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DEVIATION ESTIMATIONS FOR LOTKA–NAGAEV ESTIMATOR OF A BRANCHING PROCESS WITH IMMIGRATION

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Abstract. In a recent manuscript, Chu (2018) applied the self–normalized large deviations for i.i.d. random variables to the Lotka–Nagaev estimation of a supercritical Galton–Watson process. In this paper, we consider decay rates for the Lotka–Nagaev estimation of a supercritical branching process with immigration. We have two main contributions. On the one hand, Chu's paper considered the self–normalizing constants of second order, otherwise, we consider the maximum case. On the other hand, except for large deviations, we also studied the self–normalized moderate deviations. The classical large deviation probabilities for Lotka–Negaev estimation show three different decay rates according to the degree of heavy tail of offspring distribution, but our results show that there is only one decay rate in the self–normalized version.

1. Introduction

In the last two decades, large deviation theory has been used as an important tool to measure deviations between the offspring mean and its Lotka–Nagaev estimator for a supercritical branching process with or without immigration, see [1], [4]–[9] and [12]. In a recent manuscript [2], Chu studied the self–normalized large deviations for the Lotka–Nagaev estimator of a supercritical Galton–Watson process.

Formally, let $\{Z_n, n \geqslant 0\}$ be a supercritical Galton–Watson process with offspring distribution $\{p_k, k \geqslant 0\}$ and $Z_0 = 1$, $p_0 = 0$. The Lotka–Nagaev estimator of offspring mean $m =: \sum_k k p_k$ is defined by $R_n = Z_{n+1}/Z_n$. If $\mathbb{E}(Z_1 \log Z_1) < \infty$, $p_1 > 0$, for any $x > 0$, Chu showed that

$$\mathbb{P}\left(\sqrt{\frac{Z_n}{\sum_{i=1}^{Z_n}(X_{ni} - R_n)^2}}|R_n - m| \geqslant x\right) \sim C(x) p_1^n$$

as $n \to \infty$, where $C(x)$ is some positive constant, X_{ni} denotes the number of offsprings of the ith individual in the nth generation and $f_n \sim g_n$ means $f_n/g_n \to 1$ as $n \to \infty$.

In this manuscript, we consider the Lotka–Nagaev estimator for a supercritical Galton–Watson process with immigration defined by $X_0 = Z_0 = 1$ and

$$X_n = Z_n + Y_{n,1} + \cdots + Y_{n,n}, \quad n \geqslant 1,$$

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where $Y_{n,i}$ is the number of direct descendants of the individuals from the immigration at time i, $i = 1, \cdots, n-1$ and $Y_n := Y_{n,n}$ is the number of individuals immigrated at time n with distribution $\{h_k, k \geqslant 0\}$.

The Lotka–Nagaev estimator of the offspring mean is defined by $L_n = X_{n+1}/X_n$. One of the interesting topics is to consider the decay rates of deviation probabilities

$$\mathbb{P}_n(\varepsilon) = \mathbb{P}(a_n|L_n - m| > \varepsilon) \tag{1}$$

for $\varepsilon > 0$ as $n \to \infty$.

If $a_n = O(1)$ $a.s.$, then (1) belongs to large deviation probability. For $a_n \equiv 1$, under the condition that the offspring distribution and immigrating distribution satisfy the Cramér condition, Liu and Zhang (2016) showed that $\mathbb{P}_n(\varepsilon) \sim C(\varepsilon)(h_0 p_1)^n$ for some positive constant $C(\varepsilon)$. In addition, using the results on harmonic moments, Sun and Zhang (2017) showed that the large deviation probabilities have three different decay rates according to the degree of heavy tail of offspring distribution. Conditioning large deviations were given in L. Y. Li and J. P. Li (2019).

If $a_n = O(\sqrt{X_n})$ $a.s.$, then (1) belongs to normal deviation probability. For $a_n = \sqrt{X_n}$, Heyde and Seneta(1971) considered the asymptotic normality of the Lotka–Nagaev estimator.

