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GENERALIZED ELLIPTIC INTEGRALS AND GENERALIZED GRÖTZSCH FUNCTION WITH TWO PARAMETERS

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Abstract. In this paper, we mainly study some monotonicity properties for the generalized (p,q)-elliptic integrals and the generalized Grötzsch function. As applications, we obtain some sharp functional inequalities for the generalized Grötzsch function and thus get improvements and extensions of some previous results.

1. Introduction

For $p,q \in (1,\infty)$ and $x \in [0,1]$, define the function

$$\operatorname{arcsin}_{p,q} x \equiv \int_0^x \frac{dt}{(1-t^q)^{1/p}}$$

and set

$$\pi_{p,q} = 2\operatorname{arcsin}_{p,q}(1) \equiv 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}} = \frac{2}{q}B\left(1-\frac{1}{p},\frac{1}{q}\right), \tag{1.1}$$

where B is the classical beta function. The function $\operatorname{arcsin}_{p,q}$ has an inverse defined on $[0,\pi_{p,q}/2]$, which can be extended to an odd $2\pi_{p,q}$-period function, denoted by $\sin_{p,q}$, on the set $\mathbb{R} = (-\infty,+\infty)$ of real numbers by natural procedures designed to mimic the behaviour of the sine function. The function $\sin_{p,q}$ is called the generalized (p,q)-sine function which occurs as an eigenfunction of the Dirichlet problem for the (p,q)-Laplacian [7, 16]. when $p=q=2$, the generalized (p,q)-sine function reduces to the classical sine function. Recently, these functions have been studied very extensively.

In 2016, S. Takeuchi introduced a form of the generalized complete elliptic integrals as an application of generalized trigonometric functions [10, 17]. For $p,q \in (1,\infty)$ and for $r \in (0,1)$, the so-called generalized (p,q)-elliptic integrals of the first and second kinds are respectively defined as

$$\mathcal{K}_{p,q}(r) = \int_0^{\pi_{p,q}/2} \frac{\mathrm{d}t}{(1-r^q\sin_{p,q}^q t)^{1-1/p}} = \int_0^1 \frac{\mathrm{d}t}{(1-t^q)^{1/p}(1-r^qt^q)^{1-1/p}}$$

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and

$$\mathcal{E}_{p,q}(r) = \int_0^{\pi_{p,q}/2} (1 - r^q \sin_{p,q}^q t)^{1/p} \mathrm{d}t = \int_0^1 \left(\frac{1 - r^q t^q}{1 - t^q}\right)^{1/p} \mathrm{d}t.$$

For real numbers a, b, and c with $c \neq 0, -1, -2\cdots$, the Gaussian hypergeometric function is defined by

$${}_2F_1(a,b;c;x) \equiv \sum_{n=0}^{\infty} \frac{(a,n)(b,n)}{(c,n)} \frac{x^n}{n!}, \quad |x| < 1.$$

Here $(a,0) = 1$ for $a \neq 0$, and (a,n) is the shifted factorial function

$$(a,n) \equiv a(a+1)(a+2)\cdots(a+n-1)$$

for $n \in \mathbb{N} \equiv \{k : k \text{ is a positive integer}\}$.

The generalized (p,q)-elliptic integrals can be represented by the Gaussian hypergeometric function [10]:

$$\begin{cases} \mathcal{K}_{p,q} = \mathcal{K}_{p,q}(r) = \frac{\pi_{p,q}}{2} {}_2F_1\left(1 - \frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right) \\ \mathcal{K}'_{p,q} = \mathcal{K}'_{p,q}(r) = \mathcal{K}_{p,q}(r') \\ \mathcal{K}_{p,q}(0) = \frac{\pi_{p,q}}{2}, \quad \mathcal{K}_{p,q}(1) = \infty \end{cases}$$

and

$$\begin{cases} \mathcal{E}_{p,q} = \mathcal{E}_{p,q}(r) = \frac{\pi_{p,q}}{2} {}_2F_1\left(-\frac{1}{p}, \frac{1}{q}; 1 - \frac{1}{p} + \frac{1}{q}; r^q\right) \\ \mathcal{E}'_{p,q} = \mathcal{E}'_{p,q}(r) = \mathcal{E}_{p,q}(r') \\ \mathcal{E}_{p,q}(0) = \frac{\pi_{p,q}}{2}, \quad \mathcal{E}_{p,q}(1) = 1. \end{cases}$$

Here and hereafter, we always let $r' = (1 - r^q)^{1/q}$. It is easy to see that $\mathcal{K}_{p,q}$ is strictly increasing and $\mathcal{E}_{p,q}$ is strictly decreasing on $(0,1)$. The generalized (p,q)-elliptic integrals satisfy the following beautiful Legendre relation [10]: for $r \in (0,1)$,

$$\mathcal{K}_{p,q}(r)\mathcal{E}'_{p,q}(r) + \mathcal{K}'_{p,q}(r)\mathcal{E}_{p,q}(r) - \mathcal{K}_{p,q}(r)\mathcal{K}'_{p,q}(r) = \frac{\pi_{p,q}}{2}. \tag{1.2}$$

For $p = q = 2$, the generalized (p,q)-elliptic integrals reduce to the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$, respectively. It is well known that the complete elliptic integrals $\mathcal{K}$ and $\mathcal{E}$ have many applications in several fields of mathematics as well as in physics and engineering. Numerous properties have been obtained for $\mathcal{K}$ and $\mathcal{E}$ (for instance, cf. [1, 3, 4, 6, 15]). However, only a few basic properties of the generalized (p,q)-elliptic integrals $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$ have been revealed, see [10, 17, 18]. It is natural to ask whether the known properties of $\mathcal{K}$ and $\mathcal{E}$ can be extended to $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$.

We define three related functions $\mu_{p,q}$, $m_{p,q}$, and $M_{p,q}$ as follows: for $(p,q) \in (1,\infty)$ and $r \in (0,1)$,

$$\mu_{p,q}(r) = \frac{\pi_{p,q}}{2}\frac{\mathcal{K}'_{p,q}(r)}{\mathcal{K}_{p,q}(r)}, \tag{1.3}$$

$$m_{p,q}(r) = \frac{2}{\pi_{p,q}} r'^q \mathcal{K}_{p,q}(r)\mathcal{K}'_{p,q}(r), \tag{1.4}$$

$$M_{p,q}(r) = m_{p,q}(r) + \log r. \tag{1.5}$$

For $p = q = 2$, these functions reduce to well-known special cases. The function $\mu(r) = \mu_{2,2}(r)$ is the modulus of the Grötzsch ring domain in the plane, which has numerous applications in the conformal invariants and the theory of quasiconformal mappings [3, 11]. The function μ also appears in the classical modular equations [5, 6]. Many noteworthy monotonicity and convexity properties of functions defined in terms of the modulus of the Grötzsch ring are presented in [3]. Applications of these results lead to various sharp functional inequalities for the function μ and its generalizations [2, 3, 4, 8, 11, 12, 14, 19, 20, 21, 22].

