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THE t–MODIFICATION OF HOMOGENEOUS SYMMETRIC MEANS CONCERNING COMPLETE ELLIPTIC INTEGRALS

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Abstract. In the article, we establish a monotonicity criterion for the t-modification of homogeneous symmetric mean, and provide several novel inequalities for certain homogeneous symmetric means concerning the complete elliptic integrals. Besides, we also give the answer to a conjecture recently proposed by Nishimura.

1. Introduction

As is well known, for $r \in (0,1)$, Legendre's complete elliptic integrals $\mathscr{K}(r)$ and $\mathscr{E}(r)$ of the first and second kinds are defined by

$$\mathscr{K}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{-1/2} d\theta = \int_0^1 [(1-t^2)(1-r^2t^2)]^{-1/2} dt,$$

$$\mathscr{E}(r) = \int_0^{\pi/2} (1-r^2\sin^2\theta)^{1/2} d\theta = \int_0^1 \left(\frac{1-r^2t^2}{1-t^2}\right)^{1/2} dt$$

have always been playing an important role and wide applications in mathematics and physics as well as many other natural and human social sciences [1, 5, 6, 9, 10, 13, 15, 17, 18, 20]. In the past few years, the complete elliptic integrals have been studied deeply in the theory of mean values, and some elegant inequalities for $\mathscr{K}(r)$ and $\mathscr{E}(r)$ were derived in the literatures [2, 3, 4, 7, 8, 14, 23, 26].

Let $M : (0,\infty)\times(0,\infty) \mapsto (0,\infty)$ be a continuous real-valued function. Then M is said to be a homogeneous symmetric mean (cf. [9, Definition 8.1]) if M satisfies

$$\min\{a,b\} \leqslant M(a,b) \leqslant \max\{a,b\}, \quad M(a,b) = M(b,a), \quad M(\lambda a, \lambda b) = \lambda M(a,b)$$

for all $a,b \in (0,\infty)$ and $\lambda > 0$. M is said to be a strict homogeneous symmetric mean if $\min\{a,b\} < M(a,b) < \max\{a,b\}$ for all $a \neq b$. Many classical means, for example, the

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geometric mean $G(a,b)=\sqrt{ab}$, arithmetic mean $A(a,b)=(a+b)/2$, contra-harmonic mean $C(a,b)=\left(a^2+b^2\right)/(a+b)$, logarithmic mean

$$L(a,b)=\begin{cases}\dfrac{a-b}{\log a-\log b}, & a\neq b,\\ a, & a=b,\end{cases}$$

and arithmetic-geometric mean

$$AGM(a,b)=\lim_{n\to\infty}a_n=\lim_{n\to\infty}b_n$$

of two positive real numbers a and b are the strict homogeneous symmetric means, where the sequences $\{a_n\}_{n=0}^{\infty}$ and $\{b_n\}_{n=0}^{\infty}$ are given by

$$a_0=a,\quad b_0=b,$$

$$a_{n+1}=A(a_n,b_n)=\frac{a_n+b_n}{2},\quad b_{n+1}=G(a_n,b_n)=\sqrt{a_nb_n}.$$

It follows from the Gauss identity [5, Theorem 4.4]

$$AGM(1,r)\mathscr{K}(\sqrt{1-r^2})=\frac{\pi}{2} \tag{1.1}$$

that

$$\begin{aligned}AGM(a,b)&=\frac{\pi/2}{\int_0^{\pi/2}(a^2\cos^2\theta+b^2\sin^2\theta)^{-1/2}d\theta}\\&=\begin{cases}\pi a/\left[2\mathscr{K}\left(\sqrt{1-(b/a)^2}\right)\right], & a\geqslant b,\\ \pi b/\left[2\mathscr{K}\left(\sqrt{1-(a/b)^2}\right)\right], & a<b.\end{cases}\end{aligned}$$

Due to the formula of the circumference of an ellipse, another important bivariate homogeneous symmetric mean $T(a,b)$ (cf. [21] or [9, pp. 13]), named Toader mean, was reintroduced by Wang et al. in [24]:

$$T(a,b)=\frac{2}{\pi}\int_0^{\pi/2}(a^2\cos^2\theta+b^2\sin^2\theta)^{1/2}d\theta=\begin{cases}2a\mathscr{E}\left(\sqrt{1-(b/a)^2}\right)/\pi, & a\geqslant b,\\ 2b\mathscr{E}\left(\sqrt{1-(a/b)^2}\right)/\pi, & a<b.\end{cases}$$

By proving sharp bounds for Toader mean in terms of other classical means, Barnard, Pearce and Richards [7, 8], and Wang and Chu et. al [11, 12, 23, 25] have found many new inequalities for the complete elliptic integral $\mathscr{E}(r)$, as well as the perimeter of an ellipse.

