

## SOME REFINEMENTS OF NUMERICAL RADIUS INEQUALITIES FOR $2 \times 2$ OPERATOR MATRICES

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(Communicated by T. Burić)

*Abstract.* In this paper, new numerical radius inequalities for  $2 \times 2$  operator matrices are proved. These numerical radius inequalities refine the existing upper bounds.

### 1. Introduction

Let  $(H, \langle \cdot, \cdot \rangle)$  be a nontrivial complex Hilbert space and  $\mathcal{B}(H)$  denote the  $C^*$ -algebra of all bounded linear operators on  $H$ . For  $T \in \mathcal{B}(H)$ , recall that the numerical radius and the operator norm are denoted as

$$\omega(T) = \sup\{|\langle Tx, x \rangle| : x \in H, \|x\| = 1\},$$

and

$$\begin{aligned} \|T\| &= \sup\{\|Tx\| : x \in H, \|x\| = 1\} \\ &= \sup\{|\langle Tx, y \rangle| : x, y \in H, \|x\| = \|y\| = 1\}. \end{aligned}$$

Denote  $|T| = (T^*T)^{\frac{1}{2}}$  be the absolute value of  $T \in \mathcal{B}(H)$ . Then we have

$$\omega(|T|) = \||T|\| = \|T\|.$$

It is clear that  $\omega(\cdot)$  defines an operator norm on  $\mathcal{B}(H)$  which is equivalent to the operator norm  $\|\cdot\|$ , where we have

$$\frac{1}{2}\|T\| \leq \omega(T) \leq \|T\|, \text{ for } T \in \mathcal{B}(H). \quad (1.1)$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if  $T^2 = 0$  and the second inequality becomes an equality if  $T^*T = TT^*$ .

*Mathematics subject classification* (2020): 47A05, 47A12, 47A30.

*Keywords and phrases:* Numerical radius, inequality, operator norm, operator matrix.

The research is supported by the NNSFs of China (Grant Nos. 11761052, 11862019), NSF of Inner Mongolia (Grant No. 2020ZD01).

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For  $\omega(T)$ , an important inequality is the power inequality, which asserts that

$$\omega(T^n) \leq \omega^n(T), \text{ for } n = 1, 2, \dots$$

In [1], Dragomir refined the second inequality in (1.1), and obtained that

$$\omega^2(T) \leq \frac{1}{2}(\omega(T^2) + \|T\|^2). \quad (1.2)$$

Another improvement of the second inequality in (1.1) has been established by Kitaneh in [2]. This refinement asserts that if  $T \in \mathcal{B}(H)$ , then

$$\omega^2(T) \leq \frac{1}{2} \|\ |T|^2 + |T^*|^2 \|. \quad (1.3)$$

It is obvious that the inequality (1.3) is a special case of the following more general form in [3]:

$$\omega^{2r}(T) \leq \frac{1}{2} \|\ |T|^{2r} + |T^*|^{2r} \| \text{ for } r \geq 1. \quad (1.4)$$

In [4], Dragomir showed the following numerical radius inequality involving the product of two operators:

$$\omega^r(T^*S) \leq \frac{1}{2} \|\ |T|^{2r} + |S|^{2r} \| \text{ for } r \geq 1. \quad (1.5)$$

There are more such inequalities, we refer the readers to [5, 6, 7, 8] and the references therein.

It should be mentioned that the direct sum of two copies of  $H$  is denoted by  $H^2 = H \oplus H$ . For  $P, Q, T$  and  $S \in \mathcal{B}(H)$ , the operator matrix  $W = \begin{bmatrix} P & T \\ S & Q \end{bmatrix}$  can be considered as an operator in  $\mathcal{B}(H \oplus H)$ , which is defined by  $Wx = \begin{bmatrix} Px_1 + Tx_2 \\ Sx_1 + Qx_2 \end{bmatrix}$  for every vector  $x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in H \oplus H$ .

For operator matrix  $\begin{bmatrix} P & T \\ S & Q \end{bmatrix}$ , Bani-Domi et. al. proved the following inequality (see [11]), which asserts

$$\begin{aligned} & \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\ & \leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + 3 \max \{ \|\ |T|^4 + |S^*|^4 \|, \|\ |T^*|^4 + |S|^4 \| \} \\ & \quad + \max \{ \|\ |T|^2 + |S^*|^2 \|, \|\ |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \}. \end{aligned} \quad (1.6)$$

In [12], it was shown that

$$\begin{aligned} & \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\ & \leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + (1 + \alpha) \max \{ \|\ |T|^4 + |S^*|^4 \|, \|\ |T^*|^4 + |S|^4 \| \} \\ & \quad + (3 - \alpha) \max \{ \|\ |T|^2 + |S^*|^2 \|, \|\ |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \}. \end{aligned} \quad (1.7)$$

In this paper, we use the extensions of Schwarz inequality to present several new upper bounds for the numerical radius of  $2 \times 2$  operator matrices which refine the inequalities (1.6) and (1.7). In addition, for  $T \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ , we prove that

$$\omega^4(T) \leq \frac{(1 + \alpha)}{4} \| |T|^4 + |T^*|^4 \| + \frac{(1 - \alpha)}{2} \omega^2(T^2)$$

and

$$\omega^4(T) \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2 + \alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1 - \alpha)}{4} \omega^2(T^2).$$

It should be mentioned that our numerical radius inequalities here improve the existing ones in [10, 11, 12].

### 2. Preliminaries

To obtain the desired results of this paper, we need the following lemmas (see [13, 14, 15]).

LEMMA 2.1. *Let  $T \in \mathcal{B}(H)$  be a positive operator, and let  $x \in H$  be any unit vector. Then*

$$\langle Tx, x \rangle^r \leq \langle T^r x, x \rangle \text{ for } r \geq 1.$$

LEMMA 2.2. *Let  $f$  be a non-negative, convex function on  $[0, \infty)$ , and let  $T, S \in \mathcal{B}(H)$  be positive operators. Then*

$$\left\| f\left(\frac{T+S}{2}\right) \right\| \leq \left\| \frac{f(T)+f(S)}{2} \right\|.$$

Particularly, for  $r \geq 1$ , it holds

$$\left\| \left(\frac{T+S}{2}\right)^r \right\| \leq \left\| \frac{T^r+S^r}{2} \right\|.$$

LEMMA 2.3. *Let  $a, b, c \in H$ . Then*

$$|\langle a, b \rangle|^2 + |\langle a, c \rangle|^2 \leq \|a\|^2 \sqrt{|\langle b, b \rangle|^2 + |\langle c, c \rangle|^2 + 2|\langle b, c \rangle|^2}.$$

The proofs of the following two lemmas depend on the Buzano extension of the Schwarz inequality.

