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VECTORIAL AND METRICAL RELATIONS IN TETRAHEDRON

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Abstract. The scope of this paper is twofold. On the one hand we will study some new vector relations in triangle and tetrahedron and on the other hand we will proof this relations not just for vector case but also in some special coordinates, more precisely in Wachpress's coordinates. The investigated results from this paper are done in order to find some new metric relations in triangle and tetrahedron.

1. Introduction

Let ABC be a triangle. We will start our research from the vectorial relation:

$$\overrightarrow{AD} = \frac{1}{1+k}\left(\overrightarrow{AB} + k\overrightarrow{AC}\right),$$

where $D \in (BC)$ and $k = \frac{BD}{DC}$.

In addition, if $Q \in (AD$, then $k = \frac{\sigma[ABQ]}{\sigma[ACQ]}$, where we will demote by $\sigma[XYZ]$ the area of a given triangle XYZ. In [9], we can found the following vector relation:

$$\sigma[BNC]\overrightarrow{NA} + \sigma[ANC]\overrightarrow{NB} + \sigma[ANB]\overrightarrow{NC} = \overrightarrow{0}, \tag{1}$$

where N lies in the interior of the triangle ABC.

Another scope of this paper will be to prove in a tetrahedron a similar relation with the above relation, which will allowed us to extend certain vector relations from the triangle to tetrahedron. We will investigate some of the obtained results for vector case, in Wachpress coordinates which we will see that represent an useful tool in our research. The main goal of the paper is to provide some new interesting metric relations in triangle and tetrahedron using the scalar product of two vectors but also like we present above some recent tools developed in the frame coordinates geometry theory.

Finally, we extend the inequality Ionescu-Weitzenböck from triangle to tetrahedron and we find some inequalities for tetrahedron using the Crelle triangle.

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2. Vector relations in a triangle

PROPOSITION 1. *If M is a point in the plane of the triangle ABC and N is a point inside the triangle ABC, then the following relation holds:*

$$\sigma[BNC]\overrightarrow{MA}+\sigma[ANC]\overrightarrow{MB}+\sigma[ANB]\overrightarrow{MC}=\sigma[ABC]\overrightarrow{MN}. \tag{2}$$

Proof. Using the following vectorial relations: $\overrightarrow{MA}+\overrightarrow{AN}=\overrightarrow{MN}$; $\overrightarrow{MB}+\overrightarrow{BN}=\overrightarrow{MN}$; $\overrightarrow{MC}+\overrightarrow{CN}=\overrightarrow{MN}$ and multiplying these relations with $\sigma[BNC]$; $\sigma[ANC]$ and respectively with $\sigma[ANB]$ and summing up the obtained relations, one obtains:

$$\sigma[BNC]\overrightarrow{MA}+\sigma[ANC]\overrightarrow{MB}+\sigma[ANB]\overrightarrow{MC}+\sigma[BNC]\overrightarrow{AN}+\sigma[ANC]\overrightarrow{BN}+\sigma[ANB]\overrightarrow{CN}$$
$$=\sigma[ABC]\overrightarrow{MN}.$$

In conclusion, using (1), we get the desired result. □

REMARK 1. a) If $O(0,0)$ represents the origin of the cartesian system of coordinates xOy, then (2), becomes:

$$\sigma[BNC]\overrightarrow{r_A}+\sigma[ANC]\overrightarrow{r_B}+\sigma[ANB]\overrightarrow{r_C}=\sigma[ABC]\overrightarrow{r_N},$$

which means that, for $\overrightarrow{r_A}=x_A\vec{i}+y_A\vec{j}$; $\overrightarrow{r_B}=x_B\vec{i}+y_B\vec{j}$; $\overrightarrow{r_C}=x_C\vec{i}+y_C\vec{j}$, $\overrightarrow{r_N}=x_N\vec{i}+y_N\vec{j}$; we get the cartesian coordinates of the point N:

$$x_N=\frac{\sigma[BNC]x_A+\sigma[ANC]x_B+\sigma[ANB]x_C}{\sigma[ABC]},$$

$$y_N=\frac{\sigma[BNC]y_A+\sigma[ANC]y_B+\sigma[ANB]y_C}{\sigma[ABC]};$$

b) If z_A, z_B and respectively z_C are the affixes of the points A,B and respectively C, then the affixe of the point N, is:

$$z_N=\frac{\sigma[BNC]z_A+\sigma[ANC]z_B+\sigma[ANB]z_C}{\sigma[ABC]},$$

c) The absolute barycentric coordinates of the point N, are:

$$\left(\frac{\sigma[NBC]}{\sigma[ABC]},\frac{\sigma[NCA]}{\sigma[ABC]},\frac{\sigma[NAB]}{\sigma[ABC]}\right);$$

d) If $N\in(BC)$, then the following equality holds: $\sigma[NAC]\overrightarrow{NB}+\sigma[NAB]\overrightarrow{NC}=\overrightarrow{0}$ and if the point M lie inside the plane of the triangle ABC, then we can deduce the following relation: $\sigma[NAC]\overrightarrow{MB}+\sigma[NAB]\overrightarrow{MC}=\sigma[ABC]\overrightarrow{MN}$.