It was proved in [5, Subsection 4.11] and [11, Theorem 3.1] that the inequalities

$$G(a,b)<L(a,b)<AGM(a,b)<A(a,b)<T(a,b)<C(a,b) \tag{1.2}$$

hold for all $a,b>0$, and each inequality becomes equality if and only if $a=b$.

In 1994, in order to research the asymptotic behavior for $AGM(a,b)$ and $\mathscr{K}(r)$, Vamanamurthy and Vuorinen [22] introduced the t-modification $M(a,b;t)$ (called t-th M mean) of a strict homogeneous symmetric mean $M(a,b)$ as follows

$$M(a,b;t)=\begin{cases}[M(a^t,b^t)]^{1/t}, & t\neq 0,\\ G(a,b)=\sqrt{ab}, & t=0.\end{cases} \tag{1.3}$$

It is apparent from (1.3) that $M(a,b;0)=G(a,b)$ and $M(a,b;1)=M(a,b)$ for any strict homogeneous symmetric mean M, and $G(a,b;t)=G(a,b)$ and $A(a,b;t)$ is the classical power (Hölder) mean of order t. Vamanamurthy and Vuorinen [22] also proved that

THEOREM 1.1. ([22, Theorem 1.2]) (1) *The t-th arithmetic-geometric mean $AGM(a,b;t)$ is strictly increasing with respect to $t\in\mathbb{R}$;*

(2) *The t-th logarithmic mean $L(a,b;t)$ is strictly increasing with respect to $t\in\mathbb{R}$.*

THEOREM 1.2. ([22, Theorems 1.1, 1.3, 3.4 and 3.6]) *The double inequalities*

$$A(a,b;0)<AGM(a,b)<A\left(a,b;\frac{1}{2}\right) \tag{1.4}$$

and

$$L(a,b;1)<AGM(a,b)<L\left(a,b;\frac{3}{2}\right) \tag{1.5}$$

hold for all $a,b>0$ with $a\neq b$ with the optimal parameters 0, $1/2$, 1 and $3/2$. Moreover, for $a,b>0$ with $a\neq b$, one has

$$AGM(a,b)<\frac{\pi}{2}L(a,b).$$

Ten years later, Alzer and Qiu [2] proved that the double inequality

$$A(a,b;\lambda)<T(a,b)<A(a,b;\mu) \tag{1.6}$$

holds for all $a,b>0$ with $a\neq b$ if and only if $\lambda\leqslant 3/2$ and $\mu\geqslant \log 2/\log(\pi/2)=1.5349\cdots$.

Inspired by inequalities (1.2) and (1.6) together with Theorems 1.1 and 1.2, it is natural to propose the following Question 1.3.

QUESTION 1.3. What is the simple and practical criterion for the homogeneous symmetric mean M to distinguish the monotonicity of the function $t\mapsto M(a,b;t)$ for fixed $a,b>0$ with $a\neq b$? What are the best possible parameters $\alpha,\beta\in\mathbb{R}$ such that the double inequality $C(a,b;\alpha)<T(a,b)<C(a,b;\beta)$ holds for all $a,b>0$ with $a\neq b$ if $C(a,b;t)$ is strict monotone with respect to $t\in\mathbb{R}$ for fixed $a,b>0$ with $a\neq b$?

The authors [22] also gave the Question 1.4 as follows.

QUESTION 1.4. ([22, Problems 3.6(1)]) Is it true that $AGM(a,b;t) \geqslant L(a,b)$ for some $t \in (0,1)$?

The main purpose of this paper is to answer the Questions 1.3 and 1.4. In what follows, we denote $r' = \sqrt{1-r^2}$ for $r \in (0,1)$, $\mathscr{K} = \mathscr{K}(r)$ and $\mathscr{E} = \mathscr{E}(r)$.

