# ON THE $q$–ANALOGUES FOR SOME
# KANTOROVICH TYPE LINEAR OPERATORS

NAZLIM DENIZ ARAL* AND ZEYNEP SEVINÇ

(*Communicated by V. Gupta*)

*Abstract.* In this paper, we present some Kantorovich type positive linear operators and we introduce the new modification $q$-Baskakov-Kantorovich operators. We prove the convergence of the new operators using the Korovkin criterion and establish the rate of convergence involving the modulus of continuity. Also, some numerical results to illustrate the convergence of these operators.

## 1. Introduction

The quantum calculus ($q$-calculus) which is an ordinary calculus without taking limits has a lot of applications in various areas such as quantum theory, mathematics, theory of relativity and mechanics. Jackson [20] introduced $q$-functions in the beginning of the twentieth century and developed $q$-calculus. In the approximation theory, $q$-Bernstein polynomials were first presented as the applications of $q$-calculus by Lupas [22]. $q$-analogues of the well known operators have been proposed and their approximation behavior has been discussed (see ([3]–[6], [12], [15], [18], [19], [25]–[30]). After that various generalization of the $q$-Baskakov operators were intoroduced and their approximation properties have been studied in ([3]–[6], [18]).

The Kantorovich operators $K_n : L_p([0,1]) \to L_p([0,1])$ defined by

$$K_n(f,x) = (n+1) \sum_{k=0}^{n} x^k (1-x)^{n-k} \int_{\frac{k}{n+1}}^{\frac{k+1}{n+1}} f(t)dt, \quad k = 1,2,3,\dots$$

for $n \in \mathbb{N}$, $x \in [0,1]$ and $f \in L_p[0,1]$, $1 \leqslant p < \infty$ [21]. The classical Baskakov operators is defined for $f \in C[0,\infty)$ by [7].

Kantorovich modifications of Baskakov operator was defined by Ditzian and Totik in [11]. Baskakov-Kantorovich operators and their generalizations are many studies in literature. Some of them are ([1], [2], [8], [9], [14], [16], [17], [31], [32]).

We recall some significant basic notations, definitions on the concept of $q$-calculus for the $q$-integers of the number $m$ are given as

$$[m]_q := \frac{1 - q^m}{1 - q}, \quad q \neq 1; \quad [m]_1 := m \quad \text{and} \quad [0]_q := 0$$

and

$$[m]_q! := [m]_q[m-1]_q \cdots [1]_q; \quad [0]_q! := 1.$$

The analogue of $(a+b)^m$ in $q$-analysis, are given by

$$(a \oplus b)_q^m := \sum_{v=0}^{m} \begin{bmatrix} m \\ v \end{bmatrix}_q q^{v(v-1)/2} b^v a^{n-v}$$

where $\begin{bmatrix} m \\ v \end{bmatrix}_q := \frac{[m]_q!}{[v]_q![m-v]_q!}$ for $0 \leqslant v \leqslant m$ is the $q$-binomial formula. The $q$-partial derivative of $f(x,y)$ is given by

$$\frac{\partial_q f(x,y)}{\partial_q x} = \frac{f(qx,y) - f(x,y)}{(q-1)x}, \quad x \neq 0.$$

Now, we give the $q$-analogue of integration;

$$\int_0^a f(t) d_q t = (1-q)a \sum_{v=0}^{\infty} f(q^v a) q^v, \quad a > 0$$

and suppose $0 < a < b$,

$$\int_a^b f(t) d_q t = \int_0^b f(t) d_q t - \int_0^a f(t) d_q t.$$

## 2. Main results and proofs

In the present work, inspired by the work of Şimşek and Tunç [29], we progress this studies to $q$-Kantorovich type operators and define a new type $q$-Baskakov and $q$-Baskakov-Kantorovich operators. Furthermore, we obtain local approximation properties and weighted approximation properties of $q$-Baskakov-Kantorovich operators.

We introduce a new type the Kantorovich operators as follows:

$$L_{m,q}^*(f;x) = \sum_{v=0}^{\infty} \frac{1}{[v]!} \frac{\partial_q^v \varphi_{m,q}(x,u)}{\partial_q u^v} \bigg|_{u=0} \int_0^1 f\left(\frac{[v]_q + q^v t}{\alpha_{m,v,q}}\right) d_q t$$

where $\alpha_{m,v,q}$ are positive numbers and $\{\varphi_{m,q}(x,u)\}$ generating real functions defined on $I \times [0,\infty)$ have the following conditions:

(i) For all $m \in \mathbb{N}$ and $x \in I$, $\varphi_{m,q}(x,0) \neq 0$ and $\varphi_{m,q}(x,1) = 1$.

