Journal of Mathematical Inequalities
Volume 16, Number 1 (2022), 393–412 doi:10.7153/jmi-2022-16-29

SOME NEW MULTIDIMENSIONAL HARDY–TYPE INEQUALITIES WITH GENERAL KERNELS ON TIME SCALES

E. AWWAD* AND A. I. SAIED

(*Communicated by M. Krnić*)

Abstract. This paper contains some new multidimensional Hardy-type inequalities with general kernels on time scales. Our results (when $\mathbb{T}=\mathbb{R}$) give the inequalities proved by Oguntuase and Durojaye.

1. Introduction

In [21], Opic and Kufner proved that if $1<s\leqslant\beta<\infty$, then the inequality

$$\left[\int_a^b \varpi(x)\left(\int_a^x \eta(\tau)d\tau\right)^\beta dx\right]^{\frac{1}{\beta}} \leqslant C\left(\int_a^b \xi(x)\eta^s(x)dx\right)^{\frac{1}{s}}, \tag{1}$$

holds for all measurable functions $\eta\geqslant 0$, if and only if

$$\sup_{a<x<b}\left(\int_x^b \varpi(\tau)d\tau\right)^{\frac{1}{\beta}}\left(\int_a^x \xi^{1-s'}(\tau)d\tau\right)^{\frac{1}{s'}}=K<\infty,$$

where $s'=\frac{s}{s-1}$. In addition, the estimate for constant C appearing in (1) is given by

$$K\leqslant C\leqslant\left(1+\frac{\beta}{s'}\right)^{\frac{1}{\beta}}\left(1+\frac{s'}{\beta}\right)^{\frac{1}{s'}}K. \tag{2}$$

In [18], Lai proved that if ϕ is a nonnegative kernel, $0<s\leqslant\beta\leqslant 1$ and ξ, w are nonnegative weight functions, then

$$\left[\int_0^\infty w(x)\left(\int_0^\infty \phi(x,y)\eta(y)dy\right)^\beta dx\right]^{\frac{1}{\beta}} \leqslant C\left(\int_0^\infty \xi(x)\eta^s(x)dx\right)^{\frac{1}{s}}, \tag{3}$$

Mathematics subject classification (2020): 26D10, 26D15, 34N05, 42B25, 42C10, 47B38.
Keywords and phrases: Hardy type inequality, time scales, weighted functions, kernels functions.
* Corresponding author.

Paper JMI-16-29

holds for all nonincreasing functions $\eta \geqslant 0$, if and only if

$$\left[\int_0^\infty w(x)\left(\int_0^r \phi(x,y)dy\right)^\beta dx\right]^{\frac{1}{\beta}} \leqslant C\left(\int_0^r \xi(x)\,dx\right)^{\frac{1}{s}},$$

holds for all $r > 0$. Moreover, the inequality (3) holds for nondecreasing functions $\eta \geqslant 0$ if and only if

$$\left[\int_0^\infty w(x)\left(\int_r^\infty \phi(x,y)dy\right)^\beta dx\right]^{\frac{1}{\beta}} \leqslant C\left(\int_r^\infty \xi(x)\,dx\right)^{\frac{1}{s}}, \quad r > 0,$$

holds. Many generalizations of Hardy-type inequalities have appeared in literature, we refer the reader to the papers [7, 9, 11, 12, 19, 28, 29], and the books [16, 17, 21]. Also Kaijser et al. [13] generalized the Hardy-Knopp inequality with a convex function and proved that

$$\int_0^\infty \Phi\left(\frac{1}{\zeta}\int_0^\zeta \eta(\tau)d\tau\right)\frac{d\zeta}{\zeta} \leqslant \int_0^\infty \Phi(\eta(\zeta))\frac{d\zeta}{\zeta}, \tag{4}$$

where Φ is a convex function on $\mathbb{R}^+$ and $\eta : \mathbb{R}^+ \to \mathbb{R}^+$ is a locally integrable positive function. The generalization of the Hardy-Knopp inequality (4) with two different weighted functions were proved by Čižmešija et al. [6]. In particular, it was proved that if $0 < b \leqslant \infty$, $\varpi : (0,b) \to \mathbb{R}$ is a nonnegative function such that the function $\zeta \to \varpi(\zeta)/\zeta^2$ is locally integrable on $(0,b)$ and Φ is convex on (a,c), where $-\infty \leqslant a < c \leqslant \infty$, the inequality

$$\int_0^b \varpi(\zeta)\Phi\left(\frac{1}{\zeta}\int_0^\zeta \eta(\tau)d\tau\right)\frac{d\zeta}{\zeta} \leqslant \int_0^b \xi(\zeta)\Phi(\eta(\zeta))\frac{d\zeta}{\zeta},$$

holds for all integrable functions $\eta : (0,b) \to \mathbb{R}$, such that $\eta(\zeta) \in (a,c)$ for all $\zeta \in (0,b)$ and the function ξ is defined by

$$\xi(\tau) := \tau\int_\tau^b \frac{\varpi(\zeta)}{\zeta^2}d\zeta, \quad \text{for} \quad \tau \in (0,b).$$

In 2005, Kaijser et al. [14] established an interesting generalization for inequalitiy of Hardy's type. In particular, they proved that if $0 < b \leqslant \infty$, $\varpi : (0,b) \to \mathbb{R}$ and $k : (0,b)\times(0,b) \to \mathbb{R}$ are non-negative functions, such that $0 < K(\tau) := \int_0^\tau k(\tau,\theta)d\theta < \infty$, $\tau \in (0,b)$ and

$$\xi(\zeta) := \zeta\int_\zeta^b \varpi(\tau)\frac{k(\tau,\zeta)}{K(\tau)}\frac{d\tau}{\tau} < \infty, \quad \zeta \in (0,b),$$

then

$$\int_0^b \varpi(\zeta)\Phi(A_k\eta(\zeta))\frac{d\zeta}{\zeta} \leqslant \int_0^b \xi(\zeta)\Phi(\eta(\zeta))\frac{d\zeta}{\zeta}, \tag{5}$$

where Φ is a convex function on an interval $I \subseteq \mathbb{R}$, $\eta : (0,b) \to \mathbb{R}$ is a function with values in I, and

$$A_k\eta(\zeta) := \frac{1}{K(\zeta)}\int_0^\zeta k(\zeta,\theta)\eta(\theta)d\theta, \; K(\zeta) = \int_0^\zeta k(\zeta,\theta)d\theta, \;\; \zeta \in (0,b).$$

Also, in [14] it is proved that if $1 < s \leqslant \beta < \infty$, $s \in (1,\alpha)$ and $0 < b < \infty$. Furthermore assume that Φ is a convex and strictly monotone function on (a,c), $-\infty < a < c < \infty$ and assumed that the general Hardy operator A_k defined as following

$$A_k\eta(\zeta) = \frac{1}{K(\zeta)}\int_0^{\zeta} k(\zeta,\theta)\eta(\theta)d\theta, \quad K(\zeta) = \int_0^{\zeta} k(\zeta,\theta)d\theta,$$

where $k : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a nonnegative kernel and assume that $\varpi(\zeta)$ and $\xi(\zeta)$ are nonnegative weighted functions. Then the inequality

$$\left(\int_0^b [\Phi(A_k\eta(\zeta))]^{\beta}\varpi(\zeta)\frac{d\zeta}{\zeta}\right)^{\frac{1}{\beta}} \leqslant C\left[\int_0^b \Phi^{\alpha}(\eta(\zeta))\xi(\zeta)\frac{d\zeta}{\zeta}\right]^{\frac{1}{\alpha}}, \tag{6}$$

holds for all the nonnegative functions $\eta(\zeta)$, $a < \eta(\zeta) < c$, $\zeta \in [0,b]$ and $C > 0$, if

$$A(s) := \sup_{0<\theta<b} [V(\theta)]^{\frac{s-1}{\alpha}} \left(\int_{\theta}^{b} \left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^{\beta} [V(\zeta)]^{\frac{\beta(\alpha-s)}{\alpha}} \varpi(\zeta)\frac{d\zeta}{\zeta}\right)^{\frac{1}{\beta}} < \infty,$$

where

$$V(\theta) = \int_0^{\theta} [\xi(\tau)]^{\frac{-1}{\alpha-1}} \tau^{\frac{1}{\alpha-1}} d\tau.$$