Recall that the complete elliptic integrals $\mathscr{K}$ and $\mathscr{E}$ satisfy the differential formulas and Landen identity (cf. [5, Appendix E, pp. 474–475])

$$\frac{d\mathscr{K}}{dr} = \frac{\mathscr{E} - r'^2\mathscr{K}}{rr'^2}, \qquad \frac{d\mathscr{E}}{dr} = \frac{\mathscr{E} - \mathscr{K}}{r}, \qquad \mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right) = \frac{2\mathscr{E} - r'^2\mathscr{K}}{1+r}.$$

In order to prove our main results, we need two lemmas which we present at the end of this section.

LEMMA 1.5. ([5, Theorem 1.25]) *Let $a,b \in \mathbb{R}$ with $a < b$ and $f,g : [a,b] \to \mathbb{R}$ be continuous and differentiable on (a,b) such that $g' \neq 0$ on (a,b). If the derivative ratio f'/g' is strictly increasing (decreasing) on (a,b), then the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)}, \qquad \frac{f(x)-f(b)}{g(x)-g(b)}$$

are also strictly increasing (decreasing, respectively) on (a,b).

LEMMA 1.6. ([5, Theorem 3.21(8), Exercises 3.43(13),(46)], [2, Theorem 15] and [19, Theorem 3.3]) *Let $r \in (0,1)$. Then the following statements are true:*

(1) The function $r \mapsto r'[\mathscr{K}(r) - \mathscr{E}(r)]/r^2$ is strictly decreasing on $(0,1)$;

(2) The function $r \mapsto r'^c\mathscr{E}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,\infty)$ if $c \leqslant -1/2$;

(3) The function $r \mapsto [\mathscr{E}(r) - r'^2\mathscr{K}(r)]/[r^2\mathscr{K}(r)]$ is strictly decreasing from $(0,1)$ onto $(0,1/2)$;

(4) The function $r \mapsto r'^{1/4}(\mathscr{E}(r) - r'^2\mathscr{K}(r))/r^2$ is strictly decreasing from $(0,1)$ onto $(0,\pi/4)$;

(5) The function $r \mapsto 2\mathscr{E}(r) - r'^2\mathscr{K}(r)$ is strictly increasing from $(0,1)$ onto $(\pi/2,2)$.

2. Answers to Questions 1.3 and 1.4

THEOREM 2.1. *Let $M(a,b)$ be a strict homogeneous symmetric mean of two positive numbers a and b. Then $M(a,b;t)$ is strictly increasing (decreasing) with respect to $t \in \mathbb{R}$ for fixed $a,b > 0$ with $a \neq b$ if the function $r \mapsto [\log M(1,r)]/[\log G(1,r)]$ is strictly increasing (decreasing) on $(0,1)$.*

Proof. Since $M(a,b;-t)=1/M(1/a,1/b;t)$ for $t\in\mathbb{R}$, hence it suffices to prove that the function $t\mapsto M(a,b;t)$ is strictly increasing (decreasing) on $(0,\infty)$ for fixed $a,b>0$ with $a\neq b$ if the function $r\mapsto[\log M(1,r)]/[\log G(1,r)]$ is strictly increasing (decreasing) on $(0,1)$.

Without loss of generality, we assume that $0<a<b$. Let $r=(a/b)^t$. Then the function $t\mapsto r=(a/b)^t$ is strictly decreasing from $(0,\infty)$ onto $(0,1)$, and

$$\frac{\log b-\log M(a,b;t)}{\log b-\log a}=\frac{\log b^t-\log M(a^t,b^t)}{t\log(b/a)}=\frac{\log M(1,(a/b)^t)}{\log(a/b)^t}=\frac{1}{2}\frac{\log M(1,r)}{\log G(1,r)}.$$

Therefore, Theorem 2.1 directly follows from the above identity and monotonicity of the function $t\mapsto r=(a/b)^t$. □

Making use of Theorem 2.1, we get Corollary 2.2 as follows.

COROLLARY 2.2. *Let $a,b>0$ with $a\neq b$. Then the following statements are true:*

(1) The function $t\mapsto C(a,b;t)$ is strictly increasing on $\mathbb{R}$;

(2) The function $t\mapsto L(a,b;t)$ is strictly increasing on $\mathbb{R}$;

(3) The function $t\mapsto T(a,b;t)$ is strictly increasing on $\mathbb{R}$;

(4) The function $t\mapsto AGM(a,b;t)$ is strictly increasing on $\mathbb{R}$.