(ii) $\frac{\partial_q^v \varphi_{m,q}(x,u)}{\partial_q u^v} \bigg|_{u=0}$ exists and are continuous functions of $x$ for all $v \in \mathbb{N}_0$ and $m \in \mathbb{N}$.

(iii) $\frac{\partial_q^\nu \varphi_{m,q}(x,u)}{\partial_q u^\nu} \geqslant 0$, for all $\nu \in \mathbb{N}_0$, $x, u \geqslant 0$.

It is seen that this operators are positive and linear for on the space of bounded functions on $I$.

We have some generating functions in order to generate Kantorovich type linear operators such as $q$-Bernstein-Kantorovich and $q$-Lupas-Kantorovich operators.

EXAMPLE 1. ($q$-Bernstein-Kantorovich operators)

Let us the function $\varphi_{m,q}(x,u) := (1 \boxminus x \boxplus xu)_q^m$, $x \in [0,\infty)$, $m \in \mathbb{N}$ where $(a \boxminus b \boxplus c)_q^m := \sum_{\nu=0}^m \begin{bmatrix} m \\ \nu \end{bmatrix}_q (a \ominus b)_q^{m-\nu} c^\nu$. Here, $(a \ominus b)_q^m := (a \oplus (-b))_q^m$. Using the $q$-partial derivatives, we have $\frac{\partial_q^\nu \varphi_{m,q}(x,u)}{\partial_q u^\nu} = [m]_q x^\nu (1 \boxminus x \boxplus xu)_q^{m-\nu}$, $\nu \in \mathbb{N}$ and for $u = 0$, we get

$$\frac{\partial_q^\nu \varphi_{m,q}(x,u)}{\partial_q u^\nu}\bigg|_{u=0} = [m]_q x^\nu \prod_{s=0}^{m-\nu-1} (1 - q^s x). \tag{1}$$

Writing (1) and considering $\alpha_{m,\nu,q} = [m+1]_q$ in the operators $L_{m,q}^*$, we obtain

$$L_{m,q}^*(f;x) = \sum_{\nu=0}^\infty \frac{1}{[\nu]!} \frac{\partial_q^\nu \varphi_{m,q}(x,u)}{\partial_q u^\nu}\bigg|_{u=0} \int_0^1 f\left(\frac{[\nu]_q + q^\nu t}{\alpha_{m,\nu,q}}\right) d_q t$$

$$= \sum_{\nu=0}^\infty \begin{bmatrix} m \\ \nu \end{bmatrix}_q x^\nu \prod_{s=0}^{m-\nu-1} (1 - q^s x) \int_0^1 f\left(\frac{[\nu]_q + q^\nu t}{[m+1]_q}\right) d_q t$$

and for $\nu > m$, $\begin{bmatrix} m \\ \nu \end{bmatrix}_q := 0$. Thus, we have $q$-Bernstein-Kantorovich operators $B_{n,q}^*(f,x)$ defined by Mahmudov and Sabancigil [24].

EXAMPLE 2. ($q$-Lupas-Kantorovich Operators)

If we consider the function $\varphi_{m,q}(x,u) := \frac{((1-x) \oplus xu)_q^m}{((1-x) \oplus x)_q^m}$, $x \in [0,\infty)$, $m \in \mathbb{N}$ and $\alpha_{m,\nu,q} = [m+1]_q$ in the operators $L_{m,q}^*$, then we obtain the $q$-Lupas-Kantorovich operators $\tilde{R}_n(f,q,x)$ defined by Doğru and Kanat [13],

$$L_{m,q}^*(f;x) = \frac{1}{((1-x) \oplus x)_q^m} \sum_{\nu=0}^m \begin{bmatrix} m \\ \nu \end{bmatrix}_q q^{\frac{\nu(\nu-1)}{2}} x^\nu (1-x)^{m-\nu} \int_0^1 f\left(\frac{[\nu]_q + q^\nu t}{[m+1]_q}\right) d_q t.$$

Şimşek [28] introduced a new type $q$-analogue of Baskakov operators. Inspired by that work, we define a new type $q$-Baskakov operators, which for $q \in (0,1)$, $f \in ([0,\infty))$, $x \in \mathbb{R}^+ := [0,\infty)$,

$$E_{m,q}(f;x) = ((1+x) \ominus x)_q^m \sum_{\nu=0}^\infty \begin{bmatrix} m+\nu-1 \\ \nu \end{bmatrix}_q x^\nu (1+x)^{-m-\nu} f\left(\frac{[\nu]_q}{q^{\nu-1}[m]_q}\right).$$

It is seen that the $E_{m,q}(f;x)$ are positive and linear operators.