The purpose of this paper is to study some monotonicity properties for the generalized (p,q)-elliptic integrals and the generalized Grötzsch function. As applications, we obtain some sharp functional inequalities for the generalized Grötzsch function and thus improve and extend some previous results.

2. Preliminary and derivatives

Throughout this paper, let $\gamma = 0.577215\ldots$ be the Euler constant and ψ be the classical psi function, and let

$$R(a,b) = -2\gamma - \psi(a) - \psi(b), \quad R(a) = R(a,1-a), \quad R(1/2) = \log 16.$$

The following two lemmas are very useful in proving monotonicity of the ratio of two functions or two series.

LEMMA 2.1. [3, Theorem 1.25] *Suppose that $-\infty < a < b < \infty$, f, $g : [a,b] \to \mathbb{R}$ are continuous on $[a,b]$ and differentiable on (a,b), and the derivative $g' \neq 0$ on (a,b). If f'/g' is increasing (decreasing) on (a,b), then so are the functions*

$$\frac{f(x)-f(a)}{g(x)-g(a)} \quad \text{and} \quad \frac{f(x)-f(b)}{g(x)-g(b)}. \tag{2.2}$$

If f'/g' is strictly monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.3. [13] *Let a_n and b_n for $n \in \mathbb{N}$ be real numbers and the power series*

$$f(x) = \sum_{n=0}^{\infty} a_n x^n \quad \text{and} \quad g(x) = \sum_{n=0}^{\infty} b_n x^n$$

be convergent for $|x| < 1$. *If* $b_n > 0$ *for all* $n \in \mathbb{N}$ *and if* a_n/b_n *is strictly increasing (or decreasing, respectively), then the function* f/g *is strictly increasing (or decreasing, respectively) on* $(0,1)$.

The functions $\mathcal{K}_{p,q}$ and $\mathcal{E}_{p,q}$ satisfy a system of differential equations [10, proposition 2.1]:

$$\frac{d\mathcal{K}_{p,q}}{dr} = \frac{\mathcal{E}_{p,q} - r'^q \mathcal{K}_{p,q}}{rr'^q}, \quad \frac{d\mathcal{E}_{p,q}}{dr} = \frac{q(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pr}. \tag{2.4}$$

From (2.4), we have the following derivative formulas:

$$\frac{d(\mathcal{E}_{p,q} - r'^q \mathcal{K}_{p,q})}{dr} = \frac{(p-q)(\mathcal{K}_{p,q} - \mathcal{E}_{p,q}) + p(q-1)r^q \mathcal{K}_{p,q}}{pr}, \tag{2.5}$$

$$\frac{d(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})}{dr} = \frac{(p - qr'^q)\mathcal{E}_{p,q} + (q-p)r'^q \mathcal{K}_{p,q}}{prr'^q}. \tag{2.6}$$

By the definitions (1.3), (1.4), and the derivative formula (2.4), we obtain the derivative formulas for the functions $\mu_{p,q}$ and $m_{p,q}$ as the following lemma shows.

LEMMA 2.7. *Let* $p,q > 1$, *for* $0 < r < 1$,

(1) $$\frac{d\mu_{p,q}(r)}{dr} = -\frac{\pi_{p,q}^2}{4rr'^q \mathcal{K}_{p,q}^2},$$

(2) $$\frac{dm_{p,q}(r)}{dr} = -\frac{1}{r} - \frac{4}{\pi_{p,q}} r^{q-1} \mathcal{K}_{p,q} \mathcal{K}'_{p,q} \left(\frac{q}{2} - \frac{\mathcal{E}_{p,q} - r'^q \mathcal{K}_{p,q}}{r^q \mathcal{K}_{p,q}} \right).$$

Proof. For part (1), the function $\mu_{p,q}$ is defined as in (1.3), and its derivative follows from the Legendre relation (1.2) and the derivative formula (2.4):

$$\begin{aligned} \frac{d\mu_{p,q}(r)}{dr} &= \frac{\pi_{p,q}}{2} \left(-\frac{\mathcal{E}'_{p,q} - r^q \mathcal{K}'_{p,q}}{r' r^q} \frac{r^{q-1}}{r'^{q-1}} \mathcal{K}_{p,q} - \frac{\mathcal{E}_{p,q} - r'^q \mathcal{K}_{p,q}}{rr'^q} \mathcal{K}'_{p,q} \right) \frac{1}{\mathcal{K}_{p,q}^2} \\ &= \frac{\pi_{p,q}}{2} \frac{\mathcal{K}_{p,q}\mathcal{K}'_{p,q} - \mathcal{E}'_{p,q}\mathcal{K}_{p,q} - \mathcal{K}'_{p,q}\mathcal{E}_{p,q}}{rr'^q \mathcal{K}_{p,q}^2} \\ &= -\frac{\pi_{p,q}^2}{4rr'^q \mathcal{K}_{p,q}^2}. \end{aligned}$$

For part (2), the function $m_{p,q}$ is defined as in (1.4), and its derivative follows from (1.2) and (2.4):

$$\begin{aligned} \frac{dm_{p,q}(r)}{dr} &= \frac{2}{\pi_{p,q}} \left(-qr^{q-1} \mathcal{K}_{p,q}\mathcal{K}'_{p,q} + r'^q \left(\frac{d\mathcal{K}_{p,q}}{dr} \mathcal{K}'_{p,q} + \frac{d\mathcal{K}'_{p,q}}{dr} \mathcal{K}_{p,q} \right) \right) \\ &= \frac{2}{\pi_{p,q}} \frac{\mathcal{E}_{p,q}\mathcal{K}'_{p,q} - \mathcal{K}_{p,q}\mathcal{K}'_{p,q} - \mathcal{E}'_{p,q}\mathcal{K}_{p,q} + (2-q)r^q \mathcal{K}_{p,q}\mathcal{K}'_{p,q}}{r} \end{aligned}$$

$$= \frac{2}{\pi_{p,q}} \frac{-\frac{\pi_{p,q}}{2} + 2\mathcal{E}_{p,q}\mathcal{K}'_{p,q} - 2r'^q\mathcal{K}_{p,q}\mathcal{K}'_{p,q} - qr^q\mathcal{K}_{p,q}\mathcal{K}'_{p,q}}{r}$$
$$= -\frac{1}{r} - \frac{4}{\pi_{p,q}} r^{q-1}\mathcal{K}_{p,q}\mathcal{K}'_{p,q}\left(\frac{q}{2} - \frac{\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q}}{r^q\mathcal{K}_{p,q}}\right). \quad \square$$

3. Generalized elliptic integrals

In this section, we study the monotonicity for certain combinations of the generalized elliptic integrals, thus generalizing some well-known results for the classical complete elliptic integrals.