If $a_n \to \infty$ and $a_n = o(\sqrt{X_n})$ $a.s.$, we say that (1) is a moderate deviation probability. Moderate deviations for the Lotka–Nagaev estimator of a supercritical Galton–Watson process without immigration were given in Fleischmann and Wachtel (2008), where $\{a_n\}$ is a sequence of positive and non–random constants satisfying $a_n \to \infty$ and $a_n = o(m^{n/2})$. Moderate deviations for the case with immigration are still open.

In this paper, we consider the self–normalized large and moderate deviations for L_n. For partial sum $S_n = \xi_1 + \cdots + \xi_n$, where $\{\xi_n\}$ are i.i.d., Shao(1997) use

$$n^{(p-1)/p} \left(\sum_{i=1}^{n} |\xi_i|^p \right)^{1/p} \quad (p > 1)$$

as the normalizing constant to establish a self–normalized large deviation without any moment conditions. Rozovsky (2009) derived the self–normalized large deviations for the case that $p = \infty$, that is, $\max_{i=1}^{n} |\xi_i|$ is used as the normalizing constant.

The main task of this manuscript is the estimation of the decay rates for

$$\mathbb{P}(V_n X_n^{\delta} |L_n - m| \geqslant x)$$

as $n \to \infty$, where $\delta \in [0, 1/2)$ and

$$V_n = \frac{1}{\max_{1 \leqslant i \leqslant X_n} |X_{ni} - L_n|}.$$

Let $\{\xi, \xi_n, n \geqslant 1\}$ be a sequence of i.i.d. random variables with offspring distribution $\{p_k\}$ and η be a random variable with immigrating distribution $\{h_k\}$. In addition, we assume that $\{\xi, \xi_n\}$ is independent of η. Define

$$\bar{S}_k = (\xi_1 + \cdots + \xi_k)/k + \eta/k \quad \text{and} \quad \tilde{S}_k = (\xi_1 + \cdots + \xi_k)/k. \tag{2}$$

Through out this paper, we assume that $m \in (1,\infty)$, $p_0 = 0$, $p_1 h_0 > 0$ and $p_k < 1$ for all k.

THEOREM 1. (Large deviation) *If* $\delta = 0$ *and there exists a positive constant* θ *such that* $\mathbb{E}(\exp(\theta\eta)) < \infty$*, then for all* $x > 0$*, we have*

$$\lim_{n\to\infty} \frac{1}{(h_0 p_1)^n} \mathbb{P}(V_n|L_n - m| \geqslant x) = \sum_{k=1}^{\infty} u_k \varphi(k,x) < \infty,$$

where $u_k = \sum_{i=0}^{k} v_i q_{k-i}$, $\{v_i\},\{q_i\}$ *are defined in Lemma 2 and*

$$\varphi(k,x) = \mathbb{P}\left(|\bar{S}_k - m| \geqslant x \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|\right).$$

Sun and Zhang (2017) showed that the probability of classical large deviation $\mathbb{P}(|L_n - m| \geqslant \varepsilon)$ presents three different decay rates according to the degree of heavy tail of ξ. From Theorem 1, one can see there is only one decay rate for the self–normalized version.

Now, we consider the decay rates of moderate deviation probabilities.

THEOREM 2. (Moderate deviation) *If* $\delta \in (0, 0.5)$ *and there exists a positive constant* θ *such that* $\mathbb{E}(\exp(\theta\xi)) < \infty$ *and* $\mathbb{E}(\exp(\theta\eta)) < \infty$*, then for all* $x > 0$*, we have*

$$\lim_{n\to\infty} \frac{1}{(h_0 p_1)^n} \mathbb{P}(V_n X_n^{\delta} |L_n - m| \geqslant x) = \sum_{k=1}^{\infty} u_k \phi(k,x) < \infty,$$

where $u_k = \sum_{i=0}^{k} v_i q_{k-i}$, $\{v_i\},\{q_i\}$ *are defined in Lemma 2 and*

$$\phi(k,x) = \mathbb{P}\left(|\bar{S}_k - m| \geqslant x k^{-\delta} \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|\right).$$

The rest of this article is organized as follows. In Section 2, we deal with the self–normalized large deviations. The proofs of moderate deviations are given in Section 3.

Throughout this manuscript, we denote C an absolute positive constant which may vary from line to line.