We use notations $e_r(t) := t^r$, $r \in \mathbb{N}_0$. By simple calculations we have the following lemma from Corollary 2.2. in [28].

LEMMA 1. *Let $x \in \mathbb{R}^+$ and $q \in (0,1)$, we obtain*

$$E_{m,q}(e_0,x) = 1,$$

$$E_{m,q}(e_1;x) = \frac{x}{q^{m-1}(1+x(1-q^m))},$$

$$E_{m,q}(e_2;x) = \frac{[m+1]_q}{q^{2m-3}[m]_q} \frac{x^2}{(1+x(1-q^m))(1+x(1-q^{m+1}))}$$

$$+ \frac{1}{q^{2m-2}[m]_q} \frac{x}{(1+x(1-q^m))}.$$

Next we define a new type $q$-Baskakov-Kantorovich operators as:

$$E_{m,q}^*(f;x) = ((1+x) \ominus x)_q^m \sum_{v=0}^{\infty} \begin{bmatrix} m+v-1 \\ v \end{bmatrix}_q x^v (1+x)^{-m-v} \int_0^1 f\left(\frac{[k]_q + q^v t}{q^{v-1}[m]_q}\right) d_q t \quad (2)$$

where $q \in (0,1)$. By simply calculations we can get

$$E_{m,q}^*(f;x) = ((1+x) \ominus x)_q^m [m]_q \sum_{v=0}^{\infty} \begin{bmatrix} m+v-1 \\ v \end{bmatrix}_q x^v (1+x)^{-m-v} q^{-1} \int_{\frac{[v]_q}{q^{v-1}[m]_q}}^{\frac{[v+1]_q}{q^{v-1}[m]_q}} f(t) \, d_q t.$$

LEMMA 2. *The following equalities are true;*

$$E_{m,q}^*(e_0;x) = 1;$$

$$E_{m,q}^*(e_1;x) = \frac{x}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q};$$

$$E_{m,q}^*(e_2;x) = \frac{[m+1]_q}{q^{2m-3}[m]_q} \frac{x^2}{(1+x(1-q^m))(1+x(1-q^{m+1}))}$$

$$+ \left(\frac{1}{q^{2m-2}[m]_q} + \frac{2q}{(1+q)[m]_q}\right) \frac{x}{(1+x(1-q^m))} + \frac{q^2}{[3]_q[m]_q^2}.$$

*Proof.* By using the Lemma 2.2. in [23], we obtain

$$E_{m,q}^*(e_r;x) = \sum_{\ell=0}^{r} [{}^r_\ell]_q \left(\frac{q}{[m]_q}\right)^{r-\ell} \frac{1}{[r-\ell+1]_q} E_{m,q}(e_\ell;x).$$

Thus, we have

$$E_{m,q}^*(e_0;x) = E_{m,q}(e_0;x) = 1,$$

$$E_{m,q}^*(e_1;x) = \frac{q}{[2]_q[m]_q} E_{m,q}(e_0;x) + E_{m,q}(e_1;x),$$

$$E_{m,q}^*(e_2;x) = \frac{q^2}{(1+q+q^2)[m]_q^2} E_{m,q}(e_0;x)$$

$$+ \frac{2q}{(1+q)[m]_q} E_{m,q}(e_1;x) + E_{m,q}(e_2;x). \quad \square$$

COROLLARY 1. *Central moments are given by*

$$E_{m,q}^*((t-x);x) = \frac{x}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} - x;$$

$$E_{m,q}^*((t-x)^2;x)$$
$$= \left( \frac{[m+1]_q}{q^{2m-3}[m]_q(1+x(1-q^m))(1+x(1-q^{m+1}))} - \frac{2}{q^{m-1}(1+x(1-q^m))} + 1 \right) x^2$$
$$+ \left( \left( \frac{1}{q^{2m-2}[m]_q} + \frac{2q}{(1+q)[m]_q} \right) \frac{1}{(1+x(1-q^m))} - \frac{2q}{[2]_q[m]_q} \right) x$$
$$+ \frac{q^2}{[3]_q[m]_q^2}.$$

*Proof.* We obtain first and second order central moments of the $q$-Baskakov Kantorovich operators by Lemma 2. We obtain,

$$E_{m,q}^*((t-x);x) = E_{m,q}^*(t;x) - xE_{m,q}^*(1;x) = \frac{x}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} - x$$

and

$$E_{m,q}^*((t-x)^2;x)$$
$$= E_{m,q}^*(t^2;x) - 2xE_{m,q}^*(t;x) + x^2E_{m,q}^*(1;x)$$
$$= \left( \frac{[m+1]_q}{q^{2m-3}[m]_q(1+x(1-q^m))(1+x(1-q^{m+1}))} - \frac{2}{q^{m-1}(1+x(1-q^m))} + 1 \right) x^2$$
$$+ \left( \left( \frac{1}{q^{2m-2}[m]_q} + \frac{2q}{(1+q)[m]_q} \right) \frac{1}{(1+x(1-q^m))} - \frac{2q}{[2]_q[m]_q} \right) x + \frac{q^2}{[3]_q[m]_q^2}. \quad \square$$