Proof. (1) Let $r\in(0,1)$ and

$$f(r)=\frac{1}{2}\frac{\log C(1,r)}{\log G(1,r)}=\frac{\log(1+r^2)-\log(1+r)}{\log r}. \tag{2.1}$$

Then it follows from (2.1) that

$$\begin{aligned} f'(r)=&\frac{[2r/(1+r^2)-1/(1+r)]\log r-(1/r)[\log(1+r^2)-\log(1+r)]}{(\log r)^2}\\ =&\frac{1}{r(\log r)^2}f_1(r), \end{aligned} \tag{2.2}$$

where

$$f_1(r)=\frac{r(r^2+2r-1)}{(1+r^2)(1+r)}\log r+\log\left(\frac{1+r}{1+r^2}\right).$$

It is easy to verify that

$$f_1(0^+)=f_1(1^-)=0, \tag{2.3}$$

$$f_1'(r)=\frac{\log(1/r)}{(1+r^2)^2(1+r)^2}f_2(r), \tag{2.4}$$

$$f_2(r)=r^4-4r^3-6r^2-4r+1,$$

$$f_2(0^+) = 1, \quad f_2(1^-) = -12, \tag{2.5}$$

and

$$f_2'(r) = 4r^3 - 12r^2 - 12r - 4 < 0 \tag{2.6}$$

for all $r \in (0,1)$.

Equations (2.5) and (2.6) imply that exists $r_0 \in (0,1)$ such that $f_2(r) > 0$ for $r \in (0, r_0)$ and $f_2(r) < 0$ for $r \in (r_0, 1)$, so that $f_1(r)$ is strictly increasing on $(0, r_0)$ and strictly decreasing on $(r_0, 1)$ by (2.4). This together with (2.3) leads to the conclusion that $f_1(r) > 0$ for all $r \in (0,1)$. Then from (2.2) we known that $f(r)$ is strictly increasing on $(0,1)$. Therefore, part (1) follows from the monotonicity of $f(r)$ and Theorem 2.1.

(2) Let $r \in (0,1)$, and

$$g(r) = \frac{1}{2}\frac{\log L(1,r)}{\log G(1,r)} = \frac{\log[(r-1)/\log r]}{\log r}, \tag{2.7}$$

$$g_1(r) = \log\left[\frac{\log(1/r)}{1-r}\right], \quad g_2(r) = \log\frac{1}{r}, \quad g_3(r) = \frac{g_1'(r)}{g_2'(r)}.$$

Then $g(r) = g_1(r)/g_2(r)$, $g_1(1^-) = g_2(1^-) = 0$ and

$$g_3(r) = \frac{r\log r - (r-1)}{(r-1)\log r}.$$

Differentiating g_3 leads to

$$g_3'(r) = \frac{-r[\log r]^2 + r^2 - 2r + 1}{r(r-1)^2(\log r)^2} = \frac{[(1-r)/\log(1/r)]^2 - r}{r(1-r)^2} > 0$$

due to $L(1,r) = (1-r)/[\log(1/r)] > G(1,r) = \sqrt{r}$ for all $r \in (0,1)$. Hence $g_3(r)$ is strictly increasing on $(0,1)$, so is g by application of Lemma 1.7. Therefore, part (2) directly follows from (2.7) and Theorem 2.1.

(3) Let $r \in (0,1)$, and

$$h(r) = \frac{1}{2}\frac{\log T(1,r)}{\log G(1,r)} = \frac{\log(2/\pi) + \log\mathscr{E}(r')}{\log r},$$

$$h_1(r) = -\left[\log(2/\pi) + \log\mathscr{E}(r')\right], \quad h_2(r) = \log(1/r).$$

Then $h(r) = h_1(r)/h_2(r)$, $h_1(1^-) = h_2(1^-) = 0$, and

$$\frac{h_1'(r)}{h_2'(r)} = \frac{r^2[\mathscr{K}(r') - \mathscr{E}(r')]}{r'^2\mathscr{E}(r')} = \frac{r[\mathscr{K}(r') - \mathscr{E}(r')]}{r'^2}\frac{1}{\mathscr{E}(r')/r}.$$

It follows from Lemma 1.6(1) and (2) that $h_1'(r)/h_2'(r)$ is strictly increasing on $(0,1)$, so is $h(r)$ by Lemma 1.5. Applying Theorem 2.1, we obtain that the function $t \mapsto T(a,b;t)$ is strictly increasing on $\mathbb{R}$.