PROPOSITION 2. *If $N \in Int\sphericalangle BAC - \{Int\Delta ABC \cup (BC)\}$, then, we have:*

$$-\sigma[NBC]\overrightarrow{NA} + \sigma[NCA]\overrightarrow{NB} + \sigma[NAB]\overrightarrow{NC} = \overrightarrow{0}, \tag{3}$$

and

$$-\sigma[NBC]\overrightarrow{MA} + \sigma[NCA]\overrightarrow{MB} + \sigma[NAB]\overrightarrow{MC} = \sigma[ABC]\overrightarrow{MN}, \tag{4}$$

Proof. Let $N \in Int\sphericalangle BAC - \{Int\Delta ABC \cup (BC)\}$, (see Figure 1). If $\{N'\} = BC \cap AN$, then the following equality holds: $\frac{BN'}{N'C} = \frac{\sigma[NAB]}{\sigma[NAC]}$, which means that:

$$\overrightarrow{AN'} = \frac{\sigma[NAC]\overrightarrow{AB} + \sigma[NAB]\overrightarrow{AC}}{\sigma[NAB] + \sigma[NAC]}. \tag{5}$$

But, $\frac{AN'}{AN} = \frac{\sigma[ABC]}{\sigma[NAB]+\sigma[NAC]}$. So:

$$\overrightarrow{AN'} = \frac{\sigma[ABC]}{\sigma[NAB] + \sigma[NAC]}\overrightarrow{AN}. \tag{6}$$

From (5) and (6), we obtain:

$$\sigma[ABC]\overrightarrow{AN} = \sigma[NAC]\left(\overrightarrow{NB} - \overrightarrow{NA}\right) + \sigma[NAB]\left(\overrightarrow{NC} - \overrightarrow{NA}\right),$$

which means that:

$$\overrightarrow{0} = (\sigma[ABC] - \sigma[NAC] - \sigma[NAB])\overrightarrow{NA} + \sigma[NAC]\overrightarrow{NB} + \sigma[NAB]\overrightarrow{NC}.$$

So,

$$\overrightarrow{0} = -\sigma[NBC]\overrightarrow{NA} + \sigma[NAC]\overrightarrow{NB} + \sigma[NAB]\overrightarrow{NC}.$$

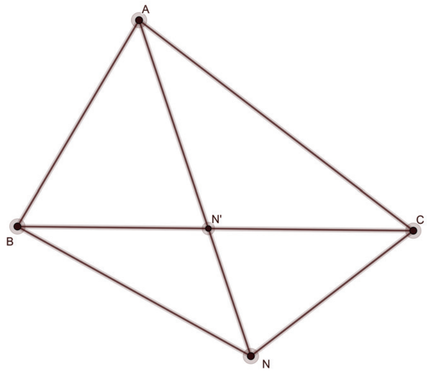

Figure 1.

Using (3) and also the fact that $\overrightarrow{MA} = \overrightarrow{MN} + \overrightarrow{NA}$; $\overrightarrow{MB} = \overrightarrow{MN} + \overrightarrow{NB}$; $\overrightarrow{MC} = \overrightarrow{MN} + \overrightarrow{NC}$, we get the relation (4). □

REMARK 2. a) If $N \equiv I_a$, where I_a represents the center of the ex-inscribed circle of the triangle ABC which correspond to the side BC of the triangle, then one obtains:

$$-\sigma[I_aBC]\overrightarrow{MA}+\sigma[I_aCA]\overrightarrow{MB}+\sigma[I_aAB]\overrightarrow{MC}=\sigma[ABC]\overrightarrow{MI_a}$$

which means:

$$-a\overrightarrow{MA}+b\overrightarrow{MB}+c\overrightarrow{MC}=(-a+b+c)\overrightarrow{MI_a}. \tag{7}$$

In a similar way, we can deduce the vectorial relations corresponding to the points I_b and respectivelly I_c, so we get:

$$\begin{aligned} a\overrightarrow{MA}+b\overrightarrow{MB}+c\overrightarrow{MC} &= (-a+b+c)\overrightarrow{MI_a}+(a-b+c)\overrightarrow{MI_b}+(a+b-c)\overrightarrow{MI_c} \\ &= (a+b+c)\overrightarrow{MI}, \end{aligned} \tag{8}$$

where I represents the center of the inscribed circle in the triangle ABC.

3. Vectorial relations in tetrahedron

In the following lines, we will continue to expand the previous vectorial relations obtined in a triangle, in a similar way, but this time for a tetrahedron.

LEMMA 1. *Let $ABCD$ be a tetrahedron and N a point inside the triangle BCD. Then, the following relation, holds:*

$$\overrightarrow{AN}=\frac{1}{\sigma[BCD]}\left(\sigma[NCD]\overrightarrow{AB}+\sigma[NBD]\overrightarrow{AC}+\sigma[NBC]\overrightarrow{AD}\right). \tag{9}$$

Proof. Let $BN \cap CD = \{M\}$, (see Figure 2).

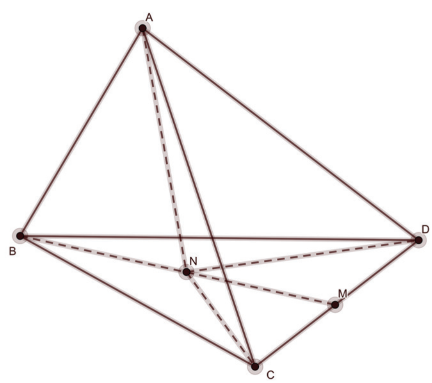

Figure 2.