The following lemma lists some known monotonicity results for the generalized elliptic integrals [9, Theorem 1] which are useful in the proof of the main results of this paper.

LEMMA 3.1. [9] *Let* $p,q > 1$, $r \in (0,1)$. *Then the function*

(1) $f_1(r) = r'^q\mathcal{K}_{p,q}/\mathcal{E}_{p,q}$ *is strictly decreasing from* $(0,1)$ *onto* $(0,1)$.

(2) $f_2(r) = (\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q})/(r^q\mathcal{K}_{p,q})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, q(p-1)/(q(p-1)+p))$.

(3) $f_3(r) = (\mathcal{K}_{p,q} - \mathcal{E}_{p,q})/(r^q\mathcal{K}_{p,q})$ *is strictly increasing from* $(0,1)$ *onto* $(p/(q(p-1)+p), 1)$.

(4) $f_4(r) = (\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q})/(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, q(p-1)/p)$.

(5) $f_5(r) = r'^q(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})/(r^q\mathcal{E}_{p,q})$ *is strictly decreasing from* $(0,1)$ *onto* $(0, p/(q(p-1)+p))$.

(6) $f_6(r) = r'^c\mathcal{K}_{p,q}(r)$ *is decreasing on* $(0,1)$ *if and only if* $c \geqslant q(p-1)/(q(p-1)+p)$ *with the range* $(0, \pi_{p,q}/2)$.

The following Lemma extends the part (1) of Lemma 3.1.

LEMMA 3.2. *The function* $f(r) = r'^q\mathcal{K}^c_{p,q}/\mathcal{E}_{p,q}$ *is strictly decreasing from* $(0,1)$ *onto* $(0, (\pi_{p,q}/2)^{c-1})$ *if and only if* $c \leqslant q+1$.

Proof. By differentiation, we have

$$f'(r) = \frac{1}{\mathcal{E}^2_{p,q}}\left(\left(-qr^{q-1}\mathcal{K}^c_{p,q} + cr'^q\mathcal{K}^{c-1}_{p,q}\frac{\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q}}{rr'^q}\right)\mathcal{E}_{p,q} - r'^q\mathcal{K}^c_{p,q}\frac{q(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pr}\right)$$
$$= \frac{1}{\mathcal{E}^2_{p,q}}\left(\frac{(-qr^q\mathcal{K}^c_{p,q} + c\mathcal{K}^{c-1}_{p,q}(\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q}))\mathcal{E}_{p,q}}{r} - \frac{qr'^q\mathcal{K}^c_{p,q}(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pr}\right)$$
$$= \frac{-qr^q\mathcal{K}^c_{p,q} + c\mathcal{K}^{c-1}_{p,q}(\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q})}{r\mathcal{E}_{p,q}} - \frac{qr'^q\mathcal{K}^c_{p,q}(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pr\mathcal{E}^2_{p,q}},$$

and

$$\frac{\mathcal{E}_{p,q}}{qr^{q-1}\mathcal{K}^c_{p,q}}f'(r) = -1 + \frac{c(\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q})}{qr^q\mathcal{K}_{p,q}} - \frac{r'^q(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pr^q\mathcal{E}_{p,q}}.$$

Then for all $r \in (0,1)$, $f'(r) < 0$ is equivalent to

$$\frac{c(\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q})}{qr^q\mathcal{K}_{p,q}} < 1 + \frac{r'^q(\mathcal{E}_{p,q} - \mathcal{K}_{p,q})}{pr^q\mathcal{E}_{p,q}},$$

i.e.

$$c < \left(1 - \frac{r'^q(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})}{pr^q\mathcal{E}_{p,q}}\right) \Big/ \left(\frac{\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q}}{qr^q\mathcal{K}_{p,q}}\right).$$

Hence it suffices to take

$$c \leqslant \inf_{r\in(0,1)} \left(1 - \frac{r'^q(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})}{pr^q\mathcal{E}_{p,q}}\right) \Big/ \left(\frac{\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q}}{qr^q\mathcal{K}_{p,q}}\right) = q+1$$

by Lemma 3.1(2) and (5). □

LEMMA 3.3. *Let* $p,q > 1$, $r \in (0,1)$. *Then*

(1) For $2 \leqslant q \leqslant p$, *the function*

$$f_1(r) = \frac{2(\mathcal{K}_{p,q} - \mathcal{E}_{p,q}) + (q-2)r^q\mathcal{K}_{p,q}}{\pi^2_{p,q}/(4r'^q\mathcal{K}^2_{p,q}) - 1}$$

is strictly decreasing from $(0,1)$ *onto* $(0, q\pi_{p,q}/2)$.

(2) For $q \geqslant \frac{2p-3}{p-1}$, *let* $C_1 = \frac{p\pi_{p,q}}{4+2p(q-1)-2q}$, *the function*

$$f_2(r) = \frac{\mathcal{K}_{p,q} - \mathcal{E}_{p,q}}{\pi^2_{p,q}/(4r'^q\mathcal{K}^2_{p,q}) - 1}$$

is strictly decreasing from $(0,1)$ *onto* $(0, C_1)$.

(3) For $q \geqslant 2$, *let* $C_2 = \frac{q(2-p-q+pq)\pi_{p,q}}{2(q(p-1)+p)}$, *the function*

$$f_3(r) = \frac{2(\mathcal{K}_{p,q} - \mathcal{E}_{p,q}) + (q-2)r^q\mathcal{K}_{p,q}}{r^q}$$

is strictly increasing from $(0,1)$ *onto* (C_2, ∞).

(4) For $q \geqslant 2$, *let* $C_3 = \frac{q(2-p-q+pq)}{q(p-1)+p}$, *the function*

$$f_4(r) = \frac{\pi^2_{p,q}/(4r'^q\mathcal{K}^2_{p,q}) - 1}{\log(1/r')}$$

is strictly increasing from $(0,1)$ *onto* (C_3, ∞).