In [20] Oguntuase and Durojaye proved some multidimensional Hardy-type inequalities. In particular, they proved that if ϕ,ψ are nonnegative functions on (c,d), $-\infty < c < d < \infty$ and ψ is a convex function such that $A\psi(\zeta) \leqslant \phi(\zeta) \leqslant B\psi(\zeta)$, where A, B are constants. Furthermore, if ϖ, ξ are nonnegative weighted functions with

$$\xi(\theta) = \left(\int_0^{b_1}\cdots\int_0^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^t \frac{\varpi(\zeta)}{\zeta_1\ldots\zeta_m}d\zeta\right)^{\frac{1}{t}}, \quad t \geqslant 1,$$

then the following inequality holds

$$\begin{aligned}&\int_0^{b_1}\cdots\int_0^{b_m}\phi^t(A_k\eta(\zeta))\varpi(\zeta)\frac{d\zeta}{\zeta_1\ldots\zeta_m}\\ &\leqslant \left(\frac{B}{A}\right)^t\left[\int_0^{b_1}\cdots\int_0^{b_m}\phi(\eta(\theta))\xi(\theta)d\theta\right]^t.\end{aligned} \tag{7}$$

where

$$A_k\eta(\zeta) := \frac{1}{K(\zeta)}\int_0^{\zeta_1}\cdots\int_0^{\zeta_m} k(\zeta,\theta)\eta(\theta)d\theta,$$

and

$$K(\zeta) := \int_0^{\zeta_1}\cdots\int_0^{\zeta_m} k(\zeta,\theta)d\theta,$$

where $\eta(\zeta) = \eta(\zeta_1,\ldots,\zeta_m)$, $k(\zeta,\theta) = k(\zeta_1,\ldots,\zeta_m,\theta_1,\ldots,\theta_m)$ and $d\theta = d\theta_1\ldots d\theta_m$.

A new theorem was appeared to unify the continuous calculus and the discrete calculus called the time scale theorem. The time scale $\mathbb{T}$ is defined as an arbitrary nonempty closed subset of real numbers $\mathbb{R}$. Many authors proved the dynamic inequalities on time scales. Fore more details, see [1, 2, 22, 24, 25, 26, 27].

In 2009, Özkan and Yildirim [23] proved the time scale version of (5) and proved that if $k(\zeta,\theta) \in C_{rd}([a,b) \times [a,b),\mathbb{R})$ and $\varpi \in C_{rd}([a,b),\mathbb{R})$ are nonnegative functions and the function ξ is defined by

$$\xi(\tau) = (\tau - a)\int_{\tau}^{b} \frac{k(\zeta,\tau)}{K(\sigma(\zeta),\zeta)}\varpi(\zeta)\frac{\Delta\zeta}{\zeta - a}, \quad \tau \in [a,b).$$

Furthermore if $\Phi : (c,d) \to \mathbb{R}$ is a continuous and convex function, where $c,\ d \in \mathbb{R}$, then the inequality

$$\int_{a}^{b} \varpi(\zeta)\Phi(A_k\eta(\sigma(\zeta),\zeta))\frac{\Delta\zeta}{\zeta - a} \leqslant \int_{a}^{b} \xi(\zeta)\Phi(\eta(\zeta))\frac{\Delta\zeta}{\zeta - a}, \tag{8}$$

holds for all delta integrable functions $\eta \in C_{rd}([a,b),\mathbb{R})$ such that $\eta(\zeta) \in (c,d)$, where

$$A_k\eta(\tau,s) := \frac{1}{K(\tau,s)}\int_{a}^{\tau} k(s,\theta)\eta(\theta)\Delta\theta, \quad K(\tau,s) := \int_{a}^{\tau} k(s,\theta)\Delta\theta.$$

Our aim in this paper is to establish some new multidimensional Hardy-type inequalities on time scales. In particular, we prove the time scale version of (7). Also, we prove some new multidimensional Hardy-type inequalities in different spaces on time scales.

The paper is organized as follows. In Section 2, we present some preliminaries concerning the time scale calculus and some basic theorems needed in Section 3 where we prove the main results. Our main results (when $\mathbb{T} = \mathbb{R}$) yield the inequality (7) proved by Oguntuase and Durojaye [20].

2. Preliminaries and basic lemmas

In this section, we show some basics of the time scale theory. For more details of time scale calculus we refer the reader to the two books [4], [5]. A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$.

The derivative of the product ηg and the quotient η/g (where $gg^{\sigma} \neq 0$) of two differentiable functions η and g are given by

$$(\eta g)^{\Delta} = \eta^{\Delta}g + \eta^{\sigma}g^{\Delta} = \eta g^{\Delta} + \eta^{\Delta}g^{\sigma}, \quad \left(\frac{\eta}{g}\right)^{\Delta} = \frac{\eta^{\Delta}g - \eta g^{\Delta}}{gg^{\sigma}}. \tag{9}$$

In this paper, we will refer to the (delta) integral which is defined as follows: If $G^{\Delta}(\tau) = g(\tau)$, then $\int_a^{\tau} g(s)\Delta s := G(\tau) - G(a)$. The integration by parts formula on time scales is given by

$$\int_{a}^{b} \varpi(\tau)\xi^{\Delta}(\tau)\Delta\tau = [\varpi(\tau)\xi(\tau)]_a^b - \int_{a}^{b} \varpi^{\Delta}(\tau)\xi^{\sigma}(\tau)\Delta\tau. \tag{10}$$

The time scales chain rule (see [4, Theorem 1.87]) is given by

$$(\eta\circ\lambda)^{\Delta}(\tau)=\eta'(\lambda(\zeta))\lambda^{\Delta}(\tau),\text{ where }\zeta\in[\tau,\sigma(\tau)]. \tag{11}$$

The Hölder inequality on time scales is given by

$$\begin{aligned}&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}|\eta(\tau)g(\tau)|\Delta\tau_1\ldots\Delta\tau_m\\&\leqslant\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}|\eta(\tau)|^{\gamma}\Delta\tau_1\ldots\Delta\tau_m\right]^{\frac{1}{\gamma}}\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}|g(\tau)|^{\nu}\Delta\tau_1\ldots\Delta\tau_m\right]^{\frac{1}{\nu}},\end{aligned} \tag{12}$$

where $a_1,\ldots,a_m,b_1,\ldots,b_m\in\mathbb{T}$, $\Delta\tau=\Delta\tau_1\ldots\Delta\tau_m$, η, $g:\mathbb{T}^m\to\mathbb{R}$ such that $\eta(\tau)=\eta(\tau_1,\tau_2,\ldots,\tau_m)$, $g(\tau)=g(\tau_1,\tau_2,\ldots,\tau_m)$, $\gamma>1$ and $1/\gamma+1/\nu=1$.