(4) Let $r\in(0,1)$. Then the identity (1.1) leads to

$$\frac{\log AGM(1,r)}{\log G(1,r)}=\frac{\log(\pi/2)-\log\mathscr{K}(r')}{\log\sqrt{r}}.$$

Let

$$J(r)=\frac{1}{2}\frac{\log AGM(1,r)}{\log G(1,r)},\quad J_1(r)=\log\mathscr{K}(r')-\log\frac{\pi}{2},\quad J_2(r)=\log\frac{1}{r}.$$

Then $J(r)=J_1(r)/J_2(r)$, $J_1(1^-)=J_2(1^-)=0$ and

$$\frac{J_1'(r)}{J_2'(r)}=\frac{\mathscr{E}(r')-r^2\mathscr{K}(r')}{r'^2\mathscr{K}(r')}$$

is strictly increasing in $(0,1)$ due to Lemma 1.6(3), so is $J(r)$ by Lemma 1.5. Therefore, by Theorem 2.1, we obtain the assented result. □

THEOREM 2.3. *Let $a,b>0$ with $a\neq b$. Then one has*

(1) $T(a,b)<C(a,b;1/2)$;

(2) $T(a,b)>(2/\pi)C(a,b)$.

Proof. Since $T(a,b)$ and $C(a,b)$ as well as $C(a,b;t)$ are strict homogeneous symmetric means, without loss of generality, we assume that $a=1+r$, $b=1-r$ for $r\in(0,1)$. Then

$$T(1+r,1-r)=\frac{2}{\pi}(1+r)\mathscr{E}\left(\frac{2\sqrt{r}}{1+r}\right)=\frac{2}{\pi}(2\mathscr{E}-r'^2\mathscr{K}),$$

$$C\left(1+r,1-r;\frac{1}{2}\right)=\left(\frac{2}{\sqrt{1+r}+\sqrt{1-r}}\right)^2=\frac{2}{1+r'},$$

$$C(1+r,1-r)=\frac{(1+r)^2+(1-r)^2}{(1+r)+(1-r)}=1+r^2.$$

Let

$$f(r)=\frac{T(1+r,1-r)}{C(1+r,1-r;1/2)}=\frac{1}{\pi}(1+r')(2\mathscr{E}-r'^2\mathscr{K}),$$

and

$$g(r)=\frac{T(1+r,1-r)}{C(1+r,1-r)}=\frac{2(2\mathscr{E}-r'^2\mathscr{K})}{\pi(1+r^2)}.$$

Then differentiating f and g lead to

$$\begin{aligned}f'(r)=&\frac{1}{\pi}\left[-\frac{r}{r'}(2\mathscr{E}-r'^2\mathscr{K})+(1+r')\frac{\mathscr{E}-r'^2\mathscr{K}}{r}\right]\\=&\frac{1}{\pi}\frac{r}{r'}\left[-(2\mathscr{E}-r'^2\mathscr{K})+\frac{r'(\mathscr{E}-r'^2\mathscr{K})}{r^2}(1+r')\right],\end{aligned}$$

$$g'(r) = \frac{2}{\pi}\frac{(1+r^2)(\mathscr{E}-r'^2\mathscr{K})/r - 2r(2\mathscr{E}-r'^2\mathscr{K})}{(1+r^2)^2} = \frac{2}{\pi}\frac{r'^2(\mathscr{E}-r'^2\mathscr{K})-2r^2\mathscr{E}}{r(1+r^2)^2}$$
$$= \frac{2}{\pi}\frac{r\mathscr{E}}{(1+r^2)^2}\left[\frac{r'^2(\mathscr{E}-r'^2\mathscr{K})}{r^2\mathscr{E}} - 2\right].$$