Proof. (1) Let

$$f_{11}(r)=2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})+(q-2)r^q\mathcal{K}_{p,q},$$

and

$$f_{12}(r)=\pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2)-1.$$

Then $f_1(r)=f_{11}(r)/f_{12}(r)$ and $f_{11}(0^+)=f_{12}(0^+)=0$. By differentiation, we have

$$\begin{aligned}f_{11}'(r)=&\frac{2\left((p-qr'^q)\mathcal{E}_{p,q}+(q-p)r'^q\mathcal{K}_{p,q}\right)}{prr'^q}\\&+(q-2)\left(qr^{q-1}\mathcal{K}_{p,q}+r^q\frac{\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q}}{rr'^q}\right)\\=&\frac{G(r)}{prr'^q},\end{aligned}$$

where

$$\begin{aligned}G(r)=&2(p-qr'^q)\mathcal{E}_{p,q}+2(q-p)r'^q\mathcal{K}_{p,q}\\&+p(q-2)r^q\mathcal{E}_{p,q}+p(q-1)(q-2)r^qr'^q\mathcal{K}_{p,q},\end{aligned}$$

and

$$\begin{aligned}f_{12}'(r)=&\frac{-4\pi_{p,q}^2}{(4r'^q\mathcal{K}_{p,q}^2)^2}\left(-qr^{q-1}\mathcal{K}_{p,q}^2+2r'^q\mathcal{K}_{p,q}\frac{\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q}}{rr'^q}\right)\\=&\frac{\pi_{p,q}^2\left(2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})+(q-2)r^q\mathcal{K}_{p,q}\right)}{4rr'^{2q}\mathcal{K}_{p,q}^3}.\end{aligned}$$

Then

$$\begin{aligned}\frac{f_{11}'(r)}{f_{12}'(r)}=&\frac{4r'^q\mathcal{K}_{p,q}^3}{p\pi_{p,q}^2}\frac{G(r)}{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})+(q-2)r^q\mathcal{K}_{p,q}}\\=&\frac{4r'^q\mathcal{K}_{p,q}^3}{p\pi_{p,q}^2}\frac{1}{q-2+\frac{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})}{r^q\mathcal{K}_{p,q}}}\left(p(q-1)(q-2)r'^q\right.\\&\left.+\frac{2(p-q)(\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q})}{r^q\mathcal{K}_{p,q}}+\frac{(pq-2(p-q))\mathcal{E}_{p,q}}{\mathcal{K}_{p,q}}\right),\end{aligned}$$

which is strictly decreasing by Lemma 3.1 (2), (3), (6) and since

$$\frac{q}{3}\geqslant\frac{q(p-1)}{q(p-1)+p}.$$

Therefore, the monotonicity of $f_1(r)$ follows from Lemma 2.1.

For the limiting values, we have

$$\lim_{r\to 0^+} f_1(r) = \frac{\pi_{p,q}}{2p}\frac{1}{q-2+\frac{2p}{q(p-1)+p}}\left(p(q-1)(q-2)+\frac{2q(p-q)(p-1)}{q(p-1)+p}+pq-2(p-q)\right)$$
$$= \frac{q\pi_{p,q}}{2}$$

and

$$\lim_{r\to 1^-} f_1(r) = \lim_{r\to 1^-}\frac{f_{11}'(r)}{f_{12}'(r)} = 0.$$

(2) Let $f_{21}(r) = \mathcal{K}_{p,q} - \mathcal{E}_{p,q}$, $f_{22}(r) = \pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2) - 1$, then $f_2(r) = f_{21}(r)/f_{22}(r)$ and $f_{21}(0^+) = f_{22}(0^+) = 0$. By differentiation,

$$f_{21}'(r) = \frac{(p-qr'^q)\mathcal{E}_{p,q} + (q-p)r'^q\mathcal{K}_{p,q}}{prr'^q},$$
$$f_{22}'(r) = \frac{\pi_{p,q}^2(2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q}) + (q-2)r^q\mathcal{K}_{p,q})}{4rr'^{2q}\mathcal{K}_{p,q}^3},$$

and

$$\frac{f_{21}'(r)}{f_{22}'(r)} = \frac{4r'^q\mathcal{K}_{p,q}^3}{p\pi_{p,q}^2}\frac{(p-qr'^q)\mathcal{E}_{p,q} + (q-p)r'^q\mathcal{K}_{p,q}}{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q}) + (q-2)r^q\mathcal{K}_{p,q}}$$
$$= \frac{4}{p\pi_{p,q}^2}\frac{r'^q\mathcal{K}_{p,q}^3}{q-2+\frac{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})}{r^q\mathcal{K}_{p,q}}}\left(\frac{(p-qr'^q)\mathcal{E}_{p,q}}{r^q\mathcal{K}_{p,q}} + \frac{(q-p)r'^q}{r^q}\right)$$
$$= \frac{4}{p\pi_{p,q}^2}\frac{r'^q\mathcal{K}_{p,q}^3}{q-2+\frac{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})}{r^q\mathcal{K}_{p,q}}}\left(\frac{p(\mathcal{E}_{p,q}-r'^q\mathcal{K}_{p,q})}{r^q\mathcal{K}_{p,q}} + \frac{qr'^q(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})}{r^q\mathcal{K}_{p,q}}\right)$$

which is strictly decreasing by Lemma 3.1 (2), (3), (5), (6) and since

$$\frac{q}{3} \geqslant \frac{q(p-1)}{q(p-1)+p}.$$

Therefore, we obtain the monotonicity of $f_1(r)$ by Lemma 2.1.

For the limiting values, we have

$$\lim_{r\to 0^+} f_2(r) = \frac{\pi_{p,q}}{2p}\frac{1}{q-2+\frac{2p}{q(p-1)+p}}\frac{qp^2}{q(p-1)+p} = \frac{p\pi_{p,q}}{4+2p(q-1)-2q}$$

and

$$\lim_{r\to 1^-} f_2(r) = \lim_{r\to 1^-}\frac{f_{21}'(r)}{f_{22}'(r)} = 0.$$

(3) It is easy to check that

$$\begin{aligned}\mathcal{K}_{p,q}-\mathcal{E}_{p,q} &= \frac{\pi_{p,q}}{2}\left(\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(a+b,n)n!}r^{qn}-\sum_{n=0}^{\infty}\frac{(a-1,n)(b,n)}{(a+b,n)n!}r^{qn}\right)\\ &= \frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\left(1-\frac{a-1}{a+n-1}\right)\frac{(a,n)(b,n)}{(a+b,n)n!}r^{qn}\\ &= \frac{\pi_{p,q}}{2}\sum_{n=1}^{\infty}\left(1-\frac{a-1}{a+n-1}\right)\frac{(a,n)(b,n)}{(a+b,n)n!}r^{qn}\\ &= \frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\frac{n+1}{a+n}\frac{(a,n+1)(b,n+1)}{(a+b,n+1)(n+1)!}r^{qn+q}\\ &= \frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\frac{(a,n)(b,n+1)}{(a+b,n+1)n!}r^{qn+q},\end{aligned}$$

where $a=1-1/p$, $b=1/q$, and

$$\frac{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})+(q-2)r^q\mathcal{K}_{p,q}}{r^q}=\frac{\pi_{p,q}}{2}\sum_{n=0}^{\infty}\left(\frac{2(b+n)}{a+b+n}+q-2\right)\frac{(a,n)(b,n)}{(a+b,n)n!}r^{qn},$$

with the coefficient

$$\left(\frac{2(b+n)}{a+b+n}+q-2\right)\frac{(a,n)(b,n)}{(a+b,n)n!}>0$$

when $q\geqslant 2$. Thus, the monotonicity of $f_3(r)$ follows immediately. The limiting values follow from Lemma 3.1(3)

$$\lim_{r\to 0^+}f_3(r)=\frac{q(2-p-q+pq)\pi_{p,q}}{2(q(p-1)+p)},\qquad \lim_{r\to 1^-}f_3(r)=\infty.$$