2. Large deviations

We need the following two lemmas to prove Theorem 1. The first one due to Rozovsky (2009) which is a generalization of Shao's self–normalized large deviation result for the maximum case.

LEMMA 1. *For* $x > 0$ *and* $V_n' = \max_{1\leqslant i\leqslant n} |\xi_i - m|$,

$$\lim_{n\to\infty} \left[\mathbb{P}\left(\frac{\tilde{S}_n}{V_n'} \geqslant x\right)\right]^{1/n} = \sup_{v\geqslant 0} \inf_{t\geqslant 0} e^{-txv} \mathbb{E} e^{t(\xi - m)} I[(\xi - m) \leqslant v],$$

where $\tilde{S}_k$ *is defined in formula* (2) *and* $I(A)$ *stands for the indictor function of set* A.

The second lemma is about the decay rates of generating functions of branching process with immigration for the case $p_1h_0>0$.

LEMMA 2. (Proposition 1 of [12]) *If* $p_1h_0>0$, *for any* $s\in(0,1)$, *one has*

$$g_n(s)\sim(p_1h_0)^nV(s)Q(s),$$

where $g_n(s)=\mathbb{E}(s^{X_n})$, $V(s)$ *and* $Q(s)$ *satisfy the following functional equations:*

$$\begin{aligned} h(s)V(f(s)) &= h_0V(s),\ \ 0\leqslant s<1,\ \ V(0)=1,\ V(1)=\infty;\\ Q(f(s)) &= p_1Q(s),\ \ 0\leqslant s<1,\ \ Q(0)=1,\ Q(1)=\infty. \end{aligned}$$

Moreover, $V(s)$, $Q(s)$ *can be represented as power series*

$$V(s)=\sum_{i=0}^{\infty}v_is^i,\quad Q(s)=\sum_{i=0}^{\infty}q_is^i$$

respectively.

The proof of Theorem 1. By the total probability, we have

$$\begin{aligned} \mathbb{P}(V_n|L_n-m|\geqslant x) &= \sum_{k=1}^{\infty}\mathbb{P}(X_n=k)\mathbb{P}\left(|\bar{S}_k-m|\geqslant x\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|\right)\\ &= \sum_{k=1}^{\infty}\mathbb{P}(X_n=k)\varphi(k,x), \end{aligned} \tag{3}$$

where $\bar{S}_k$ is defined in (2) and $\varphi(k,x)$ is defined in Theorem 1.

The main step is to prove that $\varphi(k,x)$ convergent to 0 exponentially as $k\to\infty$. In fact, we can divide $\varphi(k,x)$ into the following two parts:

$$\begin{aligned} \varphi(k,x) &= \mathbb{P}(\bar{S}_k-m\geqslant x\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|)\\ &\quad+\mathbb{P}(\bar{S}_k-m\leqslant -x\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|)\\ &=: A(k,x)+B(k,x). \end{aligned} \tag{4}$$

In addition, for any $0<\varepsilon<1$, $A(k,x)$ has the following decomposition,

$$\begin{aligned} A(k,x) &= \mathbb{P}(\bar{S}_k-m\geqslant x\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|,|\bar{S}_k-m|<\varepsilon V_k')\\ &\quad+\mathbb{P}(\bar{S}_k-m\geqslant x\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|,|\bar{S}_k-m|\geqslant\varepsilon V_k')\\ &=: A_1(k,x)+A_2(k,x), \end{aligned} \tag{5}$$

where $V_k'=\max_{1\leqslant i\leqslant k}|\xi_i-m|$.

By triangular inequality, we obtain

$$\begin{aligned}\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k| &\geqslant \max_{1\leqslant i\leqslant k}(|\xi_i-m|-|\bar{S}_k-m|)\\ &= \max_{1\leqslant i\leqslant k}|\xi_i-m|-|\bar{S}_k-m|\\ &= V_k'-|\bar{S}_k-m|. \qquad (6)\end{aligned}$$