THEOREM 1. (Jensen's inequality [8]) *Assume that $\mathbb{T}$ is a time scale with $a_i,b_i\in\mathbb{T}$, $i=1,2,\ldots,m$ and c, $d\in\mathbb{R}$. If $g:\mathbb{T}^m\to(c,d)$ is rd-continuous and $\Phi:(c,d)\to\mathbb{R}$ is continuous and convex, then*

$$\begin{aligned}&\Phi\left(\frac{1}{\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\eta(\zeta,\tau)\Delta\tau}\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\eta(\zeta,\tau)g(\tau)\Delta\tau\right)\\&\leqslant\frac{1}{\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\eta(\zeta,\tau)\Delta\tau}\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\eta(\zeta,\tau)\Phi(g(\tau))\Delta\tau,\end{aligned} \tag{13}$$

where $\Delta\tau=\Delta\tau_1\ldots\Delta\tau_m$, $\eta(\zeta,\tau)=\eta(\zeta_1,\ldots,\zeta_m,\tau_1,\ldots,\tau_m)$, $g(\tau)=g(\tau_1,\ldots,\tau_m)$.

THEOREM 2. (Minkowski's inequality [3]) *Assume that $\mathbb{T}$ is a time scale with $a_i,b_i\in\mathbb{T}$, $i=1,2,\ldots,m$ and $\gamma\geqslant1$. If $k:\mathbb{T}^m\times\mathbb{T}^m\to\mathbb{R}$, $w,h:\mathbb{T}^m\to\mathbb{R}$ are nonnegative rd-continuous functions, then*

$$\begin{aligned}&\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}w(\zeta)\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}h(\tau)k(\zeta,\tau)\Delta\tau\right)^{\gamma}\Delta\zeta\right]^{\frac{1}{\gamma}}\\&\leqslant\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}h(\tau)\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}w(\zeta)k^{\gamma}(\zeta,\tau)\Delta\zeta\right)^{\frac{1}{\gamma}}\Delta\tau,\end{aligned} \tag{14}$$

where $\Delta\zeta=\Delta\zeta_1\ldots\Delta\zeta_m$, $k(\zeta,\tau)=k(\zeta_1,\ldots,\zeta_m,\tau_1,\ldots,\tau_m)$, and $h(\tau)=h(\tau_1,\ldots,\tau_m)$.

3. Main results

Throughout the paper, we will assume that the functions (without mentioning) are positive rd-continuous functions on $[a,b]_{\mathbb{T}}$ and the integrals considered are assumed to exist (finite i.e. convergent). We define the time scale interval $[a,b]_{\mathbb{T}}$ by $[a,b]_{\mathbb{T}}:=[a,b]\cap\mathbb{T}$. Also, we define the general Hardy operator A_k as following

$$A_k\eta(\zeta):=\frac{1}{K(\zeta)}\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k(\zeta,\theta)\eta(\theta)\Delta\theta,$$

and

$$K(\zeta) := \int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\Delta\theta,$$

where $\eta(\zeta) = \eta(\zeta_1,\ldots,\zeta_m)$, $k(\zeta,\theta) = k(\zeta_1,\ldots,\zeta_m,\theta_1,\ldots,\theta_m)$ and $\Delta\theta = \Delta\theta_1 \ldots \Delta\theta_m$.

Now, we can prove the first result.

THEOREM 3. *Let $a_i, b_i \in \mathbb{T}$, $i = 1,2,\ldots m$, $t \geqslant 1$ and k, η, ϖ, ξ are positive weighted functions such that*

$$\xi(\theta) = \left(\int_{\theta_1}^{b_1} \cdots \int_{\theta_m}^{b_m} \left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^t \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta\right)^{\frac{1}{t}}.$$

Furthermore assume that ϕ,ψ are positive functions on (c,d), $-\infty < c < d < \infty$ and ψ is a convex function such that

$$A\psi(\zeta) \leqslant \phi(\zeta) \leqslant B\psi(\zeta), \tag{15}$$

where A, B are constants, then the inequality

$$\begin{aligned} &\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \phi^t\left(A_k\eta(\zeta)\right)\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)} \\ &\leqslant \left(\frac{B}{A}\right)^t \left[\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \phi(\eta(\theta))\xi(\theta)\Delta\theta\right]^t, \end{aligned} \tag{16}$$

holds for the positive function η.

Proof. Using (15) and applying Jensen's inequality (where ψ is convex), we obtain

$$\begin{aligned} &\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \phi^t\left(A_k\eta(\zeta)\right)\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)} \\ &= \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \phi^t\left(\frac{1}{K(\zeta)}\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\eta(\theta)\Delta\theta\right) \\ &\quad \times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta \\ &\leqslant B^t \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \psi^t\left(\frac{1}{K(\zeta)}\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\eta(\theta)\Delta\theta\right) \\ &\quad \times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta \\ &\leqslant B^t \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \frac{1}{K^t(\zeta)}\left(\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta\right)^t \\ &\quad \times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta. \end{aligned} \tag{17}$$

Applying Minkowski's inequality on the term

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\frac{1}{K^t(\zeta)}\left(\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta\right)^t$$
$$\times\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta,$$

with $t\geqslant 1,$ we see that

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\frac{1}{K^t(\zeta)}\left(\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta\right)^t\right.$$
$$\left.\times\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta\right)^{\frac{1}{t}}$$
$$\leqslant\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi(\eta(\theta))\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^t\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta,$$

then

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\frac{1}{K^t(\zeta)}\left(\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta\right)^t$$
$$\times\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta$$
$$\leqslant\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi(\eta(\theta))\right.$$
$$\left.\times\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^t\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta\right]^t. \tag{18}$$

Substituting (18) into (17), we have that

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^t\left(A_k\eta\left(\zeta\right)\right)\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant B^t\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi(\eta(\theta))\right.$$
$$\left.\times\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^t\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta\right)^{\frac{1}{t}}\Delta\theta\right]^t$$
$$=B^t\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi(\eta(\theta))\xi(\theta)\Delta\theta\right]^t,$$

and then by using (15), we observe that

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^t\left(A_k\eta\left(\zeta\right)\right)\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant\left(\frac{B}{A}\right)^t\left[\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi(\eta(\theta))\xi(\theta)\Delta\theta\right]^t,$$

which is (16). $\square$

REMARK 1. As a special case of Theorem 3 when $\mathbb{T}=\mathbb{R}$, we get the inequality (7) proved by Oguntuase and Durojaye [20].