(4) Let $f_{41}(r)=\pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2)-1$, $f_{42}(r)=\log(1/r')$, then $f_4(r)=f_{41}(r)/f_{42}(r)$ and $f_{41}(0^+)=f_{42}(0^+)=0$. By differentiation, we obtain

$$f_{41}'(r)=\frac{\pi_{p,q}^2(2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})+(q-2)r^q\mathcal{K}_{p,q})}{4rr'^{2q}\mathcal{K}_{p,q}^3},\quad f_{42}'(r)=\frac{r^{q-1}}{r'^q},$$

and

$$\frac{f_{41}'(r)}{f_{42}'(r)}=\frac{\pi_{p,q}^2}{4}\frac{1}{r'^q\mathcal{K}_{p,q}^3}\frac{2(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})+(q-2)r^q\mathcal{K}_{p,q}}{r^q},$$

which is strictly increasing by Lemma 3.1(6) and Lemma 3.3(3), since

$$\frac{q}{3}\geqslant\frac{q(p-1)}{q(p-1)+p}.$$

Therefore, we obtain the monotonicity of $f_4(r)$ by Lemma 2.1.

For the limiting values, we have

$$\lim_{r\to 0^+} f_4(r) = \frac{\pi_{p,q}^2}{4}\frac{8}{\pi_{p,q}^3}\frac{q\pi_{p,q}(2-p-q+pq)}{2(q(p-1)+p)} = \frac{q(2-p-q+pq)}{q(p-1)+p}$$

and

$$\lim_{r\to 1^-} f_4(r) = \infty. \quad \square$$

4. Generalized Grötzsch function

In this section, we study the monotonicity and convexity properties of the generalized Grötzsch function and obtain sharp functional inequalities which extend and sharpen some results for the modulus of the Grötzsch ring in the plane in [12, 19, 21, 22].

LEMMA 4.1. [14] *Let* $p,q>1$, $r\in(0,1)$, *and* $C=\frac{1}{q}R(1-1/p,1/q)$. *Then the function*

(1) $h_1(r)=\mu_{p,q}(r)+\log r$ *is strictly decreasing from* $(0,1)$ *onto* $(0,C)$.

(2) $h_2(r)=m_{p,q}(r)+\log r$ *is strictly decreasing from* $(0,1)$ *onto* $(0,C)$.

THEOREM 4.2. *Let* $p,q>1$.

(1) The function $f_1(x)=\mu_{p,q}(1-1/\cosh x)$ *is strictly decreasing and convex from* $(0,\infty)$ *onto* $(0,\infty)$. *In particular, for* $a,b\in(0,1)$,

$$2\mu_{p,q}\left(1-\sqrt{\frac{2ab}{1+ab+a'b'}}\right)\leqslant \mu_{p,q}(1-a)+\mu_{p,q}(1-b)$$

with equality if and only if $a=b$.

(2) The function $f_2(x)=\mu_{p,q}(1-e^{-x})$ *is strictly decreasing and convex from* $(0,\infty)$ *onto* $(0,\infty)$. *In particular, for* $a,b\in(0,1)$,

$$\mu_{p,q}\left(1-\sqrt{(1-a)(1-b)}\right)\leqslant \frac{\mu_{p,q}(a)+\mu_{p,q}(b)}{2}$$

with equality if and only if $a=b$.

(3) The function $f_3(x)=\mu_{p,q}(e^{-x})$ *is strictly increasing and concave from* $(0,\infty)$ *onto* $(0,\infty)$. *In particular, for* $a,b\in(0,1)$

$$\mu_{p,q}\left(\sqrt{ab}\right)\geqslant \frac{\mu_{p,q}(a)+\mu_{p,q}(b)}{2}$$

with equality if and only if $a=b$.

Proof. (1) Let $t = 1-1/\cosh x$, then $dt/dx = (1-t)\sqrt{2t-t^2}$. By differentiation, we have

$$\begin{aligned} f_1'(x) &= \frac{d\mu_{p,q}(t)}{dt}\frac{dt}{dx} = \frac{-\pi_{p,q}^2}{4tt'^q\mathcal{K}_{p,q}^2}(1-t)\sqrt{2t-t^2} \\ &= \frac{-\pi_{p,q}^2}{4}\frac{(1-t)\sqrt{2t-t^2}}{(1-t^q)t\mathcal{K}_{p,q}^2} \\ &= \frac{-\pi_{p,q}^2}{4}\sqrt{\frac{2}{t}-1}\frac{1}{(1-t^q)/(1-t)\mathcal{K}_{p,q}^2}, \end{aligned}$$

which is negative and strictly increasing in t, and hence increasing in x. Therefore, f_1 is decreasing and convex on $(0,\infty)$. In particular, we have $f_1((x+y)/2) \leqslant (f_1(x)+f_1(y))/2$, with equality if and only if $x=y$. Set $a = 1/\cosh x$ and $b = 1/\cosh y$, then

$$\cosh\left(\frac{x+y}{2}\right) = \sqrt{\frac{1+ab+a'b'}{2ab}}.$$

Hence

$$f_1((x+y)/2) \leqslant (f_1(x)+f_1(y))/2$$

gives

$$2\mu_{p,q}\left(1-\sqrt{\frac{2ab}{1+ab+a'b'}}\right) \leqslant \mu_{p,q}(1-a)+\mu_{p,q}(1-b)$$

with equality if and only if $a=b$.

(2) Let $r = 1-e^{-x}$, then $r \in (0,1)$. We have

$$f_2'(x) = \frac{-\pi_{p,q}^2(1-r)}{4rr'^q\mathcal{K}_{p,q}^2} = \frac{-\pi_{p,q}^2}{4r\mathcal{K}_{p,q}^2}\frac{1-r}{r'^q},$$

which is negative and increasing in r, hence in x. So $f_2(x)$ is strictly decreasing and convex in x, then

$$f_2((x+y)/2) \leqslant (f_2(x)+f_2(y))/2$$

for all $x,y \in (0,\infty)$, with equality if and only if $x=y$. Setting $1-e^{-x}=a$, $1-e^{-y}=b$, we obtain

$$\mu_{p,q}\left(1-\sqrt{(1-a)(1-b)}\right) \leqslant \frac{\mu_{p,q}(a)+\mu_{p,q}(b)}{2}$$

with equality if and only if $a=b$.