Consequently,

$$\begin{aligned}A_1(k,x) &= \mathbb{P}(\bar{S}_k-m\geqslant x\max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|,|\bar{S}_k-m|<\varepsilon V_k')\\ &\leqslant \mathbb{P}(\bar{S}_k-m\geqslant x(V_k'-|\bar{S}_k-m|),|\bar{S}_k-m|<\varepsilon V_k')\\ &\leqslant \mathbb{P}(\bar{S}_k-m\geqslant x(1-\varepsilon)V_k')\\ &= \mathbb{P}(\bar{S}_k-m\geqslant x(1-\varepsilon)V_k',\eta/k<x\varepsilon V_k')\\ &\quad +\mathbb{P}(\bar{S}_k-m\geqslant x(1-\varepsilon)V_k',\eta/k\geqslant x\varepsilon V_k')\\ &\leqslant \mathbb{P}(\tilde{S}_k-m\geqslant x(1-2\varepsilon)V_k')+\mathbb{P}(\eta/k\geqslant x\varepsilon V_k'),\end{aligned}$$

where $\tilde{S}_k$ is defined in formula (2). Applying Lemma 1, there exists a constant $\lambda_1(x,\varepsilon)\in(0,1)$ such that for k large enough, one has

$$\begin{aligned}\mathbb{P}(\tilde{S}_k-m\geqslant x(1-2\varepsilon)V_k') &= \mathbb{P}\left(\frac{\tilde{S}_k-m}{V_k'}\geqslant x(1-2\varepsilon)\right)\\ &\leqslant \lambda_1^k(x,\varepsilon). \qquad (7)\end{aligned}$$

On the other hand, by Chebyshev's inequality,

$$\begin{aligned}\mathbb{P}(\eta/k\geqslant x\varepsilon V_k') &= \int_0^\infty \mathbb{P}(\eta/k\geqslant x\varepsilon y)\,d\mathbb{P}(V_k'\leqslant y)\\ &\leqslant \int_0^\infty \mathbb{E}(\exp(\theta\eta))\exp(-kx\varepsilon y)d\mathbb{P}(V_k'\leqslant y)\\ &= \mathbb{E}(\exp(\theta\eta))\mathbb{E}(\exp(-x\varepsilon kV_k')).\end{aligned}$$

Let u be the essential supremum of $|\xi-m|$, that is,

$$u=\inf\{c:\mathbb{P}(|\xi-m|\leqslant u)=1 \text{ and for any } v<u \text{ we have } \mathbb{P}(|\xi-m|<v)<1\}.$$

Note that $p_k<1$ for all k, one has $u>0$. For any $\nu\in(0,u)$,

$$\mathbb{P}(V_k'\leqslant\nu)=(\mathbb{P}(|\xi-m|\leqslant\nu))^k.$$

Consequently,

$$\begin{aligned}\mathbb{E}(\exp(-x\varepsilon kV_k')) &= \mathbb{E}(\exp(-x\varepsilon kV_k')I(V_k'\leqslant\nu))+\mathbb{E}(\exp(-x\varepsilon kV_k')I(V_k'>\nu))\\ &\leqslant \mathbb{P}(V_k'\leqslant\nu)+\exp(-x\varepsilon k\nu)\\ &= (\mathbb{P}(|\xi-m|\leqslant\nu))^k+(\exp(-x\varepsilon\nu))^k.\end{aligned}$$

Thus, there exists a constant $\lambda_2(x,\varepsilon)\in(0,1)$ such that for k large enough, one has

$$\mathbb{P}\left(\eta/k \geqslant x\varepsilon V_k'\right) \leqslant C\lambda_2^k(x,\varepsilon). \tag{8}$$

At this point, we have shown that $A_1(k,x)$ convergent to 0 exponentially as $k\to\infty$. Next, for $A_2(k,x)$, we have

$$\begin{aligned} A_2(k,x) &= \mathbb{P}(\bar{S}_k - m \geqslant x \max_{1\leqslant i\leqslant k}|\xi_i-\bar{S}_k|, |\bar{S}_k-m|\geqslant \varepsilon V_k') \\ &\leqslant \mathbb{P}(|\bar{S}_k-m|\geqslant \varepsilon V_k') \\ &\leqslant \mathbb{P}\left(\frac{\bar{S}_k-m}{V_k'}\geqslant\varepsilon\right)+\mathbb{P}\left(\frac{-(\bar{S}_k-m)}{V_k'}\geqslant\varepsilon\right). \end{aligned}$$

