INCREASING PROPERTY AND LOGARITHMIC CONVEXITY OF TWO FUNCTIONS INVOLVING DIRICHLET ETA FUNCTION

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Dedicated to retired Professor Shi-Ying Yuan, former President of Henan Polytechnic University, China

(*Communicated by L. Mihoković*)

Abstract. In the paper, with the help of an integral representation of the Dirichlet eta function and by means of a monotonicity rule for a ratio of two integrals with a parameter, the authors find increasing property and logarithmic convexity of two functions involving the gamma function, the extended binomial coefficient, and the Dirichlet eta function.

1. Introduction

In this paper, we use the notation

$$\mathbb{N} = \{1, 2, \ldots\}, \quad \mathbb{N}_0 = \{0, 1, 2, \ldots\}, \quad \mathbb{N}_- = \{-1, -2, \ldots\}, \quad \mathbb{Z} = \{0, \pm 1, \pm 2, \ldots\}.$$

It is well known (see [1, Chapter 6], [21, Chapter 3]) that the classical Euler's gamma function $\Gamma(z)$ can be defined by

$$\Gamma(z) = \lim_{n \to \infty} \frac{n! n^z}{\prod_{k=0}^{n}(z+k)}, \quad z \in \mathbb{C} \setminus \{0, -1, -2, \ldots\}.$$

According to [6, Fact 13.3], for $z \in \mathbb{C}$ such that $\Re(z) > 1$, the Riemann zeta function $\zeta(z)$ can be defined by

$$\zeta(z) = \sum_{k=1}^{\infty} \frac{1}{k^z} = \frac{1}{1 - 2^{-z}} \sum_{k=1}^{\infty} \frac{1}{(2k-1)^z} = \frac{1}{1 - 2^{1-z}} \sum_{k=1}^{\infty} (-1)^{k-1} \frac{1}{k^z}. \tag{1.1}$$

In [21, Section 3.5, pp. 57–58], the analytic continuation of the Riemann zeta function $\zeta(z)$ into the punctured complex plane $\mathbb{C} \setminus \{1\}$ is discussed: the only singularity $z = 1$ is a simple pole with residue 1.

Basing on the last equation in (1.1), ones consider the Dirichlet eta function

$$\eta(z) = \left(1 - \frac{1}{2^{z-1}}\right) \zeta(z), \quad \Re(z) > 0.$$

It is also known as alternating zeta function. The Dirichlet eta function $\eta(z)$ has an integral representation

$$\eta(x) = \frac{1}{\Gamma(x)} \int_0^\infty \frac{t^{x-1}}{e^t + 1} \, dt, \quad x > 0 \tag{1.2}$$

in [8, p. 1046, 9.513.1] and [9, p. 604, 25.5.3]. In 1998, Wang [22] obtained that the Dirichlet eta function $\eta(x)$ is logarithmically concave on $(0, \infty)$. In 2009, Cerone and Dragomir [7] established many inequalities and properties for the Riemann eta function $\zeta(x)$ and the Dirichlet eta function $\eta(x)$. In 2015, Adell and Lekuona [2] and Alzer and Kwong [3] strengthened the logarithmic concavity in [22] to a concavity of the Dirichlet eta function $\eta(x)$ on $(0, \infty)$. In 2018, Qi [16, 18] used the logarithmic concavity of the Dirichlet eta function $\eta(x)$ in [22] to establish a double inequality of the ratio $\frac{|B_{2(n+1)}|}{|B_{2n}|}$ for $n \in \mathbb{N}$, where the Bernoulli numbers B_{2n} for $n \geqslant 0$ are generated by

$$\frac{z}{e^z - 1} = \sum_{n=0}^\infty B_n \frac{z^n}{n!} = 1 - \frac{z}{2} + \sum_{n=1}^\infty B_{2n} \frac{z^{2n}}{(2n)!}, \quad |z| < 2\pi.$$

Very recently, the double inequality established in [16, 18] was extended, sharpened, discussed in the papers [20, 25, 26].