(3) Let $r = e^{-x}$, then $r \in (0,1)$. By differentiation, we have

$$f_3'(x) = \frac{-\pi_{p,q}^2(-e^{-x})}{4rr'^q\mathcal{K}_{p,q}^2} = \frac{\pi_{p,q}^2}{4r'^q\mathcal{K}_{p,q}^2},$$

which is positive and strictly increasing in r, hence decreasing in x. So $f_3(x)$ is strictly increasing and concave in x, then

$$f_3((x+y)/2) \geqslant (f_3(x)+f_3(y))/2$$

for all $x,y \in (0,\infty)$, with equality if and only if $x=y$. Setting $e^{-x}=a$, $e^{-y}=b$, we obtain

$$\mu_{p,q}\left(\sqrt{ab}\right) \geqslant \frac{\mu_{p,q}(a)+\mu_{p,q}(b)}{2}$$

with equality if and only if $a=b$. □

REMARK 4.3. Theorem 4.2 (1) is a generalization of [22, Theorem 3.1], and Theorem 4.2 (2), (3) are generalizations of (1), (2) in [21, Theorem 3.23], and also see [3, 5.62(4), 5.68(20)].

THEOREM 4.4. *Let* $p,q>1$, $r\in(0,1)$.

(1) When $q \geqslant (3p-4)/(p-1)$, *the function* $f_1(r)=\mu_{p,q}(r')/\operatorname{arth}(r^{q/2})$ *is strictly decreasing from* $(0,1)$ *onto* $(2/q,\infty)$.

(2) When $p \leqslant 2, 2 \leqslant q \leqslant p/(p-1)$, *the function*

$$f_2(r)=\mu_{p,q}(r)-\log\left(\frac{1+r'}{r}\right)$$

is strictly decreasing from $(0,1)$ *onto* $(0,R(1-1/p,1/q)/q-\log 2)$.

Proof. (1) let $f_{11}(r)=\mu_{p,q}(r')$ and $f_{12}(r)=\operatorname{arth}(r^{q/2})$, then $f_{11}(0^+)=f_{12}(0^+)=0$. By differentiation, we have

$$f_{11}'(r)=\frac{d\mu_{p,q}(r')}{dr'}\frac{dr'}{dr}=\frac{-\pi_{p,q}^2}{4r'r^q\mathcal{K}_{p,q}'^2}\frac{-r^{q-1}}{r'^{q-1}}=\frac{\pi_{p,q}^2}{4rr'^q\mathcal{K}_{p,q}'^2}$$

and

$$f_{12}'(r)=\frac{1}{1-r^q}\frac{q}{2}r^{q/2-1}=\frac{qr^{(q-2)/2}}{2r'^q}.$$

Then

$$\frac{f_{11}'(r)}{f_{12}'(r)}=\frac{\pi_{p,q}^2}{2qr^{q/2}\mathcal{K}_{p,q}'^2},$$

which is strictly decreasing by Lemma 3.1(6) since

$$\frac{q}{4}\geqslant\frac{q(p-1)}{q(p-1)+p}.$$

Therefore, the monotonicity of f_1 follows from Lemma 2.1. It is easy to see that the limiting values are

$$f_1(0^+) = \lim_{r\to 0^+} \frac{f_{11}'(r)}{f_{12}'(r)} = \infty \quad \text{and} \quad f_1(1^-) = \frac{2}{q}.$$

(2) By differentiation, we have

$$\left(\log\left(\frac{1+r'}{r}\right)\right)' = \frac{r}{1+r'} \frac{-r(r/r')^{q-1} - (1+r')}{r^2}$$
$$= \frac{r^q + r'^{q-1} + r'^q}{-rr'^{q-1}(1+r')} = \frac{1+r'^{q-1}}{-rr'^{q-1}(1+r')},$$

and

$$f_2'(r) = \frac{1}{rr'^q}\left(\frac{1+r'^{q-1}}{1+r'} - \frac{\pi_{p,q}^2}{4r'\mathcal{K}_{p,q}^2}\right).$$

Since

$$\frac{1+r'^{q-1}}{1+r'} \leqslant 1 \qquad \text{and} \qquad \frac{1}{2} \geqslant \frac{q(p-1)}{q(p-1)+p},$$

it follows from Lemma 3.1(6) that $f_2'(r) < 0$. Hence $f_2(r)$ is strictly decreasing. We can easily get the limiting values from Lemma 4.1 (1). □

REMARK 4.5. It follows Lemma 4.1 (1) that for all $r \in (0,1)$

$$\log\frac{1}{r} < \mu_{p,q}(r) < \log\frac{1}{r} + \frac{1}{q}R\left(1-\frac{1}{p},\frac{1}{q}\right). \tag{4.6}$$

We can get the following improved inequality from Theorem 4.4(2)

$$\log\left(\frac{1+r'}{r}\right) < \mu_{p,q}(r) < \log\left(\frac{1+r'}{2r}\right) + \frac{1}{q}R\left(1-\frac{1}{p},\frac{1}{q}\right). \tag{4.7}$$

THEOREM 4.8. *Let* $p,q>1$, $r\in(0,1)$. *Then*

(1) When $2 \leqslant q \leqslant p$, *the function*

$$F(r) = (m_{p,q}(r) + \log r)/(\mu_{p,q}(r) + \log r)$$

is decreasing from $(0,1)$ *onto* $(0,1)$.

(2) When $q \geqslant \frac{2p-3}{p-1}$, *the function*

$$G(r) = (\mathcal{E}_{p,q}(r) - 1)/(\mu_{p,q}(r) + \log r)$$

is decreasing from $(0,1)$ *onto* $\left(0, \frac{q(\pi_{p,q}-2)}{2R(1-1/p,1/q)}\right)$.