For the first part of the right hand of the above inequality, one has

$$\begin{aligned} \mathbb{P}\left(\frac{\bar{S}_k-m}{V_k'}\geqslant\varepsilon\right) &\leqslant \mathbb{P}\left(\frac{\bar{S}_k-m}{V_k'}\geqslant\varepsilon, \frac{\eta}{k}<\frac{\varepsilon V_k'}{2}\right)+\mathbb{P}\left(\frac{\bar{S}_k-m}{V_k'}\geqslant\varepsilon, \frac{\eta}{k}\geqslant\frac{\varepsilon V_k'}{2}\right) \\ &\leqslant \mathbb{P}\left(\frac{\tilde{S}_k-m}{V_k'}\geqslant\frac{\varepsilon}{2}\right)+\mathbb{P}\left(\frac{\eta}{k}\geqslant\frac{\varepsilon V_k'}{2}\right). \end{aligned}$$

Similar to (7) and (8), one can obtain that $A_2(k,x)$ also has an exponential decay rate as $k\to\infty$.

According to (4) and (5),

$$\varphi(k,x)\leqslant\rho^k(x)$$

for some constant $\rho(x)\in(0,1)$ and k large enough. Define

$$h_n(k)=\frac{\mathbb{P}(X_n=k)}{(h_0p_1)^n}\varphi(k,x).$$

For k large enough, one has

$$0\leqslant h_n(k)\leqslant C\frac{\mathbb{P}(X_n=k)}{(h_0p_1)^n}\rho^k(x).$$

According to Lemma 2,

$$\begin{aligned} \frac{1}{(h_0p_1)^n}\sum_{k=1}^{\infty}\mathbb{P}(X_n=k)\rho^k(x) &= \frac{g_n(\rho(x,\varepsilon))}{(h_0p_1)^n} \\ &\to V(\rho(x))Q(\rho(x)) \\ &< \infty. \end{aligned}$$

Therefore, by the dominated convergence theorem and (3), we have

$$\begin{aligned} \lim_{n\to\infty}\frac{1}{(h_0p_1)^n}\mathbb{P}(V_n|L_n-m|\geqslant x) &= \lim_{n\to\infty}\sum_{k=1}^{\infty}h_n(k) \\ &= \sum_{k=1}^{\infty}u_k\varphi(k,x). \end{aligned}$$

We complete the proof of Theorem 1. □

3. Moderate deviations

It is turned out that the decay rates of moderate deviation probabilities are slower that that of large deviation probabilities, which means that $\phi(k,x)$ may not have an exponential decay rate as $k \to \infty$, where $\phi(k,x)$ is defined in Theorem 2. Thus, we need the following result on harmonic moments of a supercritical Galton–Watson process with immigration which was given in Sun and Zhang(2017).

LEMMA 3. *For any r such that $h_0 p_1 m^r > 1$, one has*

$$\lim_{n\to\infty} \frac{\mathbb{E}(X_n^{-r})}{(h_0 p_1)^n} = C(r) < \infty,$$

where $C(r)$ is a positive constant.

We also need the following classical moderate deviations for i.i.d. random variables, see Theorem 3.7.1 in Dembo and Zeitouni(1998).

LEMMA 4. *Assume that $\delta \in (0, 0.5)$ and $\mathbb{E}(\exp(\theta\xi)) < \infty$ for some $\theta > 0$, then for any $x > 0$, one has*

$$\lim_{n\to\infty} \frac{1}{n^{1-2\delta}} \ln \mathbb{P}\left(\bar{S}_n - m \geqslant x n^{-\delta}\right) = -\frac{x^2}{2}.$$

The proof of Theorem 2. By the total probability, we have

$$\begin{aligned}\mathbb{P}(V_n X_n^{\delta} |L_n - m| \geqslant x) &= \sum_{k=1}^{\infty} \mathbb{P}(X_n = k) \mathbb{P}\left(|\bar{S}_k - m| \geqslant x k^{-\delta} \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|\right) \\ &= \sum_{k=1}^{\infty} \mathbb{P}(X_n = k)\phi(k,x), \end{aligned} \tag{9}$$

where $\bar{S}_k$ is defined in (2) and $\phi(k,x)$ is defined in Theorem 2.