We observe that $\frac{CM}{MD} = \frac{\sigma[NBC]}{\sigma[NBD]}$, which means that $\overrightarrow{AM} = \frac{\sigma[NBD]\overrightarrow{AC}+\sigma[NBC]\overrightarrow{AD}}{\sigma[NBD]+\sigma[NBC]}$. But, $\frac{NM}{BM} = \frac{\sigma[NCD]}{\sigma[BCD]}$, which means that: $\frac{NM}{NB} = \frac{\sigma[NCD]}{\sigma[NBC]+\sigma[NBD]} = k$, so:

$$\overrightarrow{AN} = \frac{1}{k+1}\left(\overrightarrow{AM} + k\overrightarrow{AB}\right) = \frac{1}{\sigma[BCD]}\left(\sigma[NCD]\overrightarrow{AB} + \sigma[NBD]\overrightarrow{AC} + \sigma[NBC]\overrightarrow{AD}\right).$$

This conclude the proof of the lemma. □

REMARK 3. Let h_A be the distance from point A to the plane (BCD). Multiplying in (9), h_A, one obtains:

$$\overrightarrow{AN} = \frac{1}{V[ABCD]}\left(V[NACD]\overrightarrow{AB} + V[NABD]\overrightarrow{AC} + V[NABC]\overrightarrow{AD}\right), \tag{10}$$

where $V[ABCD]$ is the volume of the tetrahedron $ABCD$.

THEOREM 1. *If P is an interior point of a tetrahedron $ABCD$, then the following relation, holds:*

$$V[PBCD]\overrightarrow{PA} + V[PACD]\overrightarrow{PB} + V[PABD]\overrightarrow{PC} + V[PABC]\overrightarrow{PD} = \overrightarrow{0} \tag{11}$$

Proof. Let $AP \cap (BCD) = \{N\}$ (see Figure 3).

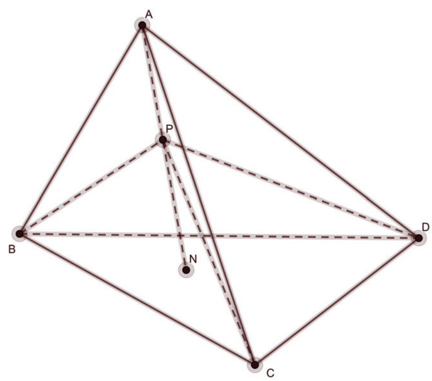

Figure 3.

We have the following equality:

$$\frac{PN}{AN} = \frac{V[PBCD]}{V[ABCD]},$$

which implies

$$\frac{PA}{AN} = \frac{V[PABC] + V[PACD] + V[PABD]}{V[ABCD]},$$

and this means that:

$$\overrightarrow{AP} = \frac{V[PABC]+V[PACD]+V[PABD]}{V[ABCD]}\overrightarrow{AN}. \tag{12}$$

Using the above relation (10), one obtains:

$$V[ABCD]\overrightarrow{AN} = V[NACD]\left(\overrightarrow{AP}+\overrightarrow{PB}\right)+V[NABD]\left(\overrightarrow{AP}+\overrightarrow{PC}\right) +V[NABC]\left(\overrightarrow{AP}+\overrightarrow{PD}\right),$$

which is equivalent with:

$$V[ABCD]\overrightarrow{AN} = V[ABCD]\overrightarrow{AP}+V[NACD]\overrightarrow{PB}+V[NABD]\overrightarrow{PC}+V[NABC]\overrightarrow{PD},$$

In conclusion,

$$V[ABCD]\overrightarrow{PN} = V[NACD]\overrightarrow{PB}+V[NABD]\overrightarrow{PC}+V[NABC]\overrightarrow{PD}.$$

Using (10), we get:

$$V[PBCD]\overrightarrow{PN} = V[NPCD]\overrightarrow{PB}+V[NPBD]\overrightarrow{PC}+V[NPBC]\overrightarrow{PD}.$$

If we make the difference between the above two relations, one obtains:

$$(V[ABCD]-V[PBCD])\overrightarrow{PN} = (V[NACD]-V[NPCD])\overrightarrow{PB}+(V[NABD]-V[NPBD])\overrightarrow{PC} +(V[NABC]-V[NPBC])\overrightarrow{PD},$$

which means that:

$$(V[PABC]+V[PACD]+V[PABD])\overrightarrow{PN} = (V[PACD])\overrightarrow{PB}+(V[PABD])\overrightarrow{PC}+(V[PABC])\overrightarrow{PD}.$$

But, we have

$$\frac{PN}{PA} = \frac{V[PBCD]}{V[PABC]+V[PACD]+V[PABD]},$$

so,

$$(V[PABC]+V[PACD]+V[PABD])\overrightarrow{PN} = (V[PBCD])\overrightarrow{AP}.$$

which completes the proof of our theorem. □

THEOREM 2. *If P is an interior point of a tetrahedron $ABCD$, and M is an arbitrary point in space, then the following relation holds:*

$$V[PBCD]\overrightarrow{MA}+V[PACD]\overrightarrow{MB}+V[PABD]\overrightarrow{MC}+V[PABC]\overrightarrow{MD} = V[ABCD]\overrightarrow{MP}. \tag{13}$$