LEMMA 2.4. *Let  $x, y, e \in H$  with  $\|e\| = 1$  and  $0 \leq \alpha \leq 1$ . Then*

$$|\langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{1}{2} \left( (1 + \alpha) \|x\|^2 \|y\|^2 + (1 - \alpha) |\langle x, y \rangle|^2 \right).$$

*Proof.* In [9], Buzano shows the following extension of the Schwarz inequality:

$$|\langle x, e \rangle \langle e, y \rangle| \leq \frac{1}{2} (\|x\| \|y\| + |\langle x, y \rangle|), \quad (2.1)$$

where  $x, y, e \in H$  with  $\|e\| = 1$ .

On the other hand, by the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle x, y \rangle| &= \alpha |\langle x, y \rangle| + (1 - \alpha) |\langle x, y \rangle| \\ &\leq \alpha \|x\| \|y\| + (1 - \alpha) |\langle x, y \rangle|. \end{aligned}$$

Then, by utilizing the Power-Mean inequality, it holds

$$\begin{aligned} |\langle x, y \rangle|^2 &\leq (\alpha \|x\| \|y\| + (1 - \alpha) |\langle x, y \rangle|)^2 \\ &\leq \alpha \|x\|^2 \|y\|^2 + (1 - \alpha) |\langle x, y \rangle|^2. \end{aligned} \quad (2.2)$$

Now, from the inequality (2.1) and by using the convexity of the function  $f(t) = t^2$ , it can be obtained

$$\begin{aligned} |\langle x, e \rangle \langle e, y \rangle|^2 &\leq \left( \frac{\|x\| \|y\| + |\langle x, y \rangle|}{2} \right)^2 \\ &\leq \frac{1}{2} (\|x\|^2 \|y\|^2 + |\langle x, y \rangle|^2). \end{aligned} \quad (2.3)$$

Thus, from the inequalities (2.2) and (2.3), one has

$$\begin{aligned} |\langle x, e \rangle \langle e, y \rangle|^2 &\leq \frac{1}{2} (\|x\|^2 \|y\|^2 + |\langle x, y \rangle|^2) \\ &\leq \frac{1}{2} (\|x\|^2 \|y\|^2 + \alpha \|x\|^2 \|y\|^2 + (1 - \alpha) |\langle x, y \rangle|^2). \end{aligned}$$

So

$$|\langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{1}{2} ((1 + \alpha) \|x\|^2 \|y\|^2 + (1 - \alpha) |\langle x, y \rangle|^2).$$

This completes the proof.  $\square$

REMARK 2.1. It was shown in [10] that for any  $x, y, e \in H$  with  $\|e\| = 1$ , it holds

$$|\langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{3}{4} \|x\|^2 \|y\|^2 + \frac{1}{4} \|x\| \|y\| |\langle x, y \rangle|. \quad (2.4)$$

We note that Lemma 2.4 is sharper than the inequality (2.4) if taking  $\alpha = \frac{1}{2}$ . As a matter of fact, when  $\alpha = \frac{1}{2}$ , we have

$$\begin{aligned} |\langle x, e \rangle \langle e, y \rangle|^2 &\leq \frac{1}{2} \left( \left( 1 + \frac{1}{2} \right) \|x\|^2 \|y\|^2 + \left( 1 - \frac{1}{2} \right) |\langle x, y \rangle|^2 \right) \\ &= \frac{3}{4} \|x\|^2 \|y\|^2 + \frac{1}{4} |\langle x, y \rangle|^2 \\ &\leq \frac{3}{4} \|x\|^2 \|y\|^2 + \frac{1}{4} \|x\| \|y\| |\langle x, y \rangle|. \end{aligned}$$

LEMMA 2.5. Let  $x, y, e \in H$  with  $\|e\| = 1$  and  $0 \leq \alpha \leq 1$ . Then

$$|\langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{1}{4} \left( \|x\|^2 \|y\|^2 + (2 + \alpha) \|x\| \|y\| |\langle x, y \rangle| + (1 - \alpha) |\langle x, y \rangle|^2 \right).$$

*Proof.* By using the Cauchy-Schwarz inequality, we have

$$\begin{aligned} |\langle x, y \rangle|^2 &= \alpha |\langle x, y \rangle|^2 + (1 - \alpha) |\langle x, y \rangle|^2 \\ &\leq \alpha \|x\| \|y\| |\langle x, y \rangle| + (1 - \alpha) |\langle x, y \rangle|^2. \end{aligned} \tag{2.5}$$

Then, from the inequality (2.1), it can be established

$$\begin{aligned} |\langle x, e \rangle \langle e, y \rangle|^2 &\leq \left( \frac{\|x\| \|y\| + |\langle x, y \rangle|}{2} \right)^2 \\ &= \frac{1}{4} \left( \|x\|^2 \|y\|^2 + 2 \|x\| \|y\| |\langle x, y \rangle| + |\langle x, y \rangle|^2 \right). \end{aligned} \tag{2.6}$$

It follows from the inequalities (2.5) and (2.6) that

$$\begin{aligned} |\langle x, e \rangle \langle e, y \rangle|^2 &\leq \frac{1}{4} \left( \|x\|^2 \|y\|^2 + 2 \|x\| \|y\| |\langle x, y \rangle| + |\langle x, y \rangle|^2 \right) \\ &\leq \frac{1}{4} \left( \|x\|^2 \|y\|^2 + (2 + \alpha) \|x\| \|y\| |\langle x, y \rangle| + (1 - \alpha) |\langle x, y \rangle|^2 \right). \end{aligned}$$

This completes the proof.  $\square$

REMARK 2.2. It should be mentioned that Lemma 2.5 refine the inequality [12],

$$|\langle x, e \rangle \langle e, y \rangle|^2 \leq \frac{1}{4} \left( (1 + \alpha) \|x\|^2 \|y\|^2 + (3 - \alpha) \|x\| \|y\| |\langle x, y \rangle| \right).$$

