu}{4}}(a,b). \tag{3.8}$$

Proof. Let $\alpha = 1-2\tau$, $\beta = 1-2\mu$ and $x=\frac{1}{2}(\ln a-\ln b)$. Then

$$\cos h\left(\frac{\beta x}{2}\right) = \frac{e^{\left(\frac{\beta x}{2}\right)}+e^{-\left(\frac{\beta x}{2}\right)}}{2} = \frac{\left(\frac{a}{b}\right)^{\frac{\beta}{4}}+\left(\frac{a}{b}\right)^{\frac{-\beta}{4}}}{2}$$
$$= \frac{a^{\left(\frac{1}{2}-\frac{\beta}{4}\right)}b^{\left(\frac{1}{2}+\frac{\beta}{4}\right)}+a^{\left(\frac{1}{2}+\frac{\beta}{4}\right)}b^{\left(\frac{1}{2}-\frac{\beta}{4}\right)}}{2\sqrt{ab}} = \frac{1}{\sqrt{ab}}H_{\frac{1+2\mu}{4}}(a,b).$$

Since $\beta x < \alpha x$, inequality (2.8) implies

$$\cos h\left(\frac{\beta x}{2}\right) < \cos h\left(\frac{\alpha x}{2}\right) < \frac{\sin hax}{ax} < \cos h^2\left(\frac{ax}{2}\right)$$

Thus, by using inequality (2.1)

$$H_{\frac{1+2\mu}{4}}\frac{1}{\sqrt{ab}} < \frac{a^{1-\tau}b^{\tau}-a^{\tau}b^{1-\tau}}{(1-2\tau)(\ln a-\ln b)\sqrt{ab}} < \frac{1}{ab}H^2_{\frac{1+2\mu}{4}}(a,b). \quad \square$$

4. Operator results

Based on inequality (3.1) and by virtue of the monotonicity principle, we obtain a new upper and lower bounds for the Heinz operator means.

THEOREM 4.1. *Let $A, B \in B(\mathscr{H})$ be such that $0 < A < B$ and $0 < \mu < \tau < 1$. Then*

$$A\#_{2\mu-\tau}B + A\#_{1-\tau}B < 2H_{\tau}(A,B) < A\#_{\tau}B + A\#_{1-(2\mu-\tau)}B. \tag{4.1}$$

In particular, if $\mu = \frac{1}{2}$, then we have

$$A\#_{1-\tau}B < H_{\tau}(A,B) < A\#_{\tau}B. \tag{4.2}$$

Proof. Let $T = A^{-\frac{1}{2}}BA^{-\frac{1}{2}}$. Then $T > I$, where I is the identity operator, and if $t \in \sigma(T)$, we have

$$\left(\frac{1}{t}\right)^{\tau-\mu}H_{\mu}(1,t) = t^{\mu-\tau}\left(\frac{t^{\mu}+t^{1-\mu}}{2}\right) < \frac{t^{\tau}+t^{1-\tau}}{2} < t^{\tau-\mu}\left(\frac{t^{\mu}+t^{1-\mu}}{2}\right)$$
$$= \left(\frac{1}{t}\right)^{\mu-\tau}H_{\mu}(1,t).$$

Now, monotonicity principle implies

$$T^{\mu-\tau}\left(\frac{T^{\mu}+T^{1-\tau}}{2}\right) < \frac{T^{\tau}+T^{1-\tau}}{2} < T^{\tau-\mu}\left(\frac{T^{\mu}+T^{1-\tau}}{2}\right)$$

By multiplying both sides by $A^{\frac{1}{2}}$, we get

$$\frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{2\mu-\tau}A^{\frac{1}{2}}+A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\tau}A^{\frac{1}{2}}}{2}$$
$$< \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau}A^{\frac{1}{2}}+A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-\tau}A^{\frac{1}{2}}}{2}$$
$$< \frac{A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\tau}A^{\frac{1}{2}}+A^{\frac{1}{2}}(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{1-(2\mu-\tau)}A^{\frac{1}{2}}}{2}$$

Hence,

$$A\#_{2\mu-\tau}B + A\#_{1-\tau}B < 2H_{\tau}(A,B) < A\#_{\tau}B + A\#_{1-(2\mu-\tau)}B. \quad \square$$

Finally, the operator versions of the inequalities (3.3), (3.6), (3.5) and (3.8) are obtained by using the monotonicity principle in a similar way as in the proof of Theorem 4.1. Here, we regard the function F_μ as introduced in [7],

$$F_\mu = \begin{cases} \frac{x^\mu - x^{1-\mu}}{\ln x}, & x>0,\ x\neq 1 \\ 2\mu-1, & x=1 \end{cases},$$

where $0 \leqslant \mu \leqslant 1$.

Now we are ready to present new inequalities for Heinz, geometric and arithmetic means. The proofs are passed on the monotonicity property for matrices and Theorem 3.2 Theorem 3.3 (iii), Theorem 3.3 (ii) and Theorem (3.4), respectively.

THEOREM 4.2. *Let $A,\ B \in B(\mathscr{H})$ be such that $0 < A < B$. Then the following inequalities hold:*

i. For $0 < \tau < \frac{1}{2}$,

$$2^{1-\tau^2} A^{\frac{1}{2}} \left((A^{-\frac{1}{2}} B A^{-\frac{1}{2}})^2 + I \right)^{\tau^2} A^{\frac{1}{2}} < A\#_{\tau(\tau-1)}B + A\#_{\tau(\tau+1)}B. \tag{4.3}$$

ii. For $0 < \tau < 1$, $\tau \neq \frac{1}{2}$ and $q \geqslant 3$,

$$A\#_{\frac{1}{2}(q-1)+\tau}B + A\#_{\frac{1}{2}(q+1)-\tau}B < \frac{2}{(2\tau-1)^q} A^{\frac{1}{2}} \left(F_\tau(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) \right)^q A^{\frac{1}{2}}. \tag{4.4}$$

iii. For $0 \leqslant \tau \leqslant 1$,

$$\frac{1}{2\tau-1} A^{\frac{1}{2}} F_\tau(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} < \frac{2}{3} A\#B + \frac{1}{3} H_\tau(A,B). \tag{4.5}$$

iv. For $\frac{1}{2} < \mu \leqslant \tau < 1$,

$$H_{\frac{1+2\mu}{4}}(A,B) < \frac{1}{(2\tau-1)} A^{\frac{1}{2}} F_\tau(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}} < \frac{1}{2}[H_\mu(A,B) + A\#B]. \tag{4.6}$$

It should be noticed here inequality (4.5) was obtained by Liang and Shi in [9]. Also, inequality (4.6) is a refinement of Heinz operator mean inequality since

$$A\#B < H_{\frac{1+2\mu}{4}}(A,B) < \frac{1}{(2\tau-1)} A^{\frac{1}{2}} F_\tau(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$$
$$< \frac{1}{2}[H_\mu(A,B) + A\#B] < H_\mu(A,B)$$

and this refinment has been explored in [7].

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