In this paper, we consider

1. the functions

$$x \mapsto \binom{x + \alpha + \ell}{\alpha} \frac{\eta(x + \alpha)}{\eta(x)}, \quad \ell = 0, 1 \tag{1.3}$$

and their monotonicity on $(0, \infty)$, where $\alpha > 0$ is a constant,

$$\binom{z}{w} = \begin{cases} \dfrac{\Gamma(z+1)}{\Gamma(w+1)\Gamma(z-w+1)}, & z \notin \mathbb{N}_-, \quad w, z-w \notin \mathbb{N}_- \\ 0, & z \notin \mathbb{N}_-, \quad w \in \mathbb{N}_- \text{ or } z-w \in \mathbb{N}_- \\ \dfrac{\langle z \rangle_w}{w!}, & z \in \mathbb{N}_-, \quad w \in \mathbb{N}_0 \\ \dfrac{\langle z \rangle_{z-w}}{(z-w)!}, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_0 \\ 0, & z, w \in \mathbb{N}_-, \quad z-w \in \mathbb{N}_- \\ \infty, & z \in \mathbb{N}_-, \quad w \notin \mathbb{Z} \end{cases} \tag{1.4}$$

denotes the extended binomial coefficient, and

$$\langle \beta \rangle_n = \prod_{k=0}^{n-1} (\beta - k) = \begin{cases} \beta(\beta-1)\cdots(\beta-n+1), & n \in \mathbb{N} \\ 1, & n = 0 \end{cases}$$

for $\beta \in \mathbb{C}$ is called the falling factorial;

2. the functions $\Gamma(x+\ell)\eta(x)$ on $(0, \infty)$ for $\ell = 1, 2$ and their logarithmic convexity.

2. A monotonicity rule

For proving our main results in this paper, we need the following lemma, a monotonicity rule for the ratio of two integrals with a parameter, which can be found in [17, Lemma 2.7 and Remark 6.3] and [19, Remark 7.2].

LEMMA 2.1. *Let $U(t)$, $V(t) > 0$, and $W(t,x) > 0$ be integrable in $t \in (a,b)$,*

1. *if the ratios $\frac{\partial W(t,x)/\partial x}{W(t,x)}$ and $\frac{U(t)}{V(t)}$ are both increasing or both decreasing in $t \in (a,b)$, then the ratio*

$$R(x) = \frac{\int_a^b W(t,x)U(t)\,\mathrm{d}t}{\int_a^b W(t,x)V(t)\,\mathrm{d}t}$$

   is increasing in x;

2. *if one of the ratios $\frac{\partial W(t,x)/\partial x}{W(t,x)}$ and $\frac{U(t)}{V(t)}$ is increasing and another is decreasing in $t \in (a,b)$, then the ratio $R(x)$ is decreasing in x.*

REMARK 2.1. Lemma 4 in [24] states that, if the ratio $\frac{A(x)}{B(x)}$ is increasing on $(0,\infty)$, then the ratio $\frac{\int_0^\infty A(x)e^{-xt}\,\mathrm{d}x}{\int_0^\infty B(x)e^{-xt}\,\mathrm{d}x}$ is decreasing in $t \in (0,\infty)$. This monotonicity rule for the ratio of two Laplace transforms can be deduced from Lemma 2.1 by setting $(a,b) = (0,\infty)$, $U(x) = A(x)$, $V(x) = B(x)$, and $W(x,t) = e^{-xt}$. This means that Lemma 2.1 is a generalization of [24, Lemma 4].

There have been a number of literature, such as [4, pp. 10–11, Theorem 1.25] and [5, 10, 11, 12, 13, 14, 15, 23], dedicated to investigation and application of various monotonicity rules for ratios of two functions, of two integrals, of two Laplace transforms, and the like.

3. Increasing property and logarithmic convexity

We are now in a position to state and prove our main results in this paper.