Proof. Using (11), we get:

$$V[PBCD](\overrightarrow{PM}+\overrightarrow{MA}) + V[PACD](\overrightarrow{PM}+\overrightarrow{MB}) + V[PABD](\overrightarrow{PM}+\overrightarrow{MC}) + V[PABC](\overrightarrow{PM}+\overrightarrow{MD}) = \overrightarrow{0}$$

which concludes the proof of the theorem. □

REMARK 4. If $O(0,0,0)$ is the origin of the cartesian system $Oxyz$, then (13) becomes: $V[PBCD]\overrightarrow{r_A}+V[PACD]\overrightarrow{r_B}+V[PABD]\overrightarrow{r_C}+V[PABC]\overrightarrow{r_D}=V[ABCD]\overrightarrow{r_P}$, which means that, for $\overrightarrow{r_A}=\overrightarrow{OA}=x_A\overrightarrow{i}+y_A\overrightarrow{j}+z_A\overrightarrow{k}$, $\overrightarrow{OB}=x_B\overrightarrow{i}+y_B\overrightarrow{j}+z_B\overrightarrow{k}$, $\overrightarrow{OC}=x_C\overrightarrow{i}+y_C\overrightarrow{j}+z_C\overrightarrow{k}$ and respectively $\overrightarrow{OD}=x_D\overrightarrow{i}+y_D\overrightarrow{j}+z_D\overrightarrow{k}$, we obtain the cartesian coordinates of the point P and these coordinates are the following ones:

$$x_P=\frac{V[PBCD]x_A+V[PACD]x_B+V[PABD]x_C+V[PABC]x_D}{V[ABCD]};$$

$$y_P=\frac{V[PBCD]y_A+V[PACD]y_B+V[PABD]y_C+V[PABC]y_D}{V[ABCD]};$$

$$z_P=\frac{V[PBCD]z_A+V[PACD]z_B+V[PABD]z_C+V[PABC]z_D}{V[ABCD]}.$$

Next, we will use the following notations: $V[PBCD]=V_A$; $V[PACD]=V_B$; $V[PABD]=V_C$, $V[PABC]=V_D$ and respectively, $V[ABCD]=V$.

THEOREM 3. *If P is an interior point of a tetrahedron $ABCD$, and M is an arbitrary point in space, then the following relation holds:*

$$V\left(V_A\cdot MA^2+V_B\cdot MB^2+V_C\cdot MC^2+V_D\cdot MD^2\right) - (V_AV_BAB^2+V_AV_CAC^2+V_AV_DAD^2+V_BV_CBC^2+V_BV_DBD^2+V_CV_DCD^2) = V^2MP^2. \tag{14}$$

Proof. Using (13) and the scalar product of two vectors, one obtains:

$$V^2\cdot MP^2 = V_A^2MA^2+V_B^2MB^2+V_C^2MC^2+V_D^2MD^2+2V_AV_B\overrightarrow{MA}\overrightarrow{MB}+2V_AV_C\overrightarrow{MA}\overrightarrow{MC} +2V_AV_D\overrightarrow{MA}\overrightarrow{MD}+2V_BV_C\overrightarrow{MB}\overrightarrow{MC}+2V_BV_D\overrightarrow{MB}\overrightarrow{MD}+2V_CV_D\overrightarrow{MC}\overrightarrow{MD}.$$

But, replacing in the above relations $2\overrightarrow{MA}\overrightarrow{MB}=MA^2+MB^2-AB^2$, etc., we obtain the relation:

$$V^2\cdot MP^2 = V\left(V_AMA^2+V_BMB^2+V_CMC^2+V_DMD^2\right) - \left(V_AV_BAB^2+V_AV_CAC^2+V_AV_DAD^2+V_BV_CBC^2+V_BV_DBD^2+V_CV_DCD^2\right)$$

In conclusion, the theorem is proved. □

4. Proofs in Wachpress's coordinates

In the following lines using the Wachpress's coordinates, we will give a proof to some of the previous results. First of all let us recall some aspects regarding the Wachpress's coordinates in respect with triangle areas as are presented in the paper [8]. In that paper, Meyer et all., found an easy way to introduce the Wachpress's coordinates using the triangle areas:

$$A_i = A_i(x) = A(x, v_i, v_{i+1}); \quad C_i = A(v_{i-1}, v_i, v_{i+1}), \tag{15}$$

as follows:

$$\phi_i(x) = \frac{w_i(x)}{\sum_{j=1}^{n} w_j(x)}$$

where:

$$w_i(x) = \frac{C_i}{A_{i-1}(x)A_i(x)}.$$

The above introduced coordinates by Meyer et all. are barycentric because it can be proved easily the following equality:

$$\sum_{i=1}^{n} w_i(x)(v_i - x) = 0.$$

Also it can be observed that any points $x \in P$, can be expressed in barycentric coordinates as follows:

$$x = \frac{A_i}{C_i} v_{i-1} + \frac{C_i - A_{i-1} - A_i}{C_i} v_i + \frac{A_{i-1}}{C_i} v_{i+1},$$

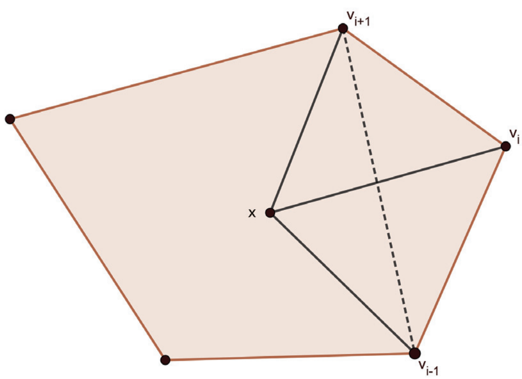

Figure 4.