The space of of all functions $f$ defined on $[0,\infty)$ and satisfying $|f(x)| \leqslant M(1+x^2)$ is called the weighted space and denoted by $B_2[0,\infty)$ where $M > 0$ is constant. Then, $B_2[0,\infty)$ is a linear normed space with the norm $\|f\|_2 = \sup_{x \geqslant 0} \frac{|f(x)|}{1+x^2}$.

Let $C_2[0,\infty) = B_2[0,\infty) \cap C[0,\infty)$,

$$C_2^*[0,\infty) := \left\{ f \in C_2[0,\infty) : \lim_{x \to \infty} \frac{|f(x)|}{1+x^2} \text{is finite} \right\}$$

and the norm on this space $\|f\|_2 = \sup_{x \in [0,\infty)} \frac{f(x)}{1+x^2}$. Let $C_B[0,\infty)$ denote the space of all valued continuous and bounded functions $f$ on $[0,\infty)$ and the norm is $\|f\|_{C_B} = \sup_{x \geqslant 0} |f(x)|$.

THEOREM 1. *Let* $(q_m)_m$, $0 < q_m < 1$ *and the condition* $\lim_{m \to \infty} q_m = 1$ *is satisfy. Then for any* $f \in C_2^*[0,\infty)$, *the sequence* $\{E_{m,q_m}^*((f;x)\}$ *convergences to* $f$ *uniformly on* $[0,\infty)$.

*Proof.* By the weighted Korovkin theorem, we need to prove

$$\lim_{n\to\infty} \|E^*_{m,q_m} e_i - e_i\|_2 = 0, \quad i = 0, 1, 2.$$

i)

$$\lim_{m\to\infty} \|E^*_{m,q_m} e_0 - e_0\|_2 = \lim_{m\to\infty} \sup_{x\geqslant 0} \frac{|E^*_{m,q_m}(1;x) - 1|}{1 + x^2} = 0.$$

ii)

$$\lim_{n\to\infty} \|E^*_{m,q_m} e_1 - e_1\|_2 = \lim_{m\to\infty} \sup_{x\geqslant 0} \frac{|E^*_{m,q_m}(t;x) - x|}{1 + x^2}$$

$$= \lim_{m\to\infty} \sup_{x\geqslant 0} \frac{\left| \frac{x}{q_m^{m-1}(1+x(1-q_m^m))} + \frac{q_m}{[2]_{q_m}[m]_{q_m}} - x \right|}{1 + x^2}$$

$$\leqslant \lim_{m\to\infty} \left| \frac{1}{q_m^{m-1}(1+x(1-q_m^m))} - 1 \right| \sup_{x\geqslant 0} \frac{x}{1+x^2}.$$

By using $\frac{1}{1+x(1-q_m^m)} \leqslant 1$, we have

$$\lim_{n\to\infty} \|E^*_{m,q_m} e_1 - e_1\|_2 = 0.$$

iii)

$$\lim_{n\to\infty} \|E^*_{m,q_m} e_2 - e_2\|_2 = \lim_{m\to\infty} \sup_{x\geqslant 0} \frac{|E^*_{m,q_m}(t^2;x) - x^2|}{1 + x^2}$$

$$\leqslant \lim_{m\to\infty} \left| \frac{[m+1]_{q_m}}{q_m^{2m-3}[m]_{q_m}(1+x(1-q_m^m))(1+x(1-q_m^{m+1}))} - \frac{2}{q_m^{m-1}(1+x(1-q_m^m))} + 1 \right|$$

$$\times \sup_{x\geqslant 0} \frac{x^2}{1 + x^2}$$

$$+ \lim_{m\to\infty} \left| \left( \frac{1}{q_m^{2m-2}[m]_{q_m}} + \frac{2q_m}{(1+q_m)[m]_{q_m}} \right) \frac{1}{(1+x(1-q_m^m))} - \frac{2q_m}{[2]_{q_m}[m]_{q_m}} \right| \sup_{x\geqslant 0} \frac{x}{1 + x^2}$$

$$+ \lim_{m\to\infty} \frac{q_m^2}{[3]_{q_m}[m]_{q_m}^2} \sup_{x\geqslant 0} \frac{1}{1 + x^2}.$$