As a matter of fact, by the Cauchy-Schwarz inequality, it holds

$$\begin{aligned} &|\langle x, e \rangle \langle e, y \rangle|^2 \\ &\leq \frac{1}{4} \left( \|x\|^2 \|y\|^2 + (2 + \alpha) \|x\| \|y\| |\langle x, y \rangle| + (1 - \alpha) |\langle x, y \rangle|^2 \right) \\ &\leq \frac{1}{4} \left( \|x\|^2 \|y\|^2 + \alpha \|x\|^2 \|y\|^2 + 2 \|x\| \|y\| |\langle x, y \rangle| + (1 - \alpha) \|x\| \|y\| |\langle x, y \rangle| \right) \\ &= \frac{1}{4} \left( (1 + \alpha) \|x\|^2 \|y\|^2 + (3 - \alpha) \|x\| \|y\| |\langle x, y \rangle| \right). \end{aligned}$$

The last needed lemma is well known and it was obtained in [16, 17].

LEMMA 2.6. Let  $T, S \in \mathcal{B}(H)$ . Then

$$(i) \omega \left( \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right) = \max\{\omega(T + S), \omega(T - S)\};$$

$$(ii) \omega \left( \begin{bmatrix} T & S \\ S & T \end{bmatrix} \right) = \max\{\omega(T + S), \omega(T - S)\};$$

In particular,

$$\omega\left(\begin{bmatrix} 0 & S \\ S & 0 \end{bmatrix}\right) = \omega(S);$$

$$(iii) \left\| \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix} \right\| = \left\| \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} \right\| = \max\{\|T\|, \|S\|\}.$$

### 3. Some refinements of numerical radius for $2 \times 2$ operators matrices

The main goal of this section is to derive several upper bounds for numerical radius which are refinements of some existing ones.

**THEOREM 3.1.** *Let  $P, Q, T, S \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then*

$$\begin{aligned} \omega^4\left(\begin{bmatrix} P & T \\ S & Q \end{bmatrix}\right) &\leq 8 \max\{\omega^4(P), \omega^4(Q)\} + 4(1 - \alpha) \max\{\omega^2(TS), \omega^2(ST)\} \\ &\quad + 2(1 + \alpha) \max\{\| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \}. \end{aligned}$$

*Proof.* Let  $x \in H \oplus H$  with  $\|x\| = 1$ . Then

$$\begin{aligned} &\left| \left\langle \begin{bmatrix} P & T \\ S & Q \end{bmatrix} x, x \right\rangle \right|^4 \\ &= \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right|^4 \\ &\leq \left( \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right| + \left| \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right| \right)^4 \\ &= \left( \frac{2 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right| + 2 \left| \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right|}{2} \right)^4 \\ &\leq 8 \left( \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + \left| \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right|^4 \right) \\ &\quad (\text{by the convexity of } f(t) = t^4) \\ &\leq 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + 4((1 + \alpha)\|Yx\|^2\|Y^*x\|^2 + (1 - \alpha)|\langle Yx, Y^*x \rangle|^2) \\ &\quad (\text{by Lemma 2.4 with } Y = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) \\ &= 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + 4(1 + \alpha)\langle |Y|^2 x, x \rangle \langle |Y^*|^2 x, x \rangle + 4(1 - \alpha)|\langle Y^2 x, x \rangle|^2 \end{aligned}$$

$$\begin{aligned} &\leq 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + 2(1 + \alpha) (\langle |Y|^2 x, x \rangle^2 + \langle |Y^*|^2 x, x \rangle^2) + 4(1 - \alpha) |\langle Y^2 x, x \rangle|^2 \\ &\quad \text{(by the arithmetic-geometric mean inequality)} \\ &\leq 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + 2(1 + \alpha) (\langle |Y|^4 x, x \rangle + \langle |Y^*|^4 x, x \rangle) + 4(1 - \alpha) |\langle Y^2 x, x \rangle|^2 \\ &\quad \text{(by Lemma 2.1)} \\ &\leq 8\omega^4 \left( \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right) + 2(1 + \alpha) \| |Y|^4 + |Y^*|^4 \| + 4(1 - \alpha)\omega^2(Y^2) \\ &= 8\omega^4 \left( \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right) + 2(1 + \alpha) \left\| \begin{bmatrix} |T^*|^4 + |S|^4 & 0 \\ 0 & |T|^4 + |S^*|^4 \end{bmatrix} \right\| \\ &\quad + 4(1 - \alpha)\omega^2 \left( \begin{bmatrix} TS & 0 \\ 0 & ST \end{bmatrix} \right). \end{aligned}$$

Taking the supremum over  $x \in H \oplus H$  and by Lemma 2.6, the result can be written as

$$\begin{aligned} \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + 4(1 - \alpha) \max \{ \omega^2(TS), \omega^2(ST) \} \\ &\quad + 2(1 + \alpha) \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \}. \quad \square \end{aligned}$$

REMARK 3.1. In Theorem 3.1, if taking  $\alpha = \frac{1}{2}$ , we observe that

$$\begin{aligned} \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + 3 \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \} \\ &\quad + 2 \max \{ \omega^2(TS), \omega^2(ST) \}. \end{aligned}$$

It should be mentioned here that this inequality is sharper than the inequality (1.6). As a matter of fact, by applying inequality (1.5), we can obtain

$$2 \max \{ \omega^2(TS), \omega^2(ST) \} \leq \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \}.$$

So

$$\begin{aligned} \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + 3 \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \} \\ &\quad + 2 \max \{ \omega^2(TS), \omega^2(ST) \} \\ &\leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + 3 \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \} \\ &\quad + \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \}. \end{aligned}$$

Thus, for  $\alpha = \frac{1}{2}$ , Theorem 3.1 is a refinement of the inequality (1.6).

With the property in Lemma 2.6, it can be established the following inequality of  $\omega^4(T)$ .