Similar to (4), we can divide $\phi(k,x)$ into the following two parts:

$$\begin{aligned}\phi(k,x) &= \mathbb{P}(\bar{S}_k - m \geqslant x k^{-\delta} \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|) \\ &\quad + \mathbb{P}(\bar{S}_k - m \leqslant -x k^{-\delta} \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|) \\ &=: I(k,x) + J(k,x). \end{aligned} \tag{10}$$

In addition, for any $0 < \varepsilon < 1$, $I(k,x)$ has the following decomposition,

$$\begin{aligned} I(k,x) &= \mathbb{P}(\bar{S}_k - m \geqslant x k^{-\delta} \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|, |\bar{S}_k - m| < \varepsilon V_k') \\ &\quad + \mathbb{P}(\bar{S}_k - m \geqslant x k^{-\delta} \max_{1\leqslant i\leqslant k} |\xi_i - \bar{S}_k|, |\bar{S}_k - m| \geqslant \varepsilon V_k') \\ &=: I_1(k,x) + I_2(k,x), \end{aligned} \tag{11}$$

where $V_k' = \max_{1\leqslant i\leqslant k}|\xi_i - m|$.

By formula (6), we have

$$\begin{aligned} I_1(k,x) &= \mathbb{P}(\bar{S}_k - m \geqslant xk^{-\delta}\max_{1\leqslant i\leqslant k}|\xi_i - \bar{S}_k|, |\bar{S}_k - m| < \varepsilon V_k') \\ &\leqslant \mathbb{P}(\bar{S}_k - m \geqslant xk^{-\delta}(V_k' - |\bar{S}_k - m|), |\bar{S}_k - m| < \varepsilon V_k') \\ &\leqslant \mathbb{P}(\bar{S}_k - m \geqslant xk^{-\delta}(1-\varepsilon)V_k') \end{aligned}$$

Define $S_k = \xi_1 + \cdots + \xi_k + \eta$, one has

$$\begin{aligned} I_1(k,x) &:= \mathbb{P}(\bar{S}_k - m \geqslant xk^{-\delta}(1-\varepsilon)V_k') \\ &= \mathbb{P}(S_k - km \geqslant xk^{1-\delta}(1-\varepsilon)V_k', \cup_{i=1}^{k}\{V_k' = |\xi_i - m|\}) \\ &\leqslant k\mathbb{P}(S_{k-1} - (k-1)m \geqslant xk^{1-\delta}(1-\varepsilon)V_k' - (\xi_k - m), V_{k-1}' \leqslant |\xi_k - m|) \\ &\leqslant k\mathbb{P}(S_{k-1} - (k-1)m \geqslant xk^{1-\delta}(1-\varepsilon)|\xi_k - m| - |\xi_k - m|, V_{k-1}' \leqslant |\xi_k - m|) \\ &\leqslant \sup_{v>0} k\mathbb{P}(S_{k-1} - (k-1)m \geqslant (xk^{1-\delta}(1-\varepsilon) - 1)v, V_{k-1}' \leqslant v) \end{aligned}$$

For k large enough, we have $xk^{1-\delta}\varepsilon > 1$. Consequently, for any $\tau > 0$

$$\begin{aligned} I_1(k,x) &\leqslant \sup_{v>0} k\mathbb{P}(S_{k-1} - (k-1)m \geqslant x(k-1)^{1-\delta}(1-2\varepsilon)v, V_{k-1}' \leqslant v) \\ &= \sup_{v>0} k\mathbb{P}(\bar{S}_{k-1} - m \geqslant x(k-1)^{-\delta}(1-2\varepsilon)v, V_{k-1}' \leqslant v) \\ &= \sup_{v>\tau} k\mathbb{P}(\bar{S}_{k-1} - m \geqslant x(k-1)^{-\delta}(1-2\varepsilon)v, V_{k-1}' \leqslant v) \\ &\quad + \sup_{\tau\geqslant v>0} k\mathbb{P}(\bar{S}_{k-1} - m \geqslant x(k-1)^{-\delta}(1-2\varepsilon)v, V_{k-1}' \leqslant v) \\ &=: E(k,x) + F(k,x). \end{aligned}$$