THEOREM 4. *Let $a_i, b_i \in \mathbb{T}$, $0<\alpha$, $s_i<1$, such that $0<\alpha<s_i<\infty$, $i=1,2,\ldots m$, and $1<\beta<\infty$. Also, we assume that ϕ is a positive and convex function on (c,d), $-\infty<c<d<\infty$, and η, k, ϖ, ξ are positive weighted functions with*

$$\xi(\theta)=\xi_1(\theta_1)\ldots\xi_m(\theta_m). \tag{19}$$

Then the inequality

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right)^{\frac{\alpha}{\beta}}$$
$$\leqslant C\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\alpha}, \tag{20}$$

holds for the positive function η and $C>0$, if

$$A(s_1,\ldots,s_m)=\sup_{a_i\leqslant\theta_i\leqslant b_i}\left[V_1^{\sigma}(\theta_1)\right]^{\frac{1-s_1}{\alpha}}\ldots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{1-s_m}{\alpha}}$$
$$\times\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^{\frac{\beta}{\alpha}}\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right.$$
$$\left.\times\left[V_1^{\sigma}(\zeta_1)\right]^{\frac{\beta(s_1-\alpha)}{\alpha}}\ldots\left[V_m^{\sigma}(\zeta_m)\right]^{\frac{\beta(s_m-\alpha)}{\alpha}}\Delta\zeta\right)^{\frac{1}{\beta}}<\infty, \tag{21}$$

where

$$V_i(\theta_i)=\int_{a_i}^{\theta_i}\left[\xi_i(t_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma\left(t_i\right)-a_i\right)^{\frac{1}{1-\alpha}}\Delta t_i.$$

Proof. By applying Jensen's inequality, we get that

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$=\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(\frac{1}{K(\zeta)}\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k(\zeta,\theta)\eta(\theta)\Delta\theta\right)\right]^{\frac{\beta}{\alpha}}$$
$$\times\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta$$
$$\leqslant\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left(\frac{1}{K(\zeta)}\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k(\zeta,\theta)\phi(\eta(\theta))\Delta\theta\right)^{\frac{\beta}{\alpha}}$$
$$\times\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\Delta\zeta$$

$$
\begin{aligned}
&= \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \frac{1}{K^{\frac{\beta}{\alpha}}(\zeta)} \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\phi(\eta(\theta))\Delta\theta \right)^{\frac{\beta}{\alpha}} \\
&\quad \times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)}\Delta\zeta. \qquad (22)
\end{aligned}
$$

Define a function g such that

$$
\phi^{\frac{1}{\alpha}}(\eta(\theta)) \frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\dots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\dots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}} = \phi(g(\theta)), \qquad (23)
$$

and then we have that

$$
\begin{aligned}
&\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\phi(\eta(\theta))\Delta\theta \\
&= \int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\phi^{\alpha}(g(\theta))\left[V_1^{\sigma}(\theta_1)\right]^{1-s_1}\dots\left[V_m^{\sigma}(\theta_m)\right]^{1-s_m}\left[V_1^{\sigma}(\theta_1)\right]^{s_1-1}\dots \\
&\quad \times \left[V_m^{\sigma}(\theta_m)\right]^{s_m-1}(\xi_1(\theta_1))^{-1}\dots(\xi_m(\theta_m))^{-1}(\sigma(\theta_1)-a_1)\dots(\sigma(\theta_m)-a_m)\Delta\theta. \qquad (24)
\end{aligned}
$$

By applying the Hölder inequality (12) with $\gamma = 1/\alpha > 1$ and $\nu = 1/(1-\alpha)$, (where $0 < \alpha < 1$) on the right hand side of (24), we see that

$$
\begin{aligned}
&\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\phi^{\alpha}(g(\theta))\left[V_1^{\sigma}(\theta_1)\right]^{1-s_1}\dots \\
&\quad \times \left[V_m^{\sigma}(\theta_m)\right]^{1-s_m}\left[V_1^{\sigma}(\theta_1)\right]^{s_1-1}\dots\left[V_m^{\sigma}(\theta_m)\right]^{s_m-1} \\
&\quad \times (\xi_1(\theta_1))^{-1}\dots(\xi_m(\theta_m))^{-1}(\sigma(\theta_1)-a_1)\dots(\sigma(\theta_m)-a_m)\Delta\theta \\
&\leqslant \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\phi(g(\theta))\left[V_1^{\sigma}(\theta_1)\right]^{\frac{1-s_1}{\alpha}}\dots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{1-s_m}{\alpha}}\Delta\theta \right)^{\alpha} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} \left[V_1^{\sigma}(\theta_1)\right]^{\frac{s_1-1}{1-\alpha}}\dots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{s_m-1}{1-\alpha}}\left[\xi_1(\theta_1)\right]^{\frac{-1}{1-\alpha}}\dots\left[\xi_m(\theta_m)\right]^{\frac{-1}{1-\alpha}} \right. \\
&\quad \left. \times (\sigma(\theta_1)-a_1)^{\frac{1}{1-\alpha}}\dots(\sigma(\theta_m)-a_m)^{\frac{1}{1-\alpha}}\Delta\theta \right)^{1-\alpha}. \qquad (25)
\end{aligned}
$$

Substituting (25) into (24), we see that

$$
\begin{aligned}
&\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\phi(\eta(\theta))\Delta\theta \\
&\leqslant \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\phi(g(\theta))\left[V_1^{\sigma}(\theta_1)\right]^{\frac{1-s_1}{\alpha}}\dots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{1-s_m}{\alpha}}\Delta\theta \right)^{\alpha} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} \left[V_1^{\sigma}(\theta_1)\right]^{\frac{s_1-1}{1-\alpha}}\dots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{s_m-1}{1-\alpha}}\left[\xi_1(\theta_1)\right]^{\frac{-1}{1-\alpha}} \right. \\
&\quad \left. \times \dots\left[\xi_m(\theta_m)\right]^{\frac{-1}{1-\alpha}}(\sigma(\theta_1)-a_1)^{\frac{1}{1-\alpha}}\dots(\sigma(\theta_m)-a_m)^{\frac{1}{1-\alpha}}\Delta\theta \right)^{1-\alpha},
\end{aligned}
$$

and then we have from (22) that

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\\
&\leqslant\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)K^{\frac{\beta}{\alpha}}(\zeta)}\\
&\quad\times\left(\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k^{\frac{1}{\alpha}}(\zeta,\theta)\phi(g(\theta))\left[V_1^{\sigma}(\theta_1)\right]^{\frac{1-s_1}{\alpha}}\ldots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{1-s_m}{\alpha}}\Delta\theta\right)^{\beta}\\
&\quad\times\left(\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}\left[V_1^{\sigma}(\theta_1)\right]^{\frac{s_1-1}{1-\alpha}}\ldots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{s_m-1}{1-\alpha}}\left[\xi_1(\theta_1)\right]^{\frac{-1}{1-\alpha}}\ldots\left[\xi_m(\theta_m)\right]^{\frac{-1}{1-\alpha}}\right.\\
&\quad\left.\times\left(\sigma(\theta_1)-a_1\right)^{\frac{1}{1-\alpha}}\ldots\left(\sigma(\theta_m)-a_m\right)^{\frac{1}{1-\alpha}}\Delta\theta\right)^{\frac{\beta(1-\alpha)}{\alpha}}\Delta\zeta.
\end{aligned}\tag{26}$$

Since

$$V_i(\theta_i)=\int_{a_i}^{\theta_i}\left[\xi_i(t_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma\left(t_i\right)-a_i\right)^{\frac{1}{1-\alpha}}\Delta t_i,$$

then

$$V_i^{\Delta}(\theta_i)=\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma\left(\theta_i\right)-a_i\right)^{\frac{1}{1-\alpha}}>0. \tag{27}$$

Thus the function V_i is increasing. Applying the chain rule formula (11) on the term $V_i^{1+\frac{s_i-1}{1-\alpha}}(\theta_i)$, we observe that

$$\left[V_i^{1+\frac{s_i-1}{1-\alpha}}(\theta_i)\right]^{\Delta}=\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}=\left(\frac{s_i-\alpha}{1-\alpha}\right)V_i^{\frac{s_i-1}{1-\alpha}}(\zeta_i)V_i^{\Delta}(\theta_i), \tag{28}$$

where $\zeta_i\in[\theta_i,\sigma(\theta_i)]$. Thus by substituting (27) into (28), we see that

$$\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}=\left(\frac{s_i-\alpha}{1-\alpha}\right)V_i^{\frac{s_i-1}{1-\alpha}}(\zeta_i)\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma\left(\theta_i\right)-a_i\right)^{\frac{1}{1-\alpha}}. \tag{29}$$