(3) When $q \geqslant \frac{2p-3}{p-1}$, the function

$$J(r) = \frac{\pi_{p,q}/2 - \mathcal{E}_{p,q}}{R(1-1/p,1/q)/q - (\mu_{p,q}(r) + \log r)}$$

is decreasing from $(0,1)$ onto $\left(\frac{q(\pi_{p,q}-2)}{2R(1-1/p,1/q)}, \frac{q\pi_{p,q}}{4+2p(q-1)-2q}\right)$. Moreover, the double inequality

$$\frac{1}{q}R\left(1-\frac{1}{p},\frac{1}{q}\right) - \left(\frac{\pi_{p,q}}{2} - \mathcal{E}_{p,q}\right)\frac{2R(1-1/p,1/q)}{q(\pi_{p,q}-2)} < \mu_{p,q}(r) + \log r$$
$$< \frac{1}{q}R\left(1-\frac{1}{p},\frac{1}{q}\right) - \left(\frac{\pi_{p,q}}{2} - \mathcal{E}_{p,q}\right)\frac{4+2p(q-1)-2q}{q\pi_{p,q}}.$$

Proof. (1) Let $F_1(r) = m_{p,q}(r) + \log r$, $F_2(r) = \mu_{p,q}(r) + \log r$, then $F(r) = F_1(r)/F_2(r)$, $F_1(1^-) = F_2(1^-) = 0$, and $F(0^+) = 1$ by Lemma 4.1 . We have the derivatives

$$F_1'(r) = -\frac{2}{\pi_{p,q}} r^{q-1}\mathcal{K}_{p,q}\mathcal{K}_{p,q}'\left(q - \frac{2(\mathcal{E}_{p,q} - r'^q\mathcal{K}_{p,q})}{r^q\mathcal{K}_{p,q}}\right)$$
$$= -\frac{2}{\pi_{p,q}}\frac{1}{r}\mathcal{K}_{p,q}'\left(2(\mathcal{K}_{p,q} - \mathcal{E}_{p,q}) + (q-2)r^q\mathcal{K}_{p,q}\right),$$
$$F_2'(r) = \frac{-\pi_{p,q}^2}{4rr'^q\mathcal{K}_{p,q}^2} + \frac{1}{r} = \frac{1}{r}\left(1 - \frac{\pi_{p,q}^2}{4r'^q\mathcal{K}_{p,q}^2}\right),$$

and

$$\frac{F_1'(r)}{F_2'(r)} = \frac{2\mathcal{K}_{p,q}'}{\pi_{p,q}}\frac{2(\mathcal{K}_{p,q} - \mathcal{E}_{p,q}) + (q-2)r^q\mathcal{K}_{p,q}}{\pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2) - 1}.$$

which is decreasing by Lemma 3.3(1). It follows from Lemma 2.1 that $F(r)$ is strictly decreasing. By the l'Hôpital Rule, we can easily obtain the limiting values.

(2) Let $G_1(r) = \mathcal{E}_{p,q} - 1$, $G_2(r) = \mu_{p,q}(r) + \log r$, then $G(r) = G_1(r)/G_2(r)$ and $G_1(1^-) = G_2(1^-) = 0$, $G(0^+) = q(\pi_{p,q} - 2)/(2R(1-1/p,1/q))$. By calculations, one has

$$G_1'(r) = -\frac{q(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})}{pr}, \qquad G_2'(r) = \frac{1}{r}\left(1 - \frac{\pi_{p,q}^2}{4r'^q\mathcal{K}_{p,q}^2}\right).$$

Then

$$\frac{G_1'(r)}{G_2'(r)} = \frac{q(\mathcal{K}_{p,q} - \mathcal{E}_{p,q})/p}{\pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2) - 1},$$

which is decreasing by Lemma 3.3(2). Hence the monotonicity of $G(r)$ follows from Lemma 2.1, and we can easily get the limiting value $G(1^-) = 0$.

(3) Let $J_1(r)=\pi_{p,q}/2-\mathcal{E}_{p,q}$, $J_2(r)=\frac{1}{q}R(1-1/p,1/q)-(\mu_{p,q}(r)+\log r)$, then $J(r)=J_1(r)/J_2(r)$ and $J_1(0^+)=J_2(0^+)=0$, $J(1^-)=(q(\pi_{p,q}-2))/(2R(1-1/p,1/q))$. By calculations, one has

$$J_1'(r)=\frac{q(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})}{pr},\quad J_2'(r)=-F_2'(r)=\frac{1}{r}\left(\frac{\pi_{p,q}^2}{4r'^q\mathcal{K}_{p,q}^2}-1\right),$$

and

$$\frac{J_1'(r)}{J_2'(r)}=\frac{q(\mathcal{K}_{p,q}-\mathcal{E}_{p,q})/p}{\pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2)-1}.$$

Then the monotonicity of $J(r)$ follows from Lemma 3.3(2), we can easily get the limiting value $J(0^+)=(q\pi_{p,q})/(4+2p(q-1)-2q)$. □

REMARK 4.9. Theorem 4.8(1) and (2) reduce to Theorem 1.1 in [12] when $p=q=1/a$ for $a\in(0,1/2]$ and with the argument r replaced with r^{2a}. Theorem 4.8(3) is an extension of Theorem 1.2 in [12] and improves the inequality in [2, Theorem 5.5(2)].

LEMMA 4.10. *For $p>1$ and $q\geqslant 2$, the function*

$$f(r)=\frac{\log(1/r')}{(1+r^q)\,\mathrm{arth}(r^{q/2})/r^{q/2}-1}$$

is strictly increasing from $(0,1)$ onto $(3/(4q),1/q)$.

Proof. Using power series expansions

$$\log\frac{1}{r'}=-\frac{1}{q}\log(1-r^q)=\sum_{n=0}^{\infty}\frac{r^{qn+q}}{q(n+1)}$$

and

$$\mathrm{arth}r^{q/2}=\sum_{n=0}^{\infty}\frac{r^{qn+q/2}}{2n+1},$$

we obtain

$$\begin{aligned}\frac{(1+r^q)\,\mathrm{arth}r^{q/2}}{r^{q/2}}-1&=\frac{1}{r^{q/2}}\left(\sum_{n=0}^{\infty}\frac{r^{qn+q/2}}{2n+1}+\sum_{n=0}^{\infty}\frac{r^{qn+q/2+q}}{2n+1}\right)-1\\&=\sum_{n=0}^{\infty}\frac{r^{qn}}{2n+1}+\sum_{n=0}^{\infty}\frac{r^{qn+q}}{2n+1}-1\\&=1+\sum_{n=1}^{\infty}\frac{r^{qn}}{2n+1}+\sum_{n=0}^{\infty}\frac{r^{qn+q}}{2n+1}-1\\&=\sum_{n=0}^{\infty}\frac{r^{qn+q}}{2n+3}+\sum_{n=0}^{\infty}\frac{r^{qn+q}}{2n+1}\\&=\sum_{n=0}^{\infty}\frac{4(n+1)}{(2n+1)(2n+3)}r^{qn+q}\end{aligned}$$

from which it follows that

$$f(r) = \frac{\sum_{n=0}^{\infty} \alpha_1(n) r^{qn+q}}{\sum_{n=0}^{\infty} \alpha_2(n) r^{qn+q}}$$

with

$$\alpha_1(n) = \frac{1}{q(n+1)} \quad \text{and} \quad \alpha_2(n) = \frac{4(n+1)}{(2n+1)(2n+3)}.$$

Let $A(n) = \frac{\alpha_1(n)}{\alpha_2(n)}$, then

$$\frac{A(n+1)}{A(n)} = \frac{(2n+5)(n+1)^2}{(2n+1)(n+2)^2} > 1,$$

and $A(n)$ is strictly increasing. Thus, the monotonicity of $f(r)$ follows from Lemma 2.3.