Therefore,

$$\lim_{n\to\infty} \|E^*_{m,q_m} e_2 - e_2\|_2 = 0. \quad \square$$

$K$-functional is showed by

$$K_2(f;\delta) = \inf_{g\in W^2} \{\|f - g\| + \delta\|g''\|\}$$

where $W^2 = \{g \in C[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By ([10]), there exists a positive constant $D > 0$ such that $K_2(f,\delta) \leqslant D\omega_2(f,\sqrt{\delta})$, $\delta > 0$, here

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h<\sqrt{\delta},u,u+h,u+2h\in[0,\infty)} |f(u+2h) - 2f(u+h) + f(u)|$$

is the second order modulus of smoothness of function $f \in C[0,\infty)$.

$$\omega(f,\sqrt{\delta}) = \sup_{0<h<\delta,u\in[0,\infty)} |f(u+h) - f(u)|.$$

is the first order modulus of continuity.

THEOREM 2. *For all $x \in \mathbb{R}^+$ and $f \in C_B[0,\infty)$, we have*

$$|E_{m,q}^*(f,x) - f(x)| \leqslant D\omega_2(f,\delta_m(x)) + \omega(f,s_m(x)),$$

*where the constant $D > 0$, $\delta_m(x) = \{E_{m,q}^*((t-x)^2;x) + (E_{m,q}^*((t-x);x))^2\}^{\frac{1}{2}}$ and $s_m(x) = \frac{x}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q}$.*

*Proof.* We construct the auxiliary operators $\bar{E}_{m,q}^*(f;x)$ as follows:

$$\bar{E}_{m,q}^*(f;x) = E_{m,q}^*(f;x) + f(x) - f\left(\frac{q}{[2]_q[m]_q} + \frac{x(1+q^{m-1}) + q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))}\right).$$

For $x \in \mathbb{R}^+$ and $g \in W^2$, from the Taylor's formula,

$$g(t) - g(x) = (t-x)g'(x) + \int_x^t (t-u)g''(u)du.$$

Then,

$$\bar{E}_{m,q}^*(g;x) - g(x)$$
$$= g'(x)\bar{E}_{m,q}^*((t-x);x) + \bar{E}_{m,q}^*\left(\int_x^t (t-u)g''(u)du,x\right)$$
$$= E_{m,q}^*\left(\int_x^t (t-u)g''(u)du,x\right) - \int_x^{\frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))}+\frac{q}{[2]_q[m]_q}}$$
$$\times \left(\frac{x(1+q^{m-1}) + q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} - u\right)g''(u)du.$$

Also, we have

$$\left|\int_x^t (t-u)g''(u)du\right| \leqslant \|g''\| \int_x^t |t-u|du \leqslant \|g''\|(t-x)^2.$$

So,

$$\left| \int_x^{\frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))}+\frac{q}{[2]_q[m]_q}} \left( \frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} - u \right) g''(u)du \right|$$

$$\leqslant \left( \frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} - x \right)^2 \|g''\|.$$

Therefore, we can conclude that

$$\left| \bar{E}_{m,q}^*(g;x) - g(x) \right|$$

$$= \left| E_{m,q}^* \left( \int_x^t (t-u)g''(u)du, x \right) \right| + \left| \int_x^{\frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))}+\frac{q}{[2]_q[m]_q}} \right.$$

$$\left. \times \left( \frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} - u \right) g''(u)du \right|$$

$$\leqslant \|g''\| E_{m,q}^*((t-x)^2, x) + \left( \frac{q}{[2]_q[m]_q} + \frac{x}{q^{m-1}(1+x(1-q^m))} \right)^2 \|g''\|$$

$$= \delta_m^2(x)\|g''\|.$$

On the other hand,

$$|E_{m,q}^*(f, x) - f(x)|$$

$$\leqslant |\bar{E}_{m,q}^*(f-g;x) - (f-g)(x)| + \left| f\left( \frac{x(1+q^{m-1})+q^{m-1}x^2(1-q^m)}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} \right) - f(x) \right|$$

$$+ |\bar{E}_{m,q}^*(g, x) - g(x)|$$

$$\leqslant 4\|f-g\| + \omega\left( f, \left| \frac{x}{q^{m-1}(1+x(1-q^m))} + \frac{q}{[2]_q[m]_q} \right| \right) + \delta_m^2(x)\|g''\|.$$

We get the following, taking the infimum on the right side over all $g \in W^2$,

$$|E_{m,q}^*(f, x) - f(x)| \leqslant 4K_2(f, \delta_m^2(x)) + \omega(f, s_m(x)).$$

Using the definition of $K$-functional, the proof is completed. $\quad\square$

We denote the,

$$Lip_M(\gamma) = \{ f \in C_B[0, \infty) : |f(u) - f(v)| \leqslant M|u-v|^r, u \in E \subset [0, \infty), v \in \mathbb{R}^+ \}$$

where $0 < r \leqslant 1$.