Since $\zeta_i\leqslant\sigma(\theta_i)$ and V_i is increasing, we have that

$$V_i\left(\zeta_i\right)\leqslant V_i^{\sigma}(\theta_i).$$

Using the facts that $0<s_i,\ \alpha<1$ and $\alpha<s_i<\infty$, where $(s_i-1)/(1-\alpha)<0$ and $s_i-\alpha>0$, we see that

$$V_i^{\frac{s_i-1}{1-\alpha}}\left(\zeta_i\right)\geqslant\left[V_i^{\sigma}(\theta_i)\right]^{\frac{s_i-1}{1-\alpha}}. \tag{30}$$

Substituting (30) into (29), we get (note $(s_i-\alpha)/(1-\alpha)>0$) that

$$\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}\geqslant\left(\frac{s_i-\alpha}{1-\alpha}\right)\left[V_i^{\sigma}(\theta_i)\right]^{\frac{s_i-1}{1-\alpha}}\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma\left(\theta_i\right)-a_i\right)^{\frac{1}{1-\alpha}},$$

and then

$$\int_{a_i}^{\sigma(\zeta_i)} \left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta} \Delta\theta_i \geqslant \left(\frac{s_i-\alpha}{1-\alpha}\right) \int_{a_i}^{\sigma(\zeta_i)} [V_i^{\sigma}(\theta_i)]^{\frac{s_i-1}{1-\alpha}} [\xi_i(\theta_i)]^{\frac{-1}{1-\alpha}} (\sigma(\theta_i)-a_i)^{\frac{1}{1-\alpha}} \Delta\theta_i.$$

Thus (since $V_i(a_i)=0$)

$$\begin{aligned}
&\int_{a_i}^{\sigma(\zeta_i)} [V_i^{\sigma}(\theta_i)]^{\frac{s_i-1}{1-\alpha}} [\xi_i(\theta_i)]^{\frac{-1}{1-\alpha}} (\sigma(\theta_i)-a_i)^{\frac{1}{1-\alpha}} \Delta\theta_i \\
&\leqslant \left(\frac{1-\alpha}{s_i-\alpha}\right) \int_{a_i}^{\sigma(\zeta_i)} \left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta} \Delta\theta_i \\
&= \left(\frac{1-\alpha}{s_i-\alpha}\right) [V_i^{\sigma}(\zeta_i)]^{\frac{s_i-\alpha}{1-\alpha}}, \quad i=1,2,\ldots,m.
\end{aligned} \tag{31}$$

Substituting (31) into (26), we have

$$\begin{aligned}
&\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} [\phi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}} \varpi(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)} \\
&\leqslant \left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}} \cdots \left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}} \\
&\quad \times \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m) K^{\frac{\beta}{\alpha}}(\zeta)} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\phi(g(\theta)) [V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}} \ldots [V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}} \Delta\theta\right)^{\beta} \\
&\quad \times [V_1^{\sigma}(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \ldots [V_m^{\sigma}(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta
\end{aligned} \tag{32}$$

From (32) and the definition of g in (23), we obtain

$$\begin{aligned}
&\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} [\phi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}} \varpi(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)} \\
&\leqslant \left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}} \cdots \left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}} \\
&\quad \times \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m) K^{\frac{\beta}{\alpha}}(\zeta)} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\phi^{\frac{1}{\alpha}}(\eta(\theta)) \frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\right. \\
&\quad \left.\times [V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}} \ldots [V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}} \Delta\theta\right)^{\beta} [V_1^{\sigma}(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \ldots [V_m^{\sigma}(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta.
\end{aligned} \tag{33}$$

Applying the Minkowski inequality on the right hand side (33) with $\beta > 1$, we get

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)K^{\frac{\beta}{\alpha}}(\zeta)}\\
&\quad\times\left(\int_{a_1}^{\sigma(\zeta_1)}\cdots\int_{a_m}^{\sigma(\zeta_m)}k^{\frac{1}{\alpha}}(\zeta,\theta)\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\right.\\
&\quad\left.\times[V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}}\ldots[V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}}\Delta\theta\right)^{\beta}[V_1^{\sigma}(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}}\ldots[V_m^{\sigma}(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}}\Delta\zeta\\
&\leqslant\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\right.\\
&\quad\times[V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}}\ldots[V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}}\\
&\quad\times\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^{\frac{\beta}{\alpha}}\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right.\\
&\quad\left.\left.\times[V_1^{\sigma}(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}}\ldots[V_m^{\sigma}(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}}\Delta\zeta\right)^{\frac{1}{\beta}}\Delta\theta\right)^{\beta} \qquad (34)
\end{aligned}$$

Substituting (34) into (33), we get

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}[\phi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\\
&\leqslant\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\ldots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\\
&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\right.\\
&\quad\times[V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}}\ldots[V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}}\\
&\quad\times\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^{\frac{\beta}{\alpha}}\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right.\\
&\quad\left.\left.\times[V_1^{\sigma}(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}}\ldots[V_m^{\sigma}(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}}\Delta\zeta\right)^{\frac{1}{\beta}}\Delta\theta\right)^{\beta},
\end{aligned}$$

then we have from (21) that

$$\begin{aligned}
&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}[\phi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\\
&\leqslant\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\ldots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}A^{\beta}(s_1,\ldots,s_m)
\end{aligned}$$

$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\beta},$$

and then we observe from (19) that

$$\begin{aligned}&\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\\&\leqslant\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}A^{\beta}(s_1,\ldots,s_m)\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\beta},\end{aligned}$$

thus

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right)^{\frac{\alpha}{\beta}}\\&\leqslant\left(\frac{1-\alpha}{s_1-\alpha}\right)^{1-\alpha}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{1-\alpha}A^{\alpha}(s_1,\ldots,s_m)\\&\quad\times\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\alpha},\end{aligned}$$

which satisfies (20) with $C=\left(\frac{1-\alpha}{s_1-\alpha}\right)^{1-\alpha}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{1-\alpha}A^{\alpha}(s_1,\ldots,s_m)$. $\square$

In the following, we generalize Theorem 4.

THEOREM 5. *Let $a_i,b_i\in\mathbb{T}$, $0<\alpha$, $s_i<1$, $i=1,2,\ldots m$, such that $0<\alpha<s_i<\infty$ and $1<\beta<\infty$. Also, we assume that ϕ,ψ are positive functions on (c,d), $-\infty<c<d<\infty$ and ψ is a convex function such that*

$$A\psi(\zeta)\leqslant\phi(\zeta)\leqslant B\psi(\zeta),\tag{35}$$

where A, B are positive constants and η, k, ϖ, ξ are positive weighted functions such that (19) holds. Then the inequality

$$\begin{aligned}&\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right)^{\frac{\alpha}{\beta}}\\&\leqslant C\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\alpha},\end{aligned}\tag{36}$$

holds for the positive function η *and* $C > 0$, *if*

$$D(s_1,\dots,s_m) = \sup_{a_i \leqslant \theta_i \leqslant b_i} [V_1^\sigma(\theta_1)]^{\frac{1-s_1}{\alpha}} \dots [V_m^\sigma(\theta_m)]^{\frac{1-s_m}{\alpha}}$$
$$\times \left(\int_{\theta_1}^{b_1} \dots \int_{\theta_m}^{b_m} \left(\frac{k(\zeta,\theta)}{K(\zeta)} \right)^{\frac{\beta}{\alpha}} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)} \right.$$
$$\left. \times [V_1^\sigma(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \dots [V_m^\sigma(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta \right)^{\frac{1}{\beta}} < \infty. \tag{37}$$