COROLLARY 3.1. Let  $T \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then

$$\omega^4(T) \leq \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{2} \omega^2(T^2).$$

*Proof.* It follows from Lemma 2.6 and Theorem 3.1 that

$$\begin{aligned} \omega^4 \left( \begin{bmatrix} P & T \\ T & P \end{bmatrix} \right) &= \max\{\omega^4(T+P), \omega^4(T-P)\} \\ &\leq 8\omega^4(P) + 2(1+\alpha) \||T|^4 + |T^*|^4\| + 4(1-\alpha) \omega^2(T^2). \end{aligned} \quad (3.1)$$

Then, by taking  $T = P$  in the inequality (3.1), it can be obtained

$$16\omega^4(T) = \omega^4(2T) \leq 8\omega^4(T) + 2(1+\alpha) \||T|^4 + |T^*|^4\| + 4(1-\alpha) \omega^2(T^2).$$

Thus

$$\omega^4(T) \leq \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{2} \omega^2(T^2). \quad \square$$

REMARK 3.2. In [10], Omidvar et. al. proved that

$$\omega^4(T) \leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2). \quad (3.2)$$

If taking  $\alpha = \frac{1}{2}$  in Corollary 3.1, we will obtain

$$\omega^4(T) \leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{4} \omega^2(T^2),$$

which is sharper than the inequality (3.2).

In fact,

$$\begin{aligned} &\frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{4} \omega^2(T^2) \\ &\leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{4} \omega^2(T) \omega(T^2) \\ &\leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2). \end{aligned}$$

Therefore, if taking  $\alpha = \frac{1}{2}$ , it holds

$$\begin{aligned} \omega^4(T) &\leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{4} \omega^2(T^2) \\ &\leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2). \end{aligned}$$

For  $\alpha = \frac{1}{2}$ , Corollary 3.1 is an improvement of the inequality (3.2). In order to appreciate our inequality in Corollary 3.1, we give the following example to show that our inequality is a nontrivial improvement of the inequality (3.2).



EXAMPLE 3.1. Let

$$T = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then by elementary calculations, it holds

$$|T|^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}, \quad |T^*|^2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$|T|^4 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 16 \end{bmatrix}, \quad |T^*|^4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 16 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{and} \quad T^2 = \begin{bmatrix} 0 & 0 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Therefore, if taking  $\alpha = \frac{1}{2}$ , we have

$$\frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{4} \omega^2(T^2) = \frac{53}{8}$$

and

$$\frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) = 7.$$

Thus

$$\frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{4} \omega^2(T^2) < \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2).$$

REMARK 3.3. Corollary 3.1 is a refinement of the inequality (1.4) with  $r = 2$ . Indeed, by Lemma 2.2, we have

$$\begin{aligned} & \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{2} \omega^2(T^2) \\ & \leq \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{2} \omega^4(T) \\ & \leq \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{8} \||T|^2 + |T^*|^2\|^2 \\ & = \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{8} \|( |T|^2 + |T^*|^2 )^2 \| \\ & = \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{8} \left\| \left( \frac{2|T|^2 + 2|T^*|^2}{2} \right)^2 \right\| \\ & \leq \frac{(1+\alpha)}{4} \||T|^4 + |T^*|^4\| + \frac{(1-\alpha)}{4} \||T|^4 + |T^*|^4\| \\ & = \frac{1}{2} \||T|^4 + |T^*|^4\|. \end{aligned}$$

Thus, it holds

$$\begin{aligned} \omega^4(T) &\leq \frac{(1+\alpha)}{4} \| |T|^4 + |T^*|^4 \| + \frac{(1-\alpha)}{2} \omega^2(T^2) \\ &\leq \frac{1}{2} \| |T|^4 + |T^*|^4 \|. \end{aligned}$$

For  $0 \leq \alpha < 1$ , the following example shows that Corollary 3.1 is a nontrivial improvement of the inequality (1.4).

EXAMPLE 3.2. Let  $T$  be the same as described in Example 3.1. Then by simple calculations, we have

$$\frac{(1+\alpha)}{4} \| |T|^4 + |T^*|^4 \| + \frac{(1-\alpha)}{2} \omega^2(T^2) = \frac{19+15\alpha}{4}$$

and

$$\frac{1}{2} \| |T|^4 + |T^*|^4 \| = \frac{17}{2}.$$

Thus, it holds

$$\frac{(1+\alpha)}{4} \| |T|^4 + |T^*|^4 \| + \frac{(1-\alpha)}{2} \omega^2(T^2) < \frac{1}{2} \| |T|^4 + |T^*|^4 \|$$

for any  $\alpha \in [0, 1)$ .

THEOREM 3.2. Let  $P, Q, T, S \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then

$$\begin{aligned} \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 2(2+\alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\ &\quad + 4(2-\alpha) \omega^2 \left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right). \end{aligned}$$

*Proof.* Let  $x \in H \oplus H$  with  $\|x\| = 1$ , and  $W = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ ,  $Y = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}$ . Then

$$\begin{aligned} &\left| \left\langle \begin{bmatrix} P & T \\ S & Q \end{bmatrix} x, x \right\rangle \right|^2 \\ &= | \langle (W+Y)x, x \rangle |^2 \\ &\leq (| \langle Wx, x \rangle | + | \langle Yx, x \rangle |)^2 \\ &= | \langle Wx, x \rangle |^2 + | \langle Yx, x \rangle |^2 + 2| \langle Wx, x \rangle | | \langle Yx, x \rangle | \\ &\leq \sqrt{ | \langle Wx, Wx \rangle |^2 + | \langle Y^*x, Y^*x \rangle |^2 + 2| \langle Wx, Y^*x \rangle |^2 + 2| \langle Wx, x \rangle | | \langle Yx, x \rangle | } \\ &\quad \text{(by Lemma 2.3)} \end{aligned}$$

$$\begin{aligned}
 &= \frac{1}{2} \left( 2\sqrt{|\langle Wx, Wx \rangle|^2 + |\langle Y^*x, Y^*x \rangle|^2 + 2|\langle Wx, Y^*x \rangle|^2} + 4|\langle Wx, x \rangle| |\langle Yx, x \rangle| \right) \\
 &\leq [2(|\langle Wx, Wx \rangle|^2 + |\langle Y^*x, Y^*x \rangle|^2 + 2|\langle Wx, Y^*x \rangle|^2) + 8|\langle Wx, x \rangle| |\langle Yx, x \rangle|^2]^{\frac{1}{2}}. \\
 &\quad \text{(by the Power-Mean inequality)}
 \end{aligned}$$