THEOREM 3. *For every $x \in \mathbb{R}^+$, we have*

$$\left| E_{m,q_m}^*(f;x) - f(x) \right| \leqslant M(A_1(x))^{r/2} + 2(d(x, E))^r,$$

*where $M$ is a constant and $A_1(x) = \{ E_{m,q}^*((t-x)^2;x) \}$. Here, the distance between $x$ and $E$ is defined $d(x, E) = \inf\{ |t-x| : t \in E \}$.*

*Proof.* Suppose that $y$ is in the closure of $E$ such that $|x-y| = d(x,E)$. If we use the definition of functions which belong to Lipschitz class, we can get

$$
\begin{aligned}
\left| E^*_{m,q_m}(f;x) - f(x) \right| &\leqslant E^*_{m,q_m}(|f(t) - f(y)|;x) + E^*_{m,q_m}(|f(x) - f(y)|;x) \\
&\leqslant M\{E^*_{m,q_m}(|t-y|^r;x) + |x-y|^r\} \\
&\leqslant M\{E^*_{m,q_m}(|t-x|^r + |x-y|^r;x) + |x-y|^r\} \\
&= M\{E^*_{m,q_m}(|t-x|^r;x) + 2|x-y|^r\}.
\end{aligned}
$$

Now, we use Hölder inequality with $p = 2/r$ and $q = 2/(2-r)$

$$
\begin{aligned}
\left| E^*_{m,q_m}(f;x) - f(x) \right| &\leqslant M\{[E^*_{m,q_m}(|t-x|^{rp};x)]^{1/p} + 2(d(x,E))^r\} \\
&= M\{[E^*_{m,q_m}(|t-x|^2;x)]^{r/p} + 2(d(x,E))^r\} \\
&\leqslant M\{(A_1(x))^{r/2} + 2(d(x,E))^r\}. \quad \square
\end{aligned}
$$

THEOREM 4. *Let $E^*_{m,q}(f;x)$ satisfy the conditions Theorem 1 for $0 < q_m \leqslant 1$. Then for $f \in C^*_2[0,\infty)$ such that $f'$, $f'' \in C^*_2[0,\infty)$ we have*

$$
\lim_{m \to \infty} [m]_{q_m}(E^*_{m,q_m}(f;x) - f(x)) = \left( \frac{1}{2} + \gamma(x) \right) f'(x) + \beta(x)f''(x)/2
$$

*uniformly on $[0,a]$ for any $a > 0$ where*

$$
\gamma(x) = \lim_{m \to \infty} [m]_{q_m} x\left(-1 + \frac{1}{q_m^{m-1}(1+x(1-q^m))}\right),
$$

$$
\beta(x) = \lim_{m \to \infty} [m]x^2 \left( \frac{[m+1]_q}{q^{2m-3}[m]_q(1+x(1-q^m))(1+x(1-q^{m+1}))} - \frac{2}{q^{m-1}(1+x(1-q^m))} + 1 \right)
$$
$$
+ [m]x \lim_{m \to \infty} \left( \left( \frac{1}{q^{2m-2}[m]_q} + \frac{2q}{(1+q)[m]_q} \right) \frac{1}{(1+x(1-q^m))} - \frac{2q}{[2]_q[m]_q} \right).
$$

*Proof.* For a given function $f$, $f'$, $f'' \in C^*_2[0,\infty]$, we can write by the Taylor expansion for $t,x \in [0,\infty)$

$$
f(t) - f(x) = f'(x)(t-x) + f''(x)(t-x)^2/2 + k(t,x)(t-x)^2 \tag{3}
$$

where $k(t,x)$ is the remainder term and $\lim_{t \to x} k(t,x) = 0$. Applying $E^*_{m,q_m}$ operators to (3), we obtain

$$
\begin{aligned}
E^*_{m,q_m}(f;x) - f(x) &= E^*_{m,q_m}((t-x);x)f'(x) + E^*_{m,q_m}((t-x)^2;x)f''(x)/2 \\
&\quad + E^*_{m,q_m}(k(t,x)(t-x)^2;x).
\end{aligned}
$$

According to the Cauchy-Schwarz inequality for the remainder term, we obtain

$$
E^*_{m,q_m}(k(t,x)(t-x)^2;x) \leqslant \sqrt{E^*_{m,q_m}(k^2(t,x);x)} \sqrt{E^*_{m,q_m}((t,x)^4;x)}. \tag{4}
$$