For $E(k,x)$, we have

$$\begin{aligned} E(k,x) &\leqslant \sup_{v>\tau} k\mathbb{P}\left(\sum_{i=1}^{k-1}(\xi_i - m) + \eta \geqslant x(k-1)^{1-\delta}(1-2\varepsilon)v\right) \\ &= k\mathbb{P}\left(\sum_{i=1}^{k-1}(\xi_i - m) + \eta \geqslant x(k-1)^{1-\delta}(1-2\varepsilon)\tau\right) \\ &\leqslant k\mathbb{P}\left(\sum_{i=1}^{k-1}(\xi_i - m) \geqslant x(k-1)^{1-\delta}(1-3\varepsilon)\tau\right) \\ &\quad + k\mathbb{P}\left(\eta \geqslant x(k-1)^{1-\delta}\varepsilon\tau\right). \end{aligned} \tag{12}$$

On the one hand, by Chebyshev's inequality,

$$\begin{aligned} k\mathbb{P}\left(\eta \geqslant x(k-1)^{1-\delta}\varepsilon\tau\right) &\leqslant k\exp(-\theta x(k-1)^{1-\delta}\varepsilon\tau)\mathbb{E}(\exp(\theta\eta)) \\ &\leqslant Ck^{-\alpha-\delta} \end{aligned}$$

for k large enough, where α satisfies $h_0 p_1 m^{\alpha} = 1$.

On the other hand, by Lemma 4,

$$k\mathbb{P}\left(\sum_{i=1}^{k-1}(\xi_i - m) \geqslant x(k-1)^{1-\delta}(1-3\varepsilon)\tau\right) \leqslant Ck\exp\left(-\frac{(x(1-3\varepsilon)\tau)^2}{2}k^{1-2\delta}\right) \leqslant Ck^{-\alpha-\delta}$$

for k large enough. From (12), we know that for sufficiently large k,

$$E(k,x) \leqslant Ck^{-\alpha-\delta}. \tag{13}$$

For $F(k,x)$, we have

$$\begin{aligned} F(k,x) &\leqslant \sup_{\tau \geqslant \nu > 0} k\mathbb{P}(V'_{k-1} \leqslant \nu) \\ &= k\mathbb{P}(V'_{k-1} \leqslant \tau) \\ &= k(\mathbb{P}(|\xi - m| \leqslant \tau))^{k-1}. \end{aligned}$$

Choose $\tau \in (0,u)$, where u is the essential supremum of $|\xi - m|$, we know

$$F(k,x) \leqslant Ck^{-\alpha-\delta} \tag{14}$$

for sufficiently large k.

At this point, we have shown that $I_1(k,x)$ is bounded by $Ck^{-\alpha-\delta}$. Next, similar to the estimation of $A_2(k,x)$, we have $I_2(k,x)$ has an exponential decay rate as $k \to \infty$.

Combine (10), (11), (13) and (14), we know that

$$\phi(k,x) \leqslant Ck^{-\alpha-\delta}.$$

Define

$$l_n(k) = \frac{\mathbb{P}(X_n = k)}{(h_0 p_1)^n}\phi(k,x).$$

For k large enough, one has

$$0 \leqslant h_n(k) \leqslant C\frac{\mathbb{P}(X_n = k)}{(h_0 p_1)^n}k^{-\alpha-\delta}.$$

According to Lemma 3,

$$\begin{aligned} \frac{1}{(h_0 p_1)^n}\sum_{k=1}^{\infty}\mathbb{P}(X_n = k)k^{-\alpha-\delta} &= \frac{\mathbb{E}(X_n^{-\alpha-\delta})}{(h_0 p_1)^n} \\ &\to C(\alpha+\delta) \\ &< \infty. \end{aligned}$$

Therefore, by the dominated convergence theorem and (9), we have

$$\lim_{n\to\infty}\frac{1}{(h_0p_1)^n}\mathbb{P}(V_nX_n^\delta|L_n-m|\geqslant x)=\lim_{n\to\infty}\sum_{k=1}^{\infty}l_n(k)$$
$$=\sum_{k=1}^{\infty}u_k\phi(k,x).$$

We complete the proof of Theorem 2. □

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