Proof. Using (35) and by applying Jensen's inequality, we get that

$$\int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} [\phi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}} \varpi(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant B^{\frac{\beta}{\alpha}} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} [\psi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}} \varpi(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)}$$
$$= B^{\frac{\beta}{\alpha}} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \left[\psi \left(\frac{1}{K(\zeta)} \int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\eta(\theta)\Delta\theta \right) \right]^{\frac{\beta}{\alpha}}$$
$$\times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)} \Delta\zeta$$
$$\leqslant B^{\frac{\beta}{\alpha}} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \left(\frac{1}{K(\zeta)} \int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta \right)^{\frac{\beta}{\alpha}}$$
$$\times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)} \Delta\zeta$$
$$= B^{\frac{\beta}{\alpha}} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \frac{1}{K^{\frac{\beta}{\alpha}}(\zeta)} \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta \right)^{\frac{\beta}{\alpha}}$$
$$\times \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)} \Delta\zeta. \tag{38}$$

Define a function g such that

$$\psi^{\frac{1}{\alpha}}(\eta(\theta)) \frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\dots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}} \dots (\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}} = \psi(g(\theta)), \tag{39}$$

and then we have that

$$\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta$$
$$= \int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi^\alpha(g(\theta)) [V_1^\sigma(\theta_1)]^{1-s_1} \dots [V_m^\sigma(\theta_m)]^{s_m-1} [V_1^\sigma(\theta_1)]^{s_1-1}$$
$$\times \dots [V_m^\sigma(\theta_m)]^{s_m-1} (\xi_1(\theta_1))^{-1} \dots (\xi_m(\theta_m))^{-1} (\sigma(\theta_1)-a_1)\dots(\sigma(\theta_m)-a_m)\Delta\theta. \tag{40}$$

By applying (12) with $\gamma = 1/\alpha > 1$ and $\nu = 1/(1-\alpha)$, (where $0 < \alpha < 1$) on the right hand side of (40), we see that

$$\begin{aligned}
&\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi^{\alpha}(g(\theta))\,[V_1^{\sigma}(\theta_1)]^{1-s_1} \dots [V_m^{\sigma}(\theta_m)]^{1-s_m}\,[V_1^{\sigma}(\theta_1)]^{s_1-1} \\
&\quad \times \dots [V_m^{\sigma}(\theta_m)]^{s_m-1}\,(\xi_1(\theta_1))^{-1} \dots (\xi_m(\theta_m))^{-1}\,(\sigma(\theta_1)-a_1)\dots(\sigma(\theta_m)-a_m)\Delta\theta \\
&\leqslant \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\psi(g(\theta))\,[V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}} \dots [V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}}\,\Delta\theta\right)^{\alpha} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} [V_1^{\sigma}(\theta_1)]^{\frac{s_1-1}{1-\alpha}} \dots [V_m^{\sigma}(\theta_m)]^{\frac{s_m-1}{1-\alpha}}\,[\xi_1(\theta_1)]^{\frac{-1}{1-\alpha}}\right. \\
&\quad \left. \times \dots [\xi_m(\theta_m)]^{\frac{-1}{1-\alpha}}\,(\sigma(\theta_1)-a_1)^{\frac{1}{1-\alpha}} \dots (\sigma(\theta_m)-a_m)^{\frac{1}{1-\alpha}}\,\Delta\theta\right)^{1-\alpha}.
\end{aligned} \tag{41}$$

Substituting (41) into (40), we see that

$$\begin{aligned}
&\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k(\zeta,\theta)\psi(\eta(\theta))\Delta\theta \\
&\leqslant \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\psi(g(\theta))\,[V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}} \dots [V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}}\,\Delta\theta\right)^{\alpha} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} [V_1^{\sigma}(\theta_1)]^{\frac{s_1-1}{1-\alpha}} \dots [V_m^{\sigma}(\theta_m)]^{\frac{s_m-1}{1-\alpha}}\,[\xi_1(\theta_1)]^{\frac{-1}{1-\alpha}}\right. \\
&\quad \left. \times \dots [\xi_m(\theta_m)]^{\frac{-1}{1-\alpha}}\,(\sigma(\theta_1)-a_1)^{\frac{1}{1-\alpha}} \dots (\sigma(\theta_m)-a_m)^{\frac{1}{1-\alpha}}\,\Delta\theta\right)^{1-\alpha},
\end{aligned}$$

and then we have from (38) that

$$\begin{aligned}
&\int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} [\phi\left(A_k\eta\left(\zeta\right)\right)]^{\frac{\beta}{\alpha}}\,\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)} \\
&\leqslant B^{\frac{\beta}{\alpha}} \int_{a_1}^{b_1} \dots \int_{a_m}^{b_m} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)K^{\frac{\beta}{\alpha}}(\zeta)} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\psi(g(\theta))\,[V_1^{\sigma}(\theta_1)]^{\frac{1-s_1}{\alpha}} \dots [V_m^{\sigma}(\theta_m)]^{\frac{1-s_m}{\alpha}}\,\Delta\theta\right)^{\beta} \\
&\quad \times \left(\int_{a_1}^{\sigma(\zeta_1)} \dots \int_{a_m}^{\sigma(\zeta_m)} [V_1^{\sigma}(\theta_1)]^{\frac{s_1-1}{1-\alpha}} \dots [V_m^{\sigma}(\theta_m)]^{\frac{s_m-1}{1-\alpha}}\,[\xi_1(\theta_1)]^{\frac{-1}{1-\alpha}}\right. \\
&\quad \left. \times \dots [\xi_m(\theta_m)]^{\frac{-1}{1-\alpha}}\,(\sigma(\theta_1)-a_1)^{\frac{1}{1-\alpha}} \dots (\sigma(\theta_m)-a_m)^{\frac{1}{1-\alpha}}\,\Delta\theta\right)^{\frac{\beta(1-\alpha)}{\alpha}} \Delta\zeta.
\end{aligned} \tag{42}$$

Since

$$V_i(\theta_i) = \int_{a_i}^{\theta_i} [\xi_i(t_i)]^{\frac{-1}{1-\alpha}}\,(\sigma\left(t_i\right)-a_i)^{\frac{1}{1-\alpha}}\,\Delta t_i,$$

then

$$V_i^{\Delta}(\theta_i) = [\xi_i(\theta_i)]^{\frac{-1}{1-\alpha}}\,(\sigma\left(\theta_i\right)-a_i)^{\frac{1}{1-\alpha}} > 0. \tag{43}$$

Thus the function V_i is increasing. Applying the chain rule formula (11) on the term $V_i^{1+\frac{s_i-1}{1-\alpha}}(\theta_i)$, we observe that

$$\left[V_i^{1+\frac{s_i-1}{1-\alpha}}(\theta_i)\right]^{\Delta}=\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}=\left(\frac{s_i-\alpha}{1-\alpha}\right)V_i^{\frac{s_i-1}{1-\alpha}}(\zeta_i)V_i^{\Delta}(\theta_i), \tag{44}$$

where $\zeta_i\in[\theta_i,\sigma(\theta_i)]$. Thus by substituting (43) into (44), we see that

$$\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}=\left(\frac{s_i-\alpha}{1-\alpha}\right)V_i^{\frac{s_i-1}{1-\alpha}}(\zeta_i)\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma(\theta_i)-a_i\right)^{\frac{1}{1-\alpha}}. \tag{45}$$