Thus

$$\begin{aligned}
 &\left| \left\langle \begin{bmatrix} P & T \\ S & Q \end{bmatrix} x, x \right\rangle \right|^4 \\
 &\leq 2(\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2 + 2|\langle Wx, Y^*x \rangle|^2) + 8|\langle Wx, x \rangle| |\langle Yx, x \rangle|^2 \\
 &\leq 2(\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2 + 2|\langle Wx, Y^*x \rangle|^2) \\
 &\quad + 4(1 + \alpha)\|Wx\|^2\|Y^*x\|^2 + 4(1 - \alpha)|\langle Wx, Y^*x \rangle|^2 \\
 &\quad \text{(by Lemma 2.4)} \\
 &\leq 2(\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2 + 2|\langle Wx, Y^*x \rangle|^2) \\
 &\quad + 2(1 + \alpha)(\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2) + 4(1 - \alpha)|\langle Wx, Y^*x \rangle|^2 \\
 &\quad \text{(by the arithmetic-geometric mean inequality)} \\
 &\leq 2(\langle |W|^4x, x \rangle + \langle |Y^*|^4x, x \rangle + 2|\langle Wx, Y^*x \rangle|^2) \\
 &\quad + 2(1 + \alpha)(\langle |W|^4x, x \rangle + \langle |Y^*|^4x, x \rangle) + 4(1 - \alpha)|\langle Wx, Y^*x \rangle|^2 \\
 &\quad \text{(by Lemma 2.1)} \\
 &= 2(2 + \alpha)(\langle |W|^4 + |Y^*|^4 \rangle x, x) + 4(2 - \alpha)|\langle YWx, x \rangle|^2 \\
 &\leq 2(2 + \alpha)\| |W|^4 + |Y^*|^4 \|^2 + 4(2 - \alpha)\omega^2(YW) \\
 &= 2(2 + \alpha) \left\| \begin{bmatrix} |P|^4 + |T^*|^4 & 0 \\ 0 & |Q|^4 + |S^*|^4 \end{bmatrix} \right\|^2 + 4(2 - \alpha)\omega^2 \left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right).
 \end{aligned}$$

Now, taking the supremum over all  $x \in H \oplus H$  in the last inequality and by Lemma 2.6, we get

$$\begin{aligned}
 \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) &\leq 2(2 + \alpha) \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} \\
 &\quad + 4(2 - \alpha)\omega^2 \left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right). \quad \square
 \end{aligned}$$

By Theorem 3.2, it will hold the following corollary.

**COROLLARY 3.2.** *Let  $T \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then*

$$\omega^4(T) \leq \frac{(2 + \alpha)}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2 - \alpha)}{4} \omega^2(T^2).$$

*Proof.* Let  $P = Q$  and  $T = S$  in Theorem 3.2. Then it will hold

$$\begin{aligned} \omega^4 \left( \begin{bmatrix} P & T \\ T & P \end{bmatrix} \right) &= \max\{\omega^4(T + P), \omega^4(T - P)\} \\ &\leq 2(2 + \alpha) \||P|^4 + |T^*|^4\| + 4(2 - \alpha)\omega^2(TP). \end{aligned} \tag{3.3}$$

Taking  $T = P$  in the inequality (3.3), we have

$$16\omega^4(T) = \omega^4(2T) \leq 2(2 + \alpha) \||T|^4 + |T^*|^4\| + 4(2 - \alpha)\omega^2(T^2).$$

So

$$\omega^4(T) \leq \frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(2 - \alpha)}{4} \omega^2(T^2). \quad \square$$

REMARK 3.4. Corollary 3.2 is sharper than the inequality (3.2) for any  $\alpha \in [0, 1]$ . In fact, by Lemma 2.2, we can obtain

$$\begin{aligned} &\frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(2 - \alpha)}{4} \omega^2(T^2) \\ &= \frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(1 - \alpha)}{4} \omega^2(T^2) + \frac{1}{4} \omega^2(T^2) \\ &\leq \frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(1 - \alpha)}{16} \||T|^2 + |T^*|^2\|^2 + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ &= \frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(1 - \alpha)}{16} \|( |T|^2 + |T^*|^2 )^2 \| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ &\leq \frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(1 - \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ &= \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2). \end{aligned}$$

Thus, by Corollary 3.2, we have

$$\begin{aligned} \omega^4(T) &\leq \frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(2 - \alpha)}{4} \omega^2(T^2) \\ &\leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ &\leq \frac{1}{2} \||T|^4 + |T^*|^4\|. \end{aligned} \tag{3.4}$$

The final inequality in (3.4) was proved in [10].

To show that our inequality in Corollary 3.2 is a nontrivial improvement of the inequality (3.2), we give the following example.

EXAMPLE 3.3. Let  $T$  be the same as described in Example 3.1. Then it can be checked that

$$\frac{(2 + \alpha)}{8} \||T|^4 + |T^*|^4\| + \frac{(2 - \alpha)}{4} \omega^2(T^2) = \frac{38 + 15\alpha}{8}$$

and

$$\frac{3}{8} \left\| |T|^4 + |T^*|^4 \right\| + \frac{1}{8} \left\| |T|^2 + |T^*|^2 \right\| \omega(T^2) = 7.$$

Therefore

$$\frac{(2 + \alpha)}{8} \left\| |T|^4 + |T^*|^4 \right\| + \frac{(2 - \alpha)}{4} \omega^2(T^2) < \frac{3}{8} \left\| |T|^4 + |T^*|^4 \right\| + \frac{1}{8} \left\| |T|^2 + |T^*|^2 \right\| \omega(T^2)$$

for any  $\alpha \in [0, 1]$ .