By using $k(t,x) \in C_2^*[0,\infty)$ and $\lim_{t \to x} k(t,x) = 0$, we observe that

$$\lim_{m \to \infty} E_{m,q_m}^* (k^2(t,x);x) = k^2(x,x) = 0 \tag{5}$$

uniformly for each $x \in [0,a]$. Hence, by using (4) and (5) and positivity of the operators $E_{m,q_m}^*$, we have

$$\lim_{m \to \infty} [m]_{q_m} E_{m,q_m}^* (k(t-x)(t-x)^2;x) = 0.$$

So,

$$\lim_{m \to \infty} [m]_{q_m} (E_{m,q_m}^* (f;x) - f(x)) = \lim_{m \to \infty} [m]_{q_m} f'(x) E_{m,q_m}^* ((t-x);x) \tag{6}$$

$$+ \lim_{m \to \infty} [m]_{q_m} \frac{f''(x)}{2} E_{m,q_m}^* ((t-x)^2;x).$$

Consider

$$\lim_{m \to \infty} [m]_{q_m} E_{m,q_m}^* ((t-x);x) = \gamma(x) + \frac{1}{2} \tag{7}$$

and

$$\lim_{m \to \infty} [m]_{q_m} E_{m,q_m}^* ((t-x)^2;x) = \beta(x). \tag{8}$$

Then by using (6), (7) and (8), the proof is complete. $\square$

## 3. Numerical example

Let us take $f(x) = x^2 - x$. An approximation of the operators $E_{m,q}^*$ for a function $f(x)$ is illustrated in Figure 1, taking respectively $m = 5$, $m = 10$, $m = 15$ by MATLAB.
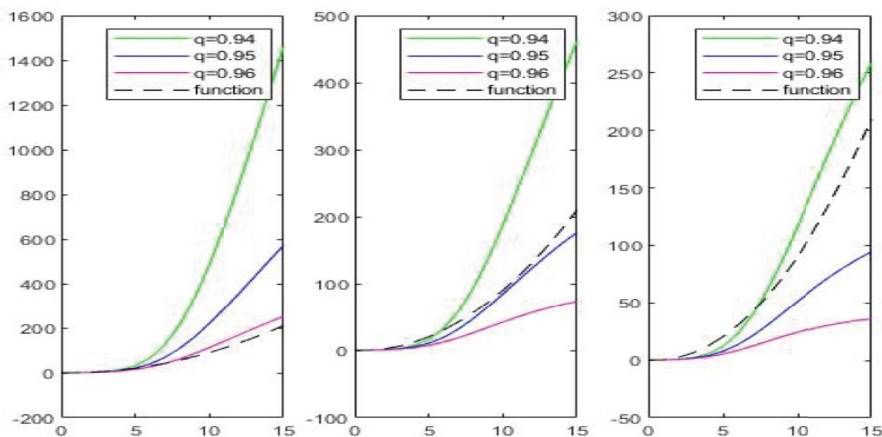


Figure 1: *Convergence of $E_{m,q}^*(f;x)$*

# REFERENCES

[1] T. ACAR, A. ARAL AND S. A. MOHIUDDINE, *On Kantorovich modification of* $(p,q)$-*Baskakov operators*, J. Inequal. Appl., (2016), Article ID 98.

[2] P. N. AGRAWAL AND M. GOYAL, *Generalized Baskakov Kantorovich Operators*, Filomat, **31**, 19 (2017), 6131–6151.

[3] A. ARAL AND V. GUPTA, *On* $q$-*Baskakov type operators*, Demons. Math., **42**, 1 (2009), 109–122.

[4] A. ARAL AND V. GUPTA, *On the Durrmeyer type modification of the* $q$-*Baskakov type operators*, Nonlinear Anal.: Theor. Meth. Appl., **72**, 3–4 (2010), 1171–1180.

[5] A. ARAL AND V. GUPTA, *Generalized* $q$-*Baskakov operators*, Math. Slovaca, **61**, 4 (2011), 619–634.

[6] A. ARAL, V. GUPTA AND R. P. AGARVAL, *Applications of* $q$-*calculus in Operator Theory*, Springer, New York, (2013).

[7] V. A. BASKAKOV, *An instance of a sequence of the linear positive operators in the space of continuous functions*, Docl. Akad. Nauk SSSR, **113**, (1957), 249–251.

[8] K. BOGALSKA, *The Voronovskaya type theorem for the Baskakov-Kantorovich operators*, Fasc. Math., **30**, (1999), 5–13.

[9] F. CAO AND C. DING, *Lp approximation by multivariate Baskakov-Kantorovich operators*, J. Math. Anal. Appl., **348**, 2 (2008), 856–861.

[10] R. A. DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer, Berlin, (1993).