Lemma 1.6(2), (4) and (5) show that the function $r \mapsto r'^2(\mathscr{E}-r'^2\mathscr{K})/(r^2\mathscr{E}) = \left[r'^{1/4}(\mathscr{E}-r'^2\mathscr{K})/r^2\right]/[r'^{-7/4}\mathscr{E}]$ is strictly decreasing from $(0,1)$ onto $(0,1/2)$, and the function $r \mapsto -(2\mathscr{E}-r'^2\mathscr{K})+(1+r')r'(\mathscr{E}-r'^2\mathscr{K})/r^2$ is strictly decreasing from $(0,1)$ onto $(-2,0)$, so that both f and g are strictly decreasing on $(0,1)$. Moreover,

$$\lim_{r\to 0^+} f(r) = \lim_{r\to 0^+} g(r) = 1, \quad \lim_{r\to 1^-} f(r) = \lim_{r\to 1^-} g(r) = \frac{2}{\pi}.$$

Consequently, for $a,b>0$ with $a \neq b$,

$$\max\left\{\frac{2}{\pi}C(a,b), \frac{2}{\pi}C(a,b;1/2)\right\} < T(a,b) < \min\{C(a,b), C(a,b;1/2)\}.$$

Therefore, Theorem 2.3 follows. □

REMARK 2.4. Inequality (1.2) shows that

$$C(a,b;0) = G(a,b) < T(a,b) < C(a,b) = C(a,b;1)$$

for all $a,b>0$ with $a\neq b$. Noting that

$$T(1,r') = \frac{2}{\pi}\mathscr{E}(r) = 1 - \frac{1}{4}r^2 - \frac{3}{64}r^4 + o(r^4)$$

and

$$C(1,r';t) = \left[\frac{1+r'^{2t}}{1+r'^t}\right]^{1/t} = 1 - \frac{1}{4}r^2 + \frac{3(t-1)}{32}r^4 + o(r^4).$$

It follows from Corollary 2.2(1) and Theorem 2.3(1) together with

$$\lim_{r\to 0^+}\frac{C(1,r';t)-T(1,r')}{r^4} = \frac{3(2t-1)}{64}$$

that the inequality $C(a,b,t) > T(a,b)$ for all $a,b>0$ with $a\neq b$ if and only if $t \geqslant 1/2$.

On the other hand, the limiting value

$$\lim_{r\to 1^-}\frac{T(1,r')}{C(1,r';t)} = \frac{2}{\pi}$$

for $t>0$ implies that there does not exist $t\in(0,1)$ such that $T(a,b) > C(a,b;t)$ for all $a,b>0$ with $a\neq b$.

By Corollary 2.2(1) and Remark 2.4, we obtain the following Corollary 2.5 immediately.

COROLLARY 2.5. *Let $\alpha,\beta \in \mathbb{R}$, then the double inequality*

$$C(a,b;\alpha) < T(a,b) < C(a,b;\beta)$$

holds for all $a,b>0$ with $a \neq b$ if and only if $\alpha \leqslant 0$ and $\beta \geqslant 1/2$.

THEOREM 2.6. *There does not exist $t \in (0,1)$ such that $AGM(a,b;t) \geqslant L(a,b)$ for all $a,b>0$.*

Proof. Let us consider the ratio $AGM(a,b;t)/L(a,b)$ for $t \in (0,1)$. Without loss of generality, we suppose that $a=1$, $b=r \in (0,1]$. Then

$$AGM(1,r;t) = \left[\frac{\pi/2}{\mathscr{K}(\sqrt{1-r^{2t}})}\right]^{1/t}, \quad L(1,r) = \frac{1-r}{\log(1/r)}.$$

Making use of the limiting behavior of the function $\mathscr{K}(r)$ (cf. [5, equation (3.4)])

$$\lim_{r\to 1^-}\left[\mathscr{K}(r) - \log\left(\frac{4}{r'}\right)\right] = 0$$