**THEOREM 3.3.** *Let  $P, Q, T, S \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then*

$$\begin{aligned} & \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\ & \leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + \max \{ \left\| |T|^4 + |S^*|^4 \right\|, \left\| |T^*|^4 + |S|^4 \right\| \} \\ & \quad + (2 + \alpha) \max \{ \left\| |T|^2 + |S^*|^2 \right\|, \left\| |T^*|^2 + |S|^2 \right\| \} \max \{ \omega(TS), \omega(ST) \} \\ & \quad + 2(1 - \alpha) \max \{ \omega^2(TS), \omega^2(ST) \}. \end{aligned}$$

*Proof.* Let  $x \in H \oplus H$  with  $\|x\| = 1$ . Then

$$\begin{aligned} & \left| \left\langle \begin{bmatrix} P & T \\ S & Q \end{bmatrix} x, x \right\rangle \right|^4 \\ & = \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle + \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right|^4 \\ & \leq \left( \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right| + \left| \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right| \right)^4 \\ & = \left( \frac{2 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right| + 2 \left| \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right|}{2} \right)^4 \\ & \leq 8 \left( \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + \left| \left\langle \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix} x, x \right\rangle \right|^4 \right) \\ & \quad (\text{by the convexity of } f(t) = t^4) \\ & \leq 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + 2 \|Yx\|^2 \|Y^*x\|^2 + 2(2 + \alpha) \|Yx\| \|Y^*x\| \left| \langle Yx, Y^*x \rangle \right| \\ & \quad + 2(1 - \alpha) |\langle Yx, Y^*x \rangle|^2 \\ & \quad (\text{by Lemma 2.5 with } Y = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}) \end{aligned}$$

$$\begin{aligned}
 &= 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + 2 \langle |Y|^2 x, x \rangle \langle |Y^*|^2 x, x \rangle \\
 &\quad + 2(2 + \alpha) \sqrt{\langle |Y|^2 x, x \rangle \langle |Y^*|^2 x, x \rangle} \left| \langle Y^2 x, x \rangle \right| + 2(1 - \alpha) |\langle Y^2 x, x \rangle|^2 \\
 &\leq 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + (\langle |Y|^2 x, x \rangle^2 + \langle |Y^*|^2 x, x \rangle^2) \\
 &\quad + (2 + \alpha) (\langle |Y|^2 x, x \rangle + \langle |Y^*|^2 x, x \rangle) \left| \langle Y^2 x, x \rangle \right| + 2(1 - \alpha) |\langle Y^2 x, x \rangle|^2 \\
 &\quad \text{(by the arithmetic-geometric mean inequality)} \\
 &\leq 8 \left| \left\langle \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} x, x \right\rangle \right|^4 + (\langle |Y|^4 x, x \rangle + \langle |Y^*|^4 x, x \rangle) \\
 &\quad + (2 + \alpha) (\langle (|Y|^2 + |Y^*|^2) x, x \rangle) \left| \langle Y^2 x, x \rangle \right| + 2(1 - \alpha) |\langle Y^2 x, x \rangle|^2 \\
 &\quad \text{(by Lemma 2.1)} \\
 &\leq 8\omega^4 \left( \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right) + \| |Y|^4 + |Y^*|^4 \| + (2 + \alpha) \| |Y|^2 + |Y^*|^2 \| \omega(Y^2) \\
 &\quad + 2(1 - \alpha) \omega^2(Y^2) \\
 &= 8\omega^4 \left( \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix} \right) + \left\| \begin{bmatrix} |T^*|^4 + |S|^4 & 0 \\ 0 & |T|^4 + |S^*|^4 \end{bmatrix} \right\| \\
 &\quad + (2 + \alpha) \left\| \begin{bmatrix} |T^*|^2 + |S|^2 & 0 \\ 0 & |T|^2 + |S^*|^2 \end{bmatrix} \right\| \omega \left( \begin{bmatrix} TS & 0 \\ 0 & ST \end{bmatrix} \right) \\
 &\quad + 2(1 - \alpha) \omega^2 \left( \begin{bmatrix} TS & 0 \\ 0 & ST \end{bmatrix} \right).
 \end{aligned}$$

Taking the supremum over  $x \in H \oplus H$  and by Lemma 2.6, the result can be written as

$$\begin{aligned}
 &\omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\
 &\leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \} \\
 &\quad + (2 + \alpha) \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \} \\
 &\quad + 2(1 - \alpha) \max \{ \omega^2(TS), \omega^2(ST) \}. \quad \square
 \end{aligned}$$

REMARK 3.5. It can be checked that Theorem 3.3 is an improvement of the inequality (1.7) for any  $\alpha \in [0, 1]$ .

To see this, we need to prove that

$$\begin{aligned}
 &\max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \} \\
 &\leq \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \}.
 \end{aligned}$$

Now, by the inequality (1.5) and Lemma 2.2, it holds

$$\begin{aligned} & \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \} \\ & \leq \frac{1}{2} \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \\ & = \frac{1}{2} \max \left\{ \| |T|^2 + |S^*|^2 \|^2, \| |T^*|^2 + |S|^2 \|^2 \right\} \\ & = \frac{1}{2} \max \left\{ \left\| \left( \frac{2|T|^2 + 2|S^*|^2}{2} \right)^2 \right\|, \left\| \left( \frac{|T^*|^2 + |S|^2}{2} \right)^2 \right\| \right\} \\ & \leq \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \}. \end{aligned}$$

Also by inequality (1.5), we can obtain

$$\max \{ \omega^2(TS), \omega^2(ST) \} \leq \frac{1}{2} \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \}.$$

Combine the above two inequalities, we observe that

$$\begin{aligned} & \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\ & \leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \} \\ & \quad + (2 + \alpha) \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \} \\ & \quad + 2(1 - \alpha) \max \{ \omega^2(TS), \omega^2(ST) \} \\ & \leq 8 \max \{ \omega^4(P), \omega^4(Q) \} + (1 + \alpha) \max \{ \| |T|^4 + |S^*|^4 \|, \| |T^*|^4 + |S|^4 \| \} \\ & \quad + (3 - \alpha) \max \{ \| |T|^2 + |S^*|^2 \|, \| |T^*|^2 + |S|^2 \| \} \max \{ \omega(TS), \omega(ST) \}. \end{aligned}$$

This indicates that Theorem 3.3 is a refinement of the inequality (1.7) for any  $\alpha \in [0, 1]$ .

By Theorem 3.3, it can be obtained the following inequality of  $\omega^4(T)$ .