[11] Z. DITZIAN AND V. TOTIK *Moduli of smoothness*, Springer-Verlag, Berlin/New York, (1987).

[12] O. DOĞRU AND O. DUMAN, *Statistical approximation of Meyer-König and Zeller operators based on the* $q$-*integers*, Publ. Math. Debrecen, **68**, (2006), 190–214.

[13] O. DOĞRU AND K. KANAT, *On statistical approximation properties of the Kantorovich type Lupas operators*, Math. Comput. Model, **55**, (2012), 1610–1621.

[14] A. ERENCIN AND S. BÜYÜKDURAKOĞLU, *A modification of generalized Baskakov-Kantorovich operators*, Stud. Univ. Babeş-Bolyai Math., **59**, 3 (2014), 351–364.

[15] T. ERNST, *A new notation for* $q$-*calculus and new* $q$-*Taylor formula*, U.U.D.M. Report 1999:25, ISSN 1101-3591, Department of Mathematics, Uppsala University, 1999.

[16] I. GADJEV, *Approximation of Functions by Baskakov-Kantorovich Operator*, Results. Math., **70**, (2016), 385–400.

[17] V. GUPTA AND C. RADU, *Statistical approximation properties of* $q$-*Baskakov-Kantorovich operators*, Cent. Eur. J. Math., **7**, 4 (2009), 809–818.

[18] Z. FINTA AND V. GUPTA, *Approximation properties of* $q$-*Baskakov Operators*, Cent. Eur. J. Math., **8**, 1 (2010), 199–211.

[19] W. HEPING, *Properties of convergence for the* $q$-*Meyer-König and Zeller operators*, J. Math. Anal. Appl., **335**, 2 (2007), 1360–1373.

[20] F. H. JACKSON, *On* $q$-*functions and a certain difference operator*, Transactions of the Royal Society Edinburgh, **46**, 2 (1909), 253–281.

[21] L. V. KANTOROVICH, *Sur certain développements suivant les polynômes de la forme de S. Bernstein, I., II.*, C.R. Acad. Sci. USSR A, (1930), 563–568, 595–600.

[22] A. LUPAS, *A* $q$-*analogue of the Bernstein operator*, In Seminar on Numerical and Statistical Calculus, University of Cluj-Napoca, **9**, (1987), 85–92.

[23] N. I. MAHMUDOV, *Statistical approximation of Baskakov and Baskakov-Kantorovich operators based on the* $q$-*integers*, Cent. Eur. J. Math., **8**, 4 (2010), 816–826.

[24] N. I. MAHMUDOV AND P. SABANCIGIL, *Approximation Theorems for* $q$-*Bernstein-Kantorovich Operators*, Filomat, **27**, 4 (2013), 721–730.

[25] S. OSTROVSKA, $q$-*Bernstein polynomials and their iterates*, J. Approx. Theory, **123**, 2 (2003), 232–255.

[26] G. M. PHILLIPS, *Bernstein polynomials based on the* $q$-*integers*, Ann. Numer. Math., **4**, (1997), 511–518.

[27] C. RADU, *On statistical approximation of a general class of positive linear operators extended in* $q$-*calculus*, Math. Comput., **215**, 6 (2009), 2317–2325.

[28] E. ŞIMŞEK, *On a New Type of* $q$-*Baskakov Operators*, Süleyman Demirel Üniversitesi Fen Bil. Derg., **22**, 1 (2018), 121–125.

[29] E. ŞIMŞEK AND T. TUNÇ, *On the construction of* $q$-*analogues for some positive linear operators*, Filomat, **31**, 13 (2017), 4287–4295.

[30] E. ŞIMŞEK AND T. TUNÇ, *On approximation properties of some class positive linear operators in q-analysis*, J. Math. Inq., **12**, 2 (2018), 4287–4295.

[31] A. WAFI AND S. KHATOON, *The Voronovskaya theorem for generalized Baskakov-Kantorovich operators in polynomial weight spaces*, Mat. Vesnik, **57**, 3–4 (2005), 87–94.

[32] C. ZHANG AND Z. ZHU, *Preservation properties of the Baskakov-Kantorovich operators*, Comput. Mat. Appl., **57**, 9 (2009), 1450–1455.

(Received June 1, 2020)

*Nazlım Deniz Aral*
*Department of Mathematics*
*Faculty of Arts and Sciences, Bitlis Eren University*
*Bitlis, 13000, Turkey*
*e-mail:* `ndaral@beu.edu.tr`

*Zeynep Sevinç*
*Bitlis Eren University*
*Mathematics Institute of Science*
*Turkey*
*e-mail:* `z.sevinc13@gmail.com`

Journal of Mathematical Inequalities
www.ele-math.com
jmi@ele-math.com