Since $\zeta_i\leqslant\sigma(\theta_i)$ and V_i is increasing, we have that

$$V_i(\zeta_i)\leqslant V_i^{\sigma}(\theta_i).$$

Using the facts that $0<s_i,\ \alpha<1$ and $\alpha<s_i<\infty$, where $(s_i-1)/(1-\alpha)<0$ and $s_i-\alpha>0$, we see that

$$V_i^{\frac{s_i-1}{1-\alpha}}(\zeta_i)\geqslant\left[V_i^{\sigma}(\theta_i)\right]^{\frac{s_i-1}{1-\alpha}}. \tag{46}$$

Substituting (46) into (45), we get (note $(s_i-\alpha)/(1-\alpha)>0$) that

$$\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}\geqslant\left(\frac{s_i-\alpha}{1-\alpha}\right)\left[V_i^{\sigma}(\theta_i)\right]^{\frac{s_i-1}{1-\alpha}}\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma(\theta_i)-a_i\right)^{\frac{1}{1-\alpha}},$$

and then

$$\begin{aligned}&\int_{a_i}^{\sigma(\zeta_i)}\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}\Delta\theta_i\\&\geqslant\left(\frac{s_i-\alpha}{1-\alpha}\right)\int_{a_i}^{\sigma(\zeta_i)}\left[V_i^{\sigma}(\theta_i)\right]^{\frac{s_i-1}{1-\alpha}}\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma(\theta_i)-a_i\right)^{\frac{1}{1-\alpha}}\Delta\theta_i.\end{aligned}$$

Thus (since $V_i(a_i)=0$)

$$\begin{aligned}&\int_{a_i}^{\sigma(\zeta_i)}\left[V_i^{\sigma}(\theta_i)\right]^{\frac{s_i-1}{1-\alpha}}\left[\xi_i(\theta_i)\right]^{\frac{-1}{1-\alpha}}\left(\sigma(\theta_i)-a_i\right)^{\frac{1}{1-\alpha}}\Delta\theta_i\\&\leqslant\left(\frac{1-\alpha}{s_i-\alpha}\right)\int_{a_i}^{\sigma(\zeta_i)}\left[V_i^{\frac{s_i-\alpha}{1-\alpha}}(\theta_i)\right]^{\Delta}\Delta\theta_i\\&=\left(\frac{1-\alpha}{s_i-\alpha}\right)\left[V_i^{\sigma}(\zeta_i)\right]^{\frac{s_i-\alpha}{1-\alpha}},\quad i=1,2,\ldots,m.\end{aligned} \tag{47}$$

Substituting (47) into (42), we have

$$\begin{aligned}&\int_{a_1}^{b_1}\ldots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta(\zeta)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\\&\leqslant B^{\frac{\beta}{\alpha}}\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\ldots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\end{aligned}$$

$$\times \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)K^{\frac{\beta}{\alpha}}(\zeta)}$$
$$\times \left(\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\psi(g(\theta))\,[V_1^\sigma(\theta_1)]^{\frac{1-s_1}{\alpha}} \ldots [V_m^\sigma(\theta_m)]^{\frac{1-s_m}{\alpha}} \Delta\theta \right)^\beta$$
$$\times [V_1^\sigma(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \ldots [V_m^\sigma(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta. \tag{48}$$

From (48) and the definition of g in (39), we obtain

$$\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} [\phi(A_k\eta(\zeta))]^{\frac{\beta}{\alpha}} \varpi(\zeta) \frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant B^{\frac{\beta}{\alpha}} \left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}} \cdots \left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}$$
$$\times \int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)K^{\frac{\beta}{\alpha}}(\zeta)}$$
$$\times \left(\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\psi^{\frac{1}{\alpha}}(\eta(\theta)) \frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}} \right.$$
$$\left. \times [V_1^\sigma(\theta_1)]^{\frac{1-s_1}{\alpha}} \ldots [V_m^\sigma(\theta_m)]^{\frac{1-s_m}{\alpha}} \Delta\theta \right)^\beta [V_1^\sigma(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \ldots [V_m^\sigma(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta. \tag{49}$$

Applying the Minkowski inequality on the right hand side (49) with $\beta > 1$, we get

$$\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)K^{\frac{\beta}{\alpha}}(\zeta)}$$
$$\times \left(\int_{a_1}^{\sigma(\zeta_1)} \cdots \int_{a_m}^{\sigma(\zeta_m)} k^{\frac{1}{\alpha}}(\zeta,\theta)\psi^{\frac{1}{\alpha}}(\eta(\theta)) \frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}} \right.$$
$$\left. \times [V_1^\sigma(\theta_1)]^{\frac{1-s_1}{\alpha}} \ldots [V_m^\sigma(\theta_m)]^{\frac{1-s_m}{\alpha}} \Delta\theta \right)^\beta [V_1^\sigma(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \ldots [V_m^\sigma(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta$$
$$\leqslant \left(\int_{a_1}^{b_1} \cdots \int_{a_m}^{b_m} \psi^{\frac{1}{\alpha}}(\eta(\theta)) \frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}} \right.$$
$$\times [V_1^\sigma(\theta_1)]^{\frac{1-s_1}{\alpha}} \ldots [V_m^\sigma(\theta_m)]^{\frac{1-s_m}{\alpha}}$$
$$\times \left(\int_{\theta_1}^{b_1} \cdots \int_{\theta_m}^{b_m} \left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^{\frac{\beta}{\alpha}} \frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)} \right.$$
$$\left.\left. \times [V_1^\sigma(\zeta_1)]^{\frac{\beta(s_1-\alpha)}{\alpha}} \ldots [V_m^\sigma(\zeta_m)]^{\frac{\beta(s_m-\alpha)}{\alpha}} \Delta\zeta \right)^{\frac{1}{\beta}} \Delta\theta \right)^\beta. \tag{50}$$

Substituting (50) into (49), we get

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant B^{\frac{\beta}{\alpha}}\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}$$
$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\right.$$
$$\times\left[V_1^{\sigma}(\theta_1)\right]^{\frac{1-s_1}{\alpha}}\ldots\left[V_m^{\sigma}(\theta_m)\right]^{\frac{1-s_m}{\alpha}}$$
$$\times\left(\int_{\theta_1}^{b_1}\cdots\int_{\theta_m}^{b_m}\left(\frac{k(\zeta,\theta)}{K(\zeta)}\right)^{\frac{\beta}{\alpha}}\frac{\varpi(\zeta)}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right.$$
$$\left.\left.\times\left[V_1^{\sigma}(\zeta_1)\right]^{\frac{\beta(s_1-\alpha)}{\alpha}}\ldots\left[V_m^{\sigma}(\zeta_m)\right]^{\frac{\beta(s_m-\alpha)}{\alpha}}\Delta\zeta\right)^{\frac{1}{\beta}}\Delta\theta\right)^{\beta},$$

and then we have from (37) that

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant B^{\frac{\beta}{\alpha}}\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}D^{\beta}(s_1,\ldots,s_m)$$
$$\times\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi_1^{\frac{1}{\alpha}}(\theta_1)\ldots\xi_m^{\frac{1}{\alpha}}(\theta_m)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\beta},$$

thus we have from (19) that

$$\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}$$
$$\leqslant B^{\frac{\beta}{\alpha}}\left(\frac{1-\alpha}{s_1-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{\frac{\beta(1-\alpha)}{\alpha}}D^{\beta}(s_1,\ldots,s_m)$$
$$\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\psi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\ldots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\beta}$$

therefore

$$\left(\int_{a_1}^{b_1}\cdots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\ldots(\sigma(\zeta_m)-a_m)}\right)^{\frac{\alpha}{\beta}}$$
$$\leqslant B\left(\frac{1-\alpha}{s_1-\alpha}\right)^{1-\alpha}\cdots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{1-\alpha}D^{\alpha}(s_1,\ldots,s_m)$$

$$\left(\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\psi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\dots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\alpha}.$$