**COROLLARY 3.3.** *Let  $T \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then*

$$\omega^4(T) \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2 + \alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1 - \alpha)}{4} \omega^2(T^2).$$

*Proof.* It follows from Lemma 2.6 and Theorem 3.3 that

$$\begin{aligned} & \omega^4 \left( \begin{bmatrix} P & T \\ T & P \end{bmatrix} \right) = \max \{ \omega^4(T + P), \omega^4(T - P) \} \\ & \leq 8\omega^4(P) + \| |T|^4 + |T^*|^4 \| + (2 + \alpha) \| |T|^2 + |T^*|^2 \| \omega(T^2) + 2(1 - \alpha) \omega^2(T^2). \end{aligned} \tag{3.5}$$

Taking  $T = P$  in the inequality (3.5), it holds

$$\begin{aligned} & 16\omega^4(T) = \omega^4(2T) \\ & \leq 8\omega^4(T) + \| |T|^4 + |T^*|^4 \| + (2 + \alpha) \| |T|^2 + |T^*|^2 \| \omega(T^2) + 2(1 - \alpha) \omega^2(T^2). \end{aligned}$$

Thus

$$\omega^4(T) \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2+\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1-\alpha)}{4} \omega^2(T^2). \quad \square$$

REMARK 3.6. In [12], it was shown that

$$\omega^4(T) \leq \frac{(1+\alpha)}{8} \| |T|^4 + |T^*|^4 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2). \quad (3.6)$$

It should be noticed that Corollary 3.3 is sharper than the inequality (3.6).

To see this, note that

$$\begin{aligned} & \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2+\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1-\alpha)}{4} \omega^2(T^2) \\ & \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2+\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1-\alpha)}{4} \omega^2(T) \omega(T^2) \\ & \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{\alpha}{16} \| |T|^2 + |T^*|^2 \|^2 + \frac{2}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) \\ & \quad + \frac{(1-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) \\ & = \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{\alpha}{16} \| (|T|^2 + |T^*|^2)^2 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) \\ & \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{\alpha}{8} \| |T|^4 + |T^*|^4 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) \\ & = \frac{(1+\alpha)}{8} \| |T|^4 + |T^*|^4 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2). \end{aligned}$$

So

$$\begin{aligned} \omega^4(T) & \leq \frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2+\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1-\alpha)}{4} \omega^2(T^2) \\ & \leq \frac{(1+\alpha)}{8} \| |T|^4 + |T^*|^4 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) \\ & \leq \frac{1}{2} \| |T|^4 + |T^*|^4 \|. \end{aligned} \quad (3.7)$$

The final inequality in (3.7) was proved in [12].

The following example shows that Corollary 3.3 is a nontrivial improvement of the inequality (3.6).

EXAMPLE 3.4. Let  $T$  is as same as the matrix in Example 3.1. Then it can be checked that

$$\frac{1}{8} \| |T|^4 + |T^*|^4 \| + \frac{(2+\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1-\alpha)}{4} \omega^2(T^2) = \frac{29+3\alpha}{8}$$

and

$$\frac{(1+\alpha)}{8} \| |T|^4 + |T^*|^4 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) = \frac{32+12\alpha}{8}.$$



Thus, we have

$$\begin{aligned} & \frac{1}{8} \left( \| |T|^4 + |T^*|^4 \| + \frac{(2+\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(1-\alpha)}{4} \omega^2(T^2) \right) \\ & < \frac{(1+\alpha)}{8} \| |T|^4 + |T^*|^4 \| + \frac{(3-\alpha)}{8} \| |T|^2 + |T^*|^2 \| \omega(T^2) \end{aligned}$$

for any  $\alpha \in [0, 1]$ .

**THEOREM 3.4.** *Let  $P, Q, T, S \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then*

$$\begin{aligned} & \omega^4 \left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\ & \leq 3 \max \{ \| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} + 2(3-\alpha) \omega^2 \left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\ & \quad + (2+\alpha) \max \{ \| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \omega \left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right). \end{aligned}$$

*Proof.* Let  $x \in H \oplus H$  with  $\|x\| = 1$ , and let  $W = \begin{bmatrix} P & 0 \\ 0 & Q \end{bmatrix}$ ,  $Y = \begin{bmatrix} 0 & T \\ S & 0 \end{bmatrix}$ . With the same argument in the proof of Theorem 3.2, we get

$$\begin{aligned} & \left| \left\langle \begin{bmatrix} P & T \\ S & Q \end{bmatrix} x, x \right\rangle \right|^2 \\ & \leq [2(|\langle Wx, Wx \rangle|^2 + |\langle Y^*x, Y^*x \rangle|^2) + 2|\langle Wx, Y^*x \rangle|^2 + 8|\langle Wx, x \rangle \langle Yx, x \rangle|^2]^{\frac{1}{2}}. \end{aligned}$$

Thus

$$\begin{aligned} & \left| \left\langle \begin{bmatrix} P & T \\ S & Q \end{bmatrix} x, x \right\rangle \right|^4 \\ & \leq 2(\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2 + 2|\langle Wx, Y^*x \rangle|^2) + 8|\langle Wx, x \rangle \langle Yx, x \rangle|^2 \\ & \leq 2(|\langle Wx, Wx \rangle|^2 + |\langle Y^*x, Y^*x \rangle|^2 + 2|\langle Wx, Y^*x \rangle|^2) + 2\|Wx\|^2 \|Y^*x\|^2 \\ & \quad + 2(2+\alpha)\|Wx\| \|Y^*x\| |\langle Wx, Y^*x \rangle| + 2(1-\alpha)|\langle Wx, Y^*x \rangle|^2 \\ & \quad \text{(by Lemma 2.5)} \\ & = 2(|\langle Wx, Wx \rangle|^2 + |\langle Y^*x, Y^*x \rangle|^2 + 2|\langle Wx, Y^*x \rangle|^2) + 2\langle Wx, Wx \rangle \langle Y^*x, Y^*x \rangle \\ & \quad + 2(2+\alpha)\sqrt{\langle Wx, Wx \rangle \langle Y^*x, Y^*x \rangle} |\langle Wx, Y^*x \rangle| + 2(1-\alpha)|\langle Wx, Y^*x \rangle|^2 \\ & \leq 2(\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2 + 2|\langle Wx, Y^*x \rangle|^2) + (\langle Wx, Wx \rangle^2 + \langle Y^*x, Y^*x \rangle^2) \\ & \quad + (2+\alpha)(\langle Wx, Wx \rangle + \langle Y^*x, Y^*x \rangle) |\langle Wx, Y^*x \rangle| + 2(1-\alpha)|\langle Wx, Y^*x \rangle|^2 \\ & \quad \text{(by the arithmetic-geometric mean inequality)} \end{aligned}$$