THEOREM 3.1. *Let $\alpha > 0$ be a constant and let $\ell \geqslant 0$ be an integer. Then the two functions defined in (1.3) are increasing from $(0,\infty)$ onto $(0,\infty)$. Consequently, for fixed $\ell = 1,2$, the functions $\Gamma(x+\ell)\eta(x)$ are logarithmically convex in $x \in (0,\infty)$.*

Proof. With the aid of the recurrence relation $\Gamma(z+1) = z\Gamma(z)$ and the integral representation (1.2), integrating by parts yields

$$\frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\eta(x+\alpha)}{\eta(x)} = \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\frac{1}{\Gamma(x+\alpha)}\int_0^\infty \frac{t^{x+\alpha-1}}{e^t+1}\,\mathrm{d}t}{\frac{1}{\Gamma(x)}\int_0^\infty \frac{t^{x-1}}{e^t+1}\,\mathrm{d}t}$$

$$= \frac{(x+\alpha)\int_0^\infty \frac{t^{x+\alpha-1}}{e^t+1}\,\mathrm{d}t}{x\int_0^\infty \frac{t^{x-1}}{e^t+1}\,\mathrm{d}t}$$

$$= \frac{\int_0^\infty \frac{1}{e^t+1}\frac{dt^{x+\alpha}}{dt}\,dt}{\int_0^\infty \frac{1}{e^t+1}\frac{dt^x}{dt}\,dt}$$

$$= \frac{\int_0^\infty \left[\frac{e^t}{(e^t+1)^2}t^\alpha\right]t^x\,dt}{\int_0^\infty \frac{e^t}{(e^t+1)^2}t^x\,dt}.$$

Applying Lemma 2.1 to

$$U(t) = \frac{e^t}{(e^t+1)^2}t^\alpha, \quad V(t) = \frac{e^t}{(e^t+1)^2} > 0, \quad W(t,x) = t^x > 0,$$

and $(a,b) = (0,\infty)$, since $\frac{U(t)}{V(t)} = t^\alpha$ and $\frac{\partial W(t,x)/\partial x}{W(t,x)} = \ln t$ are both increasing on $(0,\infty)$, we conclude by (1.4) that the function

$$\frac{\int_0^\infty \left[\frac{e^t}{(e^t+1)^2}t^\alpha\right]t^x\,dt}{\int_0^\infty \frac{e^t}{(e^t+1)^2}t^x\,dt} = \frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\eta(x+\alpha)}{\eta(x)}$$

$$= \Gamma(\alpha+1)\binom{x+\alpha}{\alpha}\frac{\eta(x+\alpha)}{\eta(x)}$$

is increasing in $x \in (0,\infty)$.

Similarly, integrating by parts, we obtain

$$\frac{\Gamma(x+\alpha+2)}{\Gamma(x+2)}\frac{\eta(x+\alpha)}{\eta(x)} = \frac{\frac{\Gamma(x+\alpha+2)}{\Gamma(x+\alpha+1)}}{\frac{\Gamma(x+2)}{\Gamma(x+1)}}\frac{\Gamma(x+\alpha+1)}{\Gamma(x+1)}\frac{\eta(x+\alpha)}{\eta(x)}$$

$$= \frac{\frac{\Gamma(x+\alpha+2)}{\Gamma(x+\alpha+1)}}{\frac{\Gamma(x+2)}{\Gamma(x+1)}}\frac{\int_0^\infty \frac{e^t}{(e^t+1)^2}t^{x+\alpha}\,dt}{\int_0^\infty \frac{e^t}{(e^t+1)^2}t^x\,dt}$$

$$= \frac{\frac{\Gamma(x+\alpha+2)}{\Gamma(x+\alpha+1)}}{\frac{\Gamma(x+2)}{\Gamma(x+1)}}\frac{\int_0^\infty \left(\frac{1}{e^t+1}\right)'t^{x+\alpha}\,dt}{\int_0^\infty \left(\frac{1}{e^t+1}\right)'t^x\,dt}$$

$$= \frac{\frac{\Gamma(x+\alpha+2)}{\Gamma(x+\alpha+2)}}{\frac{\Gamma(x+2)}{\Gamma(x+2)}}\frac{(x+\alpha+1)\int_0^\infty \left(\frac{1}{e^t+1}\right)'t^{x+\alpha}\,dt}{(x+1)\int_0^\infty \left(\frac{1}{e^t+1}\right)'t^x\,dt}$$