As we know from Proposition 2.1, we proved there the following equality:

$$\sigma[BNC]\overrightarrow{MA} + \sigma[ANC]\overrightarrow{MB} + \sigma[ANB]\overrightarrow{MC} = \sigma[ABC]\overrightarrow{MN}.$$

In the following lines we will prove the same proposition but this time in Wachpress's coordinates: We have there 3 vertices A,B,C and also two points M and N. We will associate to this points some notations for compatibility in Wachpress's coordinates, as follows: $A \to v_0$; $B \to v_1$, $C \to v_2$, $N \to x$, $M \to y$. The area of the triangle ABC, acording to the Wachpress's coordinates defined above, will be denoted by $C_1 = \sigma[ABC] = A(v_0, v_1, v_2)$. Also, the other triangle areas will be: $A_1 = \sigma[ANB]$; $A_2 = \sigma[BNC]$; $A_3 = \sigma[ANC]$. All these notations can be seen in the following figure:

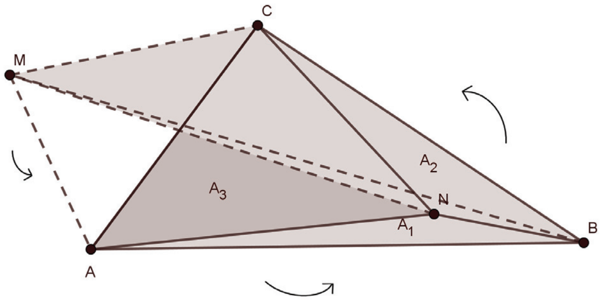

Figure 5.

Using the anticlockwise rule, with the above notations, in Wachpress's coordinates, we have:

$$C_1 y = (A_1 + A_2 + A_3)\, y$$

and respectively:

$$C_1 x = A_2 v_0 + A_3 v_1 + A_1 v_2$$

Subtracting the above 2 equalities, one obtains:

$$C_1(y-x) = A_1(y-v_2) + A_2(y-v_0) + A_3(y-v_1)$$

and this implies:

$$A_2(y-v_0) + A_3(y-v_1) + A_1(y-v_2) = C_1(y-x)$$

Finally, let us observe that the above relation is just in fact equivalent with (2):

$$\sigma[BNC]\,\overrightarrow{MA} + \sigma[ANC]\,\overrightarrow{MB} + \sigma[ANB]\,\overrightarrow{MC} = \sigma[ABC]\,\overrightarrow{MN}.$$

So, the Proposition 2.1 have been proved using the Wachpress's coordinates. In an analogous way we can now give a proof to the Proposition 2.2 but this time for Wachpress's coordinates. In the statement of Proposition 2.2, we can remark that the point N is outside the triangle ABC, so the direct counterclockwise orientation will be changed. We will need first to construct the symmetric of the point N which we will denote by N", in respect with the edge BC. The triangle BN"C is directly orientated but the triangle BNC is not directly orientated because the point N lies outside the triangle ABC. The situation is presented in the following figure:

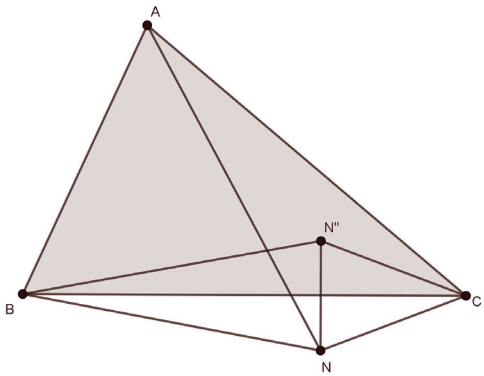

Figure 6.

In analogous way as for Proposition 2.1, we have:

$$C_1 y = (A_3 + A_2 - A_1)\, y$$

and respectively:

$$C_1 x = A_2 v_0 + A_3 v_1 + A_1 v_2$$

Substracting the above two equalities, one obtains:

$$A_2(y - v_0) + A_3(y - v_1) - A_1(y + v_2) = C_1(y - x)$$

and this conclude the proof of Propisition 2 in Wachpress's coordinates because this equality is equivalent with (4):

$$-\sigma\,[BNC]\,\overrightarrow{MA} + \sigma\,[ANC]\,\overrightarrow{MB} + \sigma\,[ANB]\,\overrightarrow{MC} = \sigma\,[ABC]\,\overrightarrow{MN}.$$

REMARK 5. In the following lines we will use Wachpress's coordinate in the three dimensional case, following the works of Wachpress [13] and respectivelly Warren [14].

So, first let us recall some results from the above mentioned works [13], [14]. In this respect, let us consider a simple polyhedron $P \subset \mathbb{R}^3$ with F faces and V vertices. For any face $f \in F$ of the polyhedron, let $n_f \in \mathbb{R}^3$ denote its unit outward normal and with h_f is denoted the perpendicular distance from x to f. This distance can be written as Floater remarked in [3]:

$$h_f(x) = (v - x)n_f,$$

for any vertex $v \in V$ from the face f. Also we will follow close the approach of the paper [3]. There are considered for each vertex $v \in V$, the three faces f_1, f_2, f_3 incident with the vertex v. For $x \in P$, let

$$w_v(x) = \frac{det\left(n_{f_1}, n_{f_2}, n_{f_3}\right)}{h_{f_1}(x) h_{f_2}(x) h_{f_3}(x)}.$$

Here, the faces f_1, f_2, f_3 are oriented such that the determinant in the numerator is positive. Also from paper [3], we know that for the polyhedron $P \in \mathbb{R}^3$, there exist its dual $\tilde{P}_x = \{y \in \mathbb{R}^3 | y(z-x) \leqslant 1, z \in P\}$, which contains the origin $y = 0$ and its vertices are the endpoints of the vectors: $p_f(x) = \frac{n_f}{h_f(x)}$, $f \in F$. Here the faces $f_1, f_2, \ldots, f_k$ are incident with the vertex v for some $k \geqslant 3$. Also from paper [3], we know that the volumes of the tetrahedrons associated with a polyhedron P, can be computed as follows:

$$w_i(x) = det\left(p_{f_i}(x), p_{f_{i+1}}(x), p_{f_k}(x)\right).$$

In this respect, the volume of the polyhedron will be: $Vol(Q_V) = w_v(x) = \sum_{i=1}^{k-2} w_{i,v}(x)$. Now, we will use all this results regarding the Wachpress's coordinates in the three dimensional case and we will proof in the following lines the equality (11), but this time using this type of coordinates.