$$\begin{aligned}
 &\leq 2(\langle |W|^4 x, x \rangle + \langle |Y^*|^4 x, x \rangle + 2|\langle Wx, Y^*x \rangle|^2) + (\langle |W|^4 x, x \rangle + \langle |Y^*|^4 x, x \rangle) \\
 &\quad + (2 + \alpha)(\langle |W|^2 x, x \rangle + \langle |Y^*|^2 x, x \rangle) |\langle Wx, Y^*x \rangle| + 2(1 - \alpha)|\langle Wx, Y^*x \rangle|^2 \\
 &\quad \text{(by Lemma 2.1)} \\
 &= 3(\langle |W|^4 + |Y^*|^4 \rangle x, x) + (2 + \alpha)\langle (|W|^2 + |Y^*|^2)x, x \rangle |\langle YWx, x \rangle| \\
 &\quad + 2(3 - \alpha)|\langle YWx, x \rangle|^2 \\
 &\leq 3\| |W|^4 + |Y^*|^4 \| + 2(3 - \alpha)\omega^2(YW) + (2 + \alpha)\| |W|^2 + |Y^*|^2 \| \omega(YW) \\
 &= 3\left\| \begin{bmatrix} |P|^4 + |T^*|^4 & 0 \\ 0 & |Q|^4 + |S^*|^4 \end{bmatrix} \right\| + 2(3 - \alpha)\omega^2\left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad + (2 + \alpha)\left\| \begin{bmatrix} |P|^2 + |T^*|^2 & 0 \\ 0 & |Q|^2 + |S^*|^2 \end{bmatrix} \right\| \omega\left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right).
 \end{aligned}$$

Taking the supremum over all  $x \in H \oplus H$  in the above inequality and by Lemma 2.6, we get

$$\begin{aligned}
 &\omega^4\left( \begin{bmatrix} P & T \\ S & Q \end{bmatrix} \right) \\
 &\leq 3\max\{\| |P|^4 + |T^*|^4 \|, \| |Q|^4 + |S^*|^4 \| \} + 2(3 - \alpha)\omega^2\left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right) \\
 &\quad + (2 + \alpha)\max\{\| |P|^2 + |T^*|^2 \|, \| |Q|^2 + |S^*|^2 \| \} \omega\left( \begin{bmatrix} 0 & TQ \\ SP & 0 \end{bmatrix} \right). \quad \square
 \end{aligned}$$

It follows from Theorem 3.4 and Lemma 2.6, it can be established the following inequality of  $\omega^4(T)$ .

**COROLLARY 3.4.** *Let  $T \in \mathcal{B}(H)$  and  $0 \leq \alpha \leq 1$ . Then*

$$\omega^4(T) \leq \frac{3}{16}\| |T|^4 + |T^*|^4 \| + \frac{(2 + \alpha)}{16}\| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{3 - \alpha}{8}\omega^2(T^2).$$

*Proof.* Let  $P = Q$  and  $T = S$ . Then the inequality in Theorem 3.4 will be

$$\begin{aligned}
 &\omega^4\left( \begin{bmatrix} P & T \\ T & P \end{bmatrix} \right) = \max\{\omega^4(T + P), \omega^4(T - P)\} \\
 &\leq 3\| |P|^4 + |T^*|^4 \| + 2(3 - \alpha)\omega^2(TP) + (2 + \alpha)\| |P|^2 + |T^*|^2 \| \omega(TP).
 \end{aligned} \tag{3.8}$$

Taking  $T = P$  in the inequality (3.8), it holds

$$16\omega^4(T) = \omega^4(2T) \leq 3\| |T|^4 + |T^*|^4 \| + (2 + \alpha)\| |T|^2 + |T^*|^2 \| \omega(T^2) + 2(3 - \alpha)\omega^2(T^2).$$

Thus, we can establish that

$$\omega^4(T) \leq \frac{3}{16}\| |T|^4 + |T^*|^4 \| + \frac{(2 + \alpha)}{16}\| |T|^2 + |T^*|^2 \| \omega(T^2) + \frac{(3 - \alpha)}{8}\omega^2(T^2). \quad \square$$

REMARK 3.7. It can be checked that Corollary 3.4 is a refinement of the inequality (3.2) for any  $\alpha \in [0, 1]$ .

To see this, we note that

$$\begin{aligned} & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{(2+\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) + \frac{(3-\alpha)}{8} \omega^2(T^2) \\ \leq & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{(2+\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) + \frac{(3-\alpha)}{8} \omega^2(T) \omega(T^2) \\ \leq & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{(2+\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) \\ & + \frac{(1-\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ \leq & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{3}{32} \||T|^2 + |T^*|^2\|^2 + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ = & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{3}{32} \|( |T|^2 + |T^*|^2 )^2 \| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ \leq & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \\ = & \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2). \end{aligned}$$

This indicates

$$\begin{aligned} \omega^4(T) & \leq \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{(2+\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) + \frac{(3-\alpha)}{8} \omega^2(T^2) \\ & \leq \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2). \end{aligned} \tag{3.9}$$

Corollary 3.4 is a nontrivial improvement of the inequality (3.2). To see this, we give the following example.

EXAMPLE 3.5. Let  $T$  be the same as described in Example 3.1. Then by elementary calculations, we have

$$\frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{(2+\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) + \frac{(3-\alpha)}{8} \omega^2(T^2) = \frac{67+3\alpha}{16}$$

and

$$\frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) = 7.$$

Therefore

$$\begin{aligned} & \frac{3}{16} \||T|^4 + |T^*|^4\| + \frac{(2+\alpha)}{16} \||T|^2 + |T^*|^2\| \omega(T^2) + \frac{(3-\alpha)}{8} \omega^2(T^2) \\ & < \frac{3}{8} \||T|^4 + |T^*|^4\| + \frac{1}{8} \||T|^2 + |T^*|^2\| \omega(T^2) \end{aligned}$$

for any  $\alpha \in [0, 1]$ .

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(Received May 26, 2021)

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