$$= \frac{\int_0^\infty \left(\frac{1}{e^t+1}\right)'\frac{dt^{x+\alpha+1}}{dt}\,dt}{\int_0^\infty \left(\frac{1}{e^t+1}\right)'\frac{dt^{x+1}}{dt}\,dt}$$

$$= \frac{\int_0^\infty \left(\frac{1}{e^t+1}\right)''t^{x+\alpha+1}\,dt}{\int_0^\infty \left(\frac{1}{e^t+1}\right)''t^{x+1}\,dt}$$

$$= \frac{\int_0^\infty \left[\frac{e^t(e^t-1)}{(e^t+1)^3}t^{\alpha+1}\right]t^x\,dt}{\int_0^\infty \left[\frac{e^t(e^t-1)}{(e^t+1)^3}t\right]t^x\,dt}.$$

Applying Lemma 2.1 to

$$U(t) = \frac{e^t(e^t-1)}{(e^t+1)^3}t^{\alpha+1}, \quad V(t) = \frac{e^t(e^t-1)}{(e^t+1)^3}t > 0, \quad W(t,x) = t^x > 0,$$

and $(a,b) = (0,\infty)$, since $\frac{U(t)}{V(t)} = t^\alpha$ and $\frac{\partial W(t,x)/\partial x}{W(t,x)} = \ln t$ are both increasing on $(0,\infty)$, we conclude by (1.4) that the function

$$\frac{\int_0^\infty \left[\frac{e^t(e^t-1)}{(e^t+1)^3}t^{\alpha+1}\right]t^x\,dt}{\int_0^\infty \left[\frac{e^t(e^t-1)}{(e^t+1)^3}t\right]t^x\,dt} = \frac{\Gamma(x+\alpha+2)}{\Gamma(x+2)}\frac{\eta(x+\alpha)}{\eta(x)}$$

$$= \Gamma(\alpha+1)\binom{x+\alpha+1}{\alpha}\frac{\eta(x+\alpha)}{\eta(x)}$$

is increasing in $x \in (0,\infty)$.

Because the function $\frac{\Gamma(x+\alpha+\ell)}{\Gamma(x+\ell)}\frac{\eta(x+\alpha)}{\eta(x)}$ for $\ell = 1,2$ is increasing in $x \in (0,\infty)$, its first derivative

$$\left[\frac{\Gamma(x+\alpha+\ell)}{\Gamma(x+\ell)}\frac{\eta(x+\alpha)}{\eta(x)}\right]'$$
$$= \frac{[\Gamma(x+\alpha+\ell)\eta(x+\alpha)]'[\Gamma(x+\ell)\eta(x)] - [\Gamma(x+\alpha+\ell)\eta(x+\alpha)][\Gamma(x+\ell)\eta(x)]'}{[\Gamma(x+\ell)\eta(x)]^2}$$

is positive for $x \in (0,\infty)$. Hence, we have

$$\frac{[\Gamma(x+\alpha+\ell)\eta(x+\alpha)]'}{\Gamma(x+\alpha+\ell)\eta(x+\alpha)} > \frac{[\Gamma(x+\ell)\eta(x)]'}{[\Gamma(x+\ell)\eta(x)]},$$

that is, the logarithmic derivative

$$(\ln[\Gamma(x+\ell)\eta(x)])' = \frac{[\Gamma(x+\ell)\eta(x)]'}{[\Gamma(x+\ell)\eta(x)]}$$

is increasing in $x \in (0,\infty)$. Consequently, for $\ell = 1,2$, the function $\Gamma(x+\ell)\eta(x)$ is logarithmically convex in $(1,\infty)$. The proof of Theorem 3.1 is complete. $\square$

REMARK 3.1. Because the third derivative

$$\left(\frac{1}{e^t+1}\right)''' = -\frac{e^t[(e^t-1)^2 - 2e^t]}{(e^t+1)^4}$$

is not monotonic and does not keep the same sign on $(0,\infty)$, we can not find any more result by the method used in this paper.

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