For the limiting values, let

$$f_1(r) = \log\frac{1}{r'} \quad \text{and} \quad f_2(r) = \frac{(1+r^q)\,\text{arth} r^{q/2}}{r^{q/2}} - 1.$$

Then $f_1'(r) = r^{q-1}/(r'^q)$ and

$$f_2'(r) = \frac{q}{2r^q}\frac{(1+r^q)r^{q-1} - r^{q/2-1}r'^{2q}\,\text{arth} r^{q/2}}{r'^q},$$

Then

$$\frac{f_1'(r)}{f_2'(r)} = \frac{2}{q}\frac{r^{2q-1}}{(1+r^q)r^{q-1} - r^{q/2-1}r'^{2q}\,\text{arth} r^{q/2}}.$$

Let $f_3(r) = r^{2q-1}$, $f_4(r) = (1+r^q)r^{q-1} - r^{q/2-1}r'^{2q}\,\text{arth} r^{q/2}$. When $q \geqslant 2$, we have $f_3(0^+) = f_4(0^+) = 0$. Differentiation yields

$$f_3'(r) = (2q-1)r^{2q-2}$$

and

$$f_4'(r) = r^{2q-2}\left(\frac{5}{2}q - 1 - \left(\frac{q}{2} - 1\right)B(r) + 2qC(r)\right)$$

where

$$B(r) = r^{-3q/2}(1-r^q)^2\,\text{arth} r^{q/2} - r^{-q}$$

and

$$C(r) = (1-r^q)r^{-q/2}\,\text{arth} r^{q/2}.$$

Let $t = r^{q/2}$, then $B(r) = ((1-t^2)^2\,\text{arth} t - t)/t^3$. It is easy to check that

$$\lim_{r\to 0^+} B(r) = \lim_{t\to 0^+}\frac{(1-t^2)^2\,\text{arth} t - t}{t^3} = \lim_{t\to 0^+}\frac{(1-t^2)^2(t + t^3/3 + o(t^5)) - t}{t^3} = -\frac{5}{3}$$

and

$$\lim_{r\to 0^+} C(r) = 1.$$

Therefore, we obtain the limiting values

$$\lim_{r\to 0^+} f(r) = \lim_{r\to 0^+} \frac{f_1'(r)}{f_2'(r)} = \lim_{r\to 0^+} \frac{2}{q}\frac{f_3'(r)}{f_4'(r)} = \frac{3}{4q}$$

and

$$\lim_{r\to 1^-} f(r) = \lim_{r\to 1^-} \frac{f_1'(r)}{f_2'(r)} = \frac{1}{q}. \quad \square$$

REMARK 4.11. This Lemma corresponds to [19, Lemma 5 (1)] when $p = q = 2$.

THEOREM 4.12. *For $p > 1$ and $q \geqslant 2$, let*

$$C = \frac{1}{q}R\left(1-\frac{1}{p},\frac{1}{q}\right) \qquad \text{and} \qquad B = \frac{3(2-p-q+pq)}{2q(q(p-1)+p)}.$$

Then the function

$$g(r) = \frac{C-(\mu_{p,q}(r)+\log r)}{1-(r'^q\,\mathrm{arth} r^{q/2})/r^{q/2}}$$

is strictly increasing from $(0,1)$ onto (B,C). In particular, the double inequalities

$$C - C\sum_{n=1}^{\infty}\frac{2}{4n^2-1}r^{qn} < \mu_{p,q}(r)+\log r < C - B\sum_{n=1}^{\infty}\frac{2}{4n^2-1}r^{qn} \tag{4.13}$$

hold.

Proof. Let $g_1(r) = C-(\mu_{p,q}(r)+\log r)$, $g_2(r) = 1-(r'^q\,\mathrm{arth} r^{q/2})/r^{q/2}$, then we have $g_1(0^+) = g_2(0^+) = 0$. By differentiation,

$$g_1'(r) = \frac{1}{r}\left(\frac{\pi_{p,q}^2}{4r'^q\mathcal{K}_{p,q}^2}-1\right)$$

and

$$g_2'(r) = \frac{q}{2r}\left(\frac{(1+r^q)\,\mathrm{arth} r^{q/2}}{r^{q/2}}-1\right).$$

Then we have

$$\frac{g_1'(r)}{g_2'(r)} = \frac{2}{q}\frac{\pi_{p,q}^2/(4r'^q\mathcal{K}_{p,q}^2)-1}{\log(1/r')}\frac{\log(1/r')}{((1+r^q)\,\mathrm{arth} r^{q/2})/r^{q/2}-1},$$

which is increasing by Lemma 3.3(4) and Lemma 4.10. The monotonicity of $g(r)$ follows from Lemma 2.1. We can easily get the limiting values

$$\lim_{r\to 0^+} g(r) = \lim_{r\to 0^+} \frac{g_1'(r)}{g_2'(r)} = \frac{3(2-p-q+pq)}{2q(q(p-1)+p)} \quad \text{and} \quad \lim_{r\to 1^-} g(r) = C.$$

As

$$\begin{aligned} 1-\frac{(1-r^q)\operatorname{arth} r^{q/2}}{r^{q/2}} &= 1-\frac{1}{r^{q/2}}\left(\sum_{n=0}^{\infty}\frac{r^{qn+q/2}}{2n+1}-\sum_{n=0}^{\infty}\frac{r^{qn+q/2+q}}{2n+1}\right) \\ &= 1-\left(\sum_{n=0}^{\infty}\frac{r^{qn}}{2n+1}-\sum_{n=0}^{\infty}\frac{r^{qn+q}}{2n+1}\right) \\ &= 1-\left(1+\sum_{n=1}^{\infty}\frac{r^{qn}}{2n+1}-\sum_{n=1}^{\infty}\frac{r^{qn}}{2n-1}\right) \\ &= \sum_{n=1}^{\infty}\frac{2}{4n^2-1}r^{qn}, \end{aligned}$$

we have the desired inequalities (4.13). □

REMARK 4.14. Theorem 4.12 extends [19, Theorem 1] and improves the following inequalities

$$0 < \mu_{p,q}(r) + \log r < \frac{1}{q} R\left(1-\frac{1}{p},\frac{1}{q}\right).$$

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