and the L'Hôpital's rule we obtain

$$\begin{aligned}\lim_{r\to 0^+}\frac{AGM(1,r;t)}{L(1,r)} &= \lim_{r\to 0^+}\frac{(\pi/2)^{1/t}\log(1/r)}{(1-r)\left[\mathscr{K}(\sqrt{1-r^{2t}})\right]^{1/t}}\\ &= \lim_{r\to 0^+}\frac{(\pi/2)^{1/t}\log(1/r)}{\left[\frac{\mathscr{K}(\sqrt{1-r^{2t}})}{\log(4/r^t)}\right]^{1/t}}\frac{1}{[\log(4/r^t)]^{1/t}} = \lim_{r\to 0^+}\frac{(\pi/2)^{1/t}\log(1/r)}{[\log(4/r^t)]^{1/t}}\\ &= \lim_{r\to 0^+}\frac{(\pi/2)^{1/t}(-1/r)}{[\log(4/r^t)]^{(1/t)-1}(-1/r)} = \lim_{r\to 0^+}\frac{(\pi/2)^{1/t}}{[\log(4/r^t)]^{(1/t)-1}} = 0.\end{aligned}$$

This completes the proof of Theorem 2.6. □

3. A note on sharp bounds for $\mathscr{K}(r)$

In 2018, by establishing a discrete monotonicity theorem for means, Nishimura [16] provided several bounds for the complete elliptic integral $\mathscr{K}(r)$ of the first kind as follows:

THEOREM 3.1. ([16, Corollay 3.2, 3.3 and Theorem 3.6]) *Inequalities*

$$\frac{1}{A(1,r;1/2)} < \frac{2}{\pi}\mathscr{K}(r'), \tag{3.1}$$

$$\frac{1}{L(1,r;2)} < \frac{2}{\pi}\mathscr{K}(r') < \frac{1}{L(1,r;1)} \tag{3.2}$$

and

$$\frac{1}{A(1,r)^{1/4}L(1,r)^{3/4}} < \frac{2}{\pi}\mathscr{K}(r') \tag{3.3}$$

hold for all $r \in (0,1)$.

Furthermore, Nishimura [16] conjectured that the inequalities (3.1)-(3.3) are the best possible in the sense of certain constants, and proposed the following Conjecture 3.2.

CONJECTURE 3.2. ([16, Conjectures 3.1, 3.2 and 3.3]) (1) The constant $1/2$ is the best possible constant such that the inequality (3.1) holds for all $r \in (0,1)$.

(2) The double inequality

$$\frac{1}{L(1,r;\alpha)} < \frac{2}{\pi}\mathscr{K}(r') < \frac{1}{L(1,r;\beta)}$$

holds for all $r \in (0,1)$ with the best possible constants $\alpha = 2$ and $\beta = 1$.

(3) The inequality

$$\frac{1}{A(1,r)^{\lambda}L(1,r)^{1-\lambda}} < \frac{2}{\pi}\mathscr{K}(r')$$

holds for all $r \in (0,1)$ with the best possible constant $\lambda = 1/4$.

REMARK 3.3. According to (1.1), inequalities (3.1), (3.2), (3.3) can be rewritten as

$$AGM(1,r) < A(1,r;1/2), \tag{3.4}$$

$$L(1,r;1) < AGM(1,r) < L(1,r;2), \tag{3.5}$$

$$AGM(1,r) < A(1,r)^{1/4}L(1,r)^{3/4}, \tag{3.6}$$

respectively. Obviously, (1.4) shows that $1/2$ in (3.4) is the optimal constant. Comparing (1.5) to (3.5), one has that, for $\alpha, \beta \in \mathbb{R}$, the double inequality

$$\frac{1}{L(1,r;\alpha)} < \frac{2}{\pi}\mathscr{K}(r') < \frac{1}{L(1,r;\beta)} \tag{3.7}$$

holds for all $r \in (0,1)$ with the best possible constant $\alpha = 3/2$ and $\beta = 1$.

For the sharpness of the constant $1/4$ in (3.6), if we let $t \in (0,1)$, then

$$AGM(1,r) = \frac{\pi/2}{\mathscr{K}(r')} = 1 - \frac{1}{2}(1-r) - \frac{1}{16}(1-r)^2 + o[(1-r)^2]$$

and

$$A(1,r)^t L(1,r)^{1-t} = 1 - \frac{1}{2}(1-r) + \frac{t-1}{12}(1-r)^2 + o(1-r)^2$$

as $r \mapsto 1$. Therefore, inequality

$$\frac{1}{A(1,r)^t L(1,r)^{1-t}} < \frac{2}{\pi}\mathscr{K}(r')$$

holds for large r which tends to 1 only for $t \geqslant 1/4$.

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