First of all, let us denote the vertices A, B, C, D in Wachpress's coordinates with v_0, v_1, v_2, v_3. The faces of the tetrahedron from (11), will be denoted in Wachpress's coordinates by $[PBC] \to f_1$, $[PCD] \to f_2$, $[PBD] \to f_3$, $[PAB] \to f_4$, $[PAD] \to f_5$, $[PAC] \to f_6$.

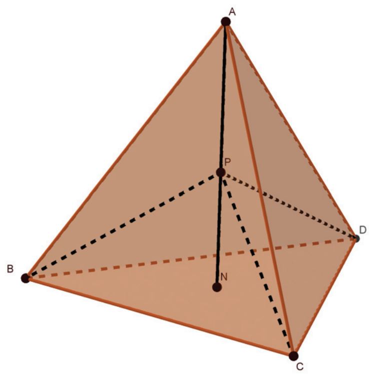

Figure 7.

Next, we will express the volumes of the tetrahedrons from (11), as follows:

$$V[PBCD] = \frac{det(n_{f_1}, n_{f_2}, n_{f_3})}{h_{f_1}(x) h_{f_2}(x) h_{f_3}(x)}$$

and $\overrightarrow{PA} \to v_0 - y$.

$$V[PACD] = \frac{det(n_{f_2}, n_{f_5}, n_{f_6})}{h_{f_2}(x) h_{f_5}(x) h_{f_6}(x)}$$

and $\overrightarrow{PB} \to v_1 - y$.

$$V[PABD] = \frac{det(n_{f_3}, n_{f_4}, n_{f_5})}{h_{f_3}(x) h_{f_4}(x) h_{f_5}(x)}$$

and $\overrightarrow{PC} \to v_2 - y$

$$V[PABC] = \frac{det(n_{f_1}, n_{f_4}, n_{f_6})}{h_{f_1}(x)h_{f_4}(x)h_{f_6}(x)}$$

and $\overrightarrow{PD} \to v_3 - y$. Next, we will take into account that $h_f(x)$ is the distance from x to the face $f \in F$, which is given by:

$$h_f(x) = (v - x)n_f.$$

Using the orthogonality of the vectors in the following equality, (after we bring to the same denominator the terms of the equality),

$$\frac{det(n_{f_1}, n_{f_2}, n_{f_3})}{h_{f_1}(x)h_{f_2}(x)h_{f_3}(x)}(v_0 - y) + \frac{det(n_{f_2}, n_{f_5}, n_{f_6})}{h_{f_2}(x)h_{f_5}(x)h_{f_6}(x)}(v_1 - y)$$

$$+\frac{det(n_{f_3}, n_{f_4}, n_{f_5})}{h_{f_3}(x)h_{f_4}(x)h_{f_5}(x)}(v_2 - y) + \frac{det(n_{f_1}, n_{f_4}, n_{f_6})}{h_{f_1}(x)h_{f_4}(x)h_{f_6}(x)}(v_3 - y) = 0,$$

we get the proof of (11), but this time using the Wachpress's coordinates in three dimensional case.

5. Some vectorial and metrical relations in tetrahedron

Let $ABCD$ be a tetrahedron with the volume V and the surface area S. Let $a = BC, b = CA, c = AB, a' = AD, b' = BD, c' = CD$ be the edge lengths of the thetrahedron $ABCD$. Let us make the following notations: $\sigma[BCD] = S_A$, $\sigma[ACD] = S_B$, $\sigma[ABD] = S_C$, $\sigma[ABC] = S_D$, R is the radius of the circumscribed sphere and r represents the radius of the inscribed sphere. It is easy to see that $S = S_A + S_B + S_C + S_D$ is the surface area.

EXAMPLE 1. If $P \equiv G$, which is the centroid of the tetrahedron $ABCD$, then $V_A = V_B = V_C = V_D = \frac{1}{4}V$ and the following relations holds:

$$\overrightarrow{GA} + \overrightarrow{GB} + \overrightarrow{GC} + \overrightarrow{GD} = \overrightarrow{0}, \tag{16}$$

$$\overrightarrow{MA} + \overrightarrow{MB} + \overrightarrow{MC} + \overrightarrow{MD} = 4\overrightarrow{MG}, \tag{17}$$

$$MA^2 + MB^2 + MC^2 + MD^2 = 4MG^2 + \frac{1}{4}\left(AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2\right). \tag{18}$$

If, in equality (18), we choose $M \equiv O$, which is the center of the circumscribed sphere over the tetrahedron $ABCD$, then, we have the relation:

$$\begin{aligned} OG^2 &= R^2 - \frac{1}{16}\left(AB^2 + AC^2 + AD^2 + BC^2 + BD^2 + CD^2\right) \\ &= R^2 - \frac{1}{16}\left(a^2 + b^2 + c^2 + a'^2 + b'^2 + c'^2\right). \end{aligned} \tag{19}$$