From (35), the last inequality becomes

$$\left(\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\left[\phi\left(A_k\eta\left(\zeta\right)\right)\right]^{\frac{\beta}{\alpha}}\varpi(\zeta)\frac{\Delta\zeta}{(\sigma(\zeta_1)-a_1)\dots(\sigma(\zeta_m)-a_m)}\right)^{\frac{\alpha}{\beta}}$$
$$\leqslant\frac{B}{A}\left(\frac{1-\alpha}{s_1-\alpha}\right)^{1-\alpha}\dots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{1-\alpha}D^{\alpha}(s_1,\dots,s_m)$$
$$\times\left(\int_{a_1}^{b_1}\dots\int_{a_m}^{b_m}\phi^{\frac{1}{\alpha}}(\eta(\theta))\frac{\xi^{\frac{1}{\alpha}}(\theta)}{(\sigma(\theta_1)-a_1)^{\frac{1}{\alpha}}\dots(\sigma(\theta_m)-a_m)^{\frac{1}{\alpha}}}\Delta\theta\right)^{\alpha},$$

which satisfies (36) with $C=(B/A)\left(\frac{1-\alpha}{s_1-\alpha}\right)^{1-\alpha}\dots\left(\frac{1-\alpha}{s_m-\alpha}\right)^{1-\alpha}D^{\alpha}(s_1,\dots,s_m)$. □

REMARK 2. If $A=B=1$, we get Theorem 4.

REFERENCES

[1] R. P. AGARWAL, D. O'REGAN, S. H. SAKER, *Dynamic Inequalities on Time Scales*, Springer Cham Heidelberg New York Dordrechet London, 2014.

[2] R. P. AGARWAL, D. O'REGAN, S. H. SAKER, *Hardy Type Inequalities on Time Scales*, Springer, 2016.

[3] R. BIBI, M. BOHNER, J. PEČARIĆ, S. VAROŠANEC, *Minkowski and Beckenbach-Dresher inequalities and functionals on time scales*, J. Math. Inequal. **7** (2013), no. 3, 299–312.

[4] M. BOHNER, A. PETERSON, *Dynamic Equations on Time Scales: An Introduction with Applications*, Birkhäuser, Boston, Mass, USA, 2001.

[5] M. BOHNER, A. PETERSON, *Advances in Dynamic Equations on Time Scales*, Birkhäuser, Boston, 2003.

[6] A. CIŽMEŠIJA, J. PEČARIĆ, L. E. PERSSON, *On strengthened Hardy and Pólya-Knopp's inequalities*, J. Approx. Theory **125**, (2003), 74–84.

[7] E. T. COPSON, *Note on series of positive terms*, J. Lond. Math. Soc. **3** (1927), 49–51.

[8] O. O. FABELURIN, J. A. OGUNTUASE, L. E. PERSSON, *Multidimensional Hardy-type inequalities on time scales with variable exponents*, J. Math. Ineq. **13** (2019), 725–736.

[9] G. H. HARDY, *Notes on some points in the integral calculus* $(LXIV)$, Mess. Math. **57** (1928), 12–16.

[10] G. H. HARDY, J. E. LITTLEWOOD, *Elementary theorems concerning power series with positive coefficents and moment constants of positive functions*, J. Für Math. **157** (1927), 141–158.

[11] H. P. HEINIG, *Weighted norm inequalities for certain integral operators II*, Proc. Amer. Math. Soc. **95** (1985), no. 3, 387–395.

[12] H. HEINIG, L. MALIGRANDA, *Weighted inequalities for monotone and concave functions*, Studia Math. **116** (1995), 133–165.

[13] S. KAIJSER, L. E. PERSSON, A. ÖBERG, *On Carleman and Knopp's inequalities*, J. Approx. Theory **117**, (2002), 140–151.

[14] S. KAIJSER, L. NIKOLOVA, L. E. PERSSON, A. WEDESTIG, *Hardy type inequalities via convexity*, Math. Inequal. Appl. **8** (2005), no. 3, 403–417.

[15] K. KNOPP, *Über Reihen mit positiven Gliedern*, J. London. Math. Soc. 3 (1928), 205–311.

[16] A. KUFNER, L. E. PERSSON, *Weighted Inequalities of Hardy Type*, World Scientific Publishing, 2003.

[17] A. KUFNER, L. MALIGRANDA, L. E. PERSSON, *The Hardy Inequalities: About its History and Some Related Results*, Vydavatelsk servis, 2007.

[18] S. LAI, *Weighted norm inequalities for general operators on monotone functions*, Transactions of the American Mathematical Society, 340 (2) (1993), 811–836.
[19] L. LEINDLER, *Generalization of inequalities of Hardy and Littlewood*, Acta Sci. Math. (Szeged), **31** (1970), 285–297.
[20] J. A. OGUNTUASE AND P. DUROJAYE, *Some new multidimensional hardy-type inequalities with kernels via convexity*, Publications de l'Institut Mathematique, 93 (107), (2013), 153–164.
[21] B. OPIC A. KUFNER, *Hardy-Type Inequalities*, Pitman Research Notes in Mathematics Series, Longman Scientific and Technical, Harlow, 1990.
[22] U. M. ÖZKAN, H. YILDIRIM, *Hardy-Knopp type inequalities on time scales*, Dyn. Syst. Appl. **17** (2008), 477–486.
[23] U. M. ÖZKAN, H. YILDIRIM, *Time scale Hardy-Knopp type integral inequalities*, Communications in Mathematical Analysis, 6 (1), (2009).
[24] S. H. SAKER, R. R. MAHMOUD, A. PETERSON, *Weighted Hardy-type inequalities on time scales with applications*, Mediterr. J. Math. **13** (2015), no. 2, 585–606.
[25] S. H. SAKER, E. AWWAD AND A. SAIED, *Some new dynamic inequalities involving monotonic functions on time scales*, Journal of Function Spaces, 2019, (2019).
[26] S. H. SAKER, A. I. SAIED AND M. KRNIĆ, *Some New Weighted Dynamic Inequalities for Monotone Functions Involving Kernels*, Mediterranean Journal of Mathematics, 17 (2), (2020), 1–18.
[27] S. H. SAKER, A. I. SAIED AND M. KRNIĆ, *Some new dynamic Hardy-type inequalities with kernels involving monotone functions*, Revista de la Real Academia de Ciencias Exactas, Fisicas y Naturales, Serie A. Matematicas, 114, (2020), 1–16.
[28] G. J. SINNAMON, WEIGHTED HARDY, *Opial-type inequalities*, J. Math. Anal. Appl. **160** (1991), 434–445.
[29] V. D. STEPANOV, *Boundedness of linear integral operators on a class of monotone functions*, Siberian Math. J. **32** (1991), 540–542.

(Received August 19, 2021)

E. Awwad
Department of Mathematics
College of Science and Arts, Jouf University
Jouf, Saudi Arabia
e-mail: emawwad@ju.edu.sa
and
Department of Mathematics
Faculty of Science, Benha University
Benha 13518, Egypt
e-mail: essam.awad@fsc.bu.edu.eg

A. I. Saied
Department of Mathematics
Faculty of Science, Benha University
Benha 13518, Egypt
e-mail: as0863289@gmail.com