EXAMPLE 2. Let $P \equiv I$, the center of the inscribed sphere in the tetrahedron $ABCD$. Then, we have the following relations:

$$V_A = \frac{1}{3}S_A r;\ \ V_B = \frac{1}{3}S_B r;\ \ V_C = \frac{1}{3}S_C r;\ \ V_D = \frac{1}{3}S_D r;$$

$$V = \frac{1}{3}\left(S_A + S_B + S_C + S_D\right) r = \frac{1}{3}Sr.$$

The following equalities holds:

$$S_A \cdot \overrightarrow{IA} + S_B \cdot \overrightarrow{IB} + S_C \cdot \overrightarrow{IC} + S_D \cdot \overrightarrow{ID} = \overrightarrow{0}, \tag{20}$$

$$S_A \cdot \overrightarrow{MA} + S_B \cdot \overrightarrow{MB} + S_C \cdot \overrightarrow{MC} + S_D \cdot \overrightarrow{MD} = \left(S_A + S_B + S_C + S_D\right)\overrightarrow{MI} = S \cdot \overrightarrow{MI}. \tag{21}$$

If we replace in (14), then we obtain:

$$S_A MA^2 + S_B MB^2 + S_C MC^2 + S_D MD^2$$

$$= S \cdot MI^2 + \frac{1}{S}\left(S_A S_B AB^2 + S_A S_C AC^2 + S_A S_D AD^2 + S_B S_C BC^2 + S_B S_D BD^2 + S_C S_D CD^2\right). \tag{22}$$

If in (22), we choose $M \equiv O$, the center of the circumscribed sphere over the tetrahedron $ABCD$, then the following relation holds:

$$OI^2 = R^2 - \frac{S_A S_B AB^2 + S_A S_C AC^2 + S_A S_D AD^2 + S_B S_C BC^2 + S_B S_D BD^2 + S_C S_D CD^2}{S^2}. \tag{23}$$

Next, let us recall some results from [10]. We will consider the tetrahedron $[ABCD]$ and the circumscribed sphere for this tetrahedron with radius R and center O, will be denoted by $S(O,R)$. The inscribed sphere in this tetrahedron will be denoted by $S(I,r)$. The half-lines $(AI, (BI, (CI, (DI$ have intersections with the sphere $S(O,R)$ in the points A', B', C', D'. We know that the following equalities holds:

$$IA \cdot IA' = IB \cdot IB' = IC \cdot IC' = ID \cdot ID' = R^2 - OI^2.$$

Using the inversion of pole I and power $OI^2 - R^2$, the plane (ABC) is transformed in the sphere $S(O_1, R_1) - \{I\}$, which represent the sphere circumscribed to the tetrahedron $[IA'B'C']$. Let E be the projection of the point I to the plane (ABC) and with E' will be denote the diametral opposite point to E, with respect to the sphere $S(O_1, R_1)$. From $IE \cdot IE' = R^2 - OI^2$, it follows:

$$2rR_1 = R^2 - OI^2 \tag{24}$$

Using this result, established in [10], we are ready to give now the following theorem:

THEOREM 4. *In any tetrahedron $[ABCD]$, with the above notations, the following equality holds:*

$$S_A S_B AB^2 + S_A S_C AC^2 + S_A S_D AD^2 + S_B S_C BC^2 + S_B S_D BD^2 + S_C S_D CD^2 = 2rR_1 S^2. \tag{25}$$

Proof. The proof can be obtained directly using (23). □

6. Some inequalities in tetrahedron

If $N \equiv K$, the Lemoine point of the triangle ABC and using relation (2), then we have:

$$a^2\overrightarrow{MA} + b^2\overrightarrow{MB} + c^2\overrightarrow{MC} = (a^2+b^2+c^2)\overrightarrow{MK},$$

where M represent a point in the plane of the triangle ABC.

Using the scalar product, we deduce the following relation:

$$a^2MA^2 + b^2MB^2 + c^2MC^2 = (a^2+b^2+c^2)MK^2 + \frac{3a^2b^2c^2}{a^2+b^2+c^2}. \tag{26}$$

If $M \equiv O$, the center of the circumscribed circle of the triangle ABC, then

$$R^2(a^2+b^2+c^2) = ((a^2+b^2+c^2))OK^2 + \frac{3a^2b^2c^2}{a^2+b^2+c^2},$$

which means that

$$R^2 - \frac{48\Delta^2R^2}{(a^2+b^2+c^2)^2} = OK^2,$$

where Δ is the area of the triangle ABC, which is equivalent with

$$OK^2 = \frac{R^2}{(a^2+b^2+c^2)^2}\left((a^2+b^2+c^2)^2 - 48\Delta^2\right). \tag{27}$$

But, $OK^2 \geqslant 0$, so we deduce:

$$(a^2+b^2+c^2)^2 \geqslant 48\Delta^2,$$

which means that:

$$a^2+b^2+c^2 \geqslant 4\sqrt{3}\Delta.$$

This inequality represents the Ionescu-Weitzenböck inequality (I-W). A refinement of the inequality (I-W) is the Finsler-Hadwiger inequality (F-H), given by:

$$a^2+b^2+c^2 \geqslant Q(a,b,c) + 4\sqrt{3}\Delta,$$

where $Q(a,b,c) = (a-b)^2 + (b-c)^2 + (c-a)^2$. In [4], Kouba presented several refinements of the Finsler-Hadwiger inequality. Andrica and Marinescu in [1] obtained refinements to some famous geometric inequalities in a triangle by constructing interpolating sequences. In [6] the inequality (F-H) is used to prove some algebraic inequalities.

We want to give a similar inequality for the tetrahedron.

Let $a = BC$, $b = CA$, $c = AB$, $a' = AD$, $b' = BD$, $c' = CD$ be the edge lengths of the tetrahedron $ABCD$ with the volume V and the surface area S. Let us make the following notations: R is the radius of the circumscribed sphere and r represents the radius of the inscribed sphere.

THEOREM 5. *In any tetrahedron ABCD, the following inequality holds:*

$$a^2+b^2+c^2+a'^2+b'^2+c'^2 \geqslant 12\sqrt[3]{9V^2}+\frac{1}{3}Q(a,b,c,a',b',c') + \frac{3}{4}\sum(m_A^2-h_A^2), \tag{28}$$

where m_A is the median from A, h_A represent the length of the altitude from A and

$$\begin{aligned} Q(a,b,c,a',b',c') = &(a-b)^2+(a-c)^2+(b-c)^2+(a-b')^2+(a-c')^2 \\ &+(b-a')^2+(b-c')^2+(c-a')^2+(c-b')^2 \\ &+(a'-b')^2+(b'-c')^2+(c'-a')^2. \end{aligned}$$

Proof. Let G_A be the centroid of the triangle BCD (see fig. 8)

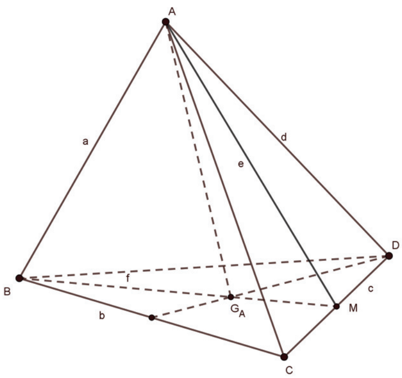

Figure 8.

Because $\frac{BG_A}{G_AM}=2$, then we have $\overrightarrow{AG_A}=\frac{1}{3}\left(\overrightarrow{AB}+2\overrightarrow{AM}\right)$, which implies the equality

$$9AG_A^2 = AB^2+4AM^2+4\overrightarrow{AB}\cdot\overrightarrow{AM} = AB^2+4AM^2+2\left(AB^2+AM^2-BM^2\right).$$

But, using the median theorem, we proved the following:

$$\begin{aligned} 9AG_A^2 &= 3AB^2+6\frac{2(AC^2+AD^2)-CD^2}{4}-\frac{2(BC^2+BD^2)-CD^2}{2} \\ &= 3\left(AB^2+AC^2+AD^2\right)-\left(BC^2+BD^2+CD^2\right), \end{aligned}$$

so we can deduce:

$$m_A^2 = AG_A^2 = \frac{1}{9}\left[3(a'^2+b^2+c^2)-(a^2+b'^2+c'^2)\right]$$

Starting with the first term of the inequality from the above statement, we have:

$$\begin{aligned}
& a^2+b^2+c^2+a'^2+b'^2+c'^2 \\
= \; & \frac{1}{3}\left[3(a'^2+b^2+c^2)-(a^2+b'^2+c'^2)\right]+\frac{a^2+b'^2+c'^2}{3}+a^2+b'^2+c'^2 \\
= \; & 3m_A^2+\frac{4}{3}(a^2+b'^2+c'^2) \\
\underset{(F-H)}{\geqslant} \; & 3m_A^2+\frac{16\sqrt{3}}{3}S_A+\frac{4}{3}\{(a-b')^2+(b'-c')^2+(c'-a)^2\} \\
= \; & 3h_A^2+\frac{16\sqrt{3}}{3}S_A+\frac{4}{3}\{(a-b')^2+(b'-c')^2+(c'-a)^2\}+3(m_A^2-h_A^2),
\end{aligned}$$

where S_A is the area of the triangle BCD.

Using the inequality between the geometric mean and the arithmetic mean, we have:

$$\begin{aligned}
3h_A^2+\frac{16\sqrt{3}}{3}S_A &= 3h_A^2+\frac{8\sqrt{3}}{3}S_A+\frac{8\sqrt{3}}{3}S_A \geqslant 3\sqrt[3]{3h_A^2\left(\frac{8\sqrt{3}}{3}\right)^2 S_A^2} \\
&= 12\sqrt[3]{(h_AS_A)^2}=12\sqrt[3]{9V^2}.
\end{aligned}$$

Therefore, we have

$$\begin{aligned}
a^2+b^2+c^2+a'^2+b'^2+c'^2 \geqslant 12\sqrt[3]{9V^2}+\frac{4}{3}\{(a-b')^2+(b'-c')^2+(c'-a)^2\} \quad (29) \\
+3(m_A^2-h_A^2). \quad (29)
\end{aligned}$$

We apply the same method for faces (ABC), (ACD), (ABD) and by adding the four inequalities similar to inequality (29), we obtain the inequality from the statement. □

REMARK 6. 1) Because $\sum(m_A^2-h_A^2)\geqslant 0$, we found the inequality given by Chen and Ma (see [17]):

$$a^2+b^2+c^2+a'^2+b'^2+c'^2 \geqslant 12\sqrt[3]{9V^2}+\frac{1}{3}Q(a,b,c,a',b',c'), \tag{30}$$

which is an inequality of Finsler-Hadwiger type for tetrahedron. This proved the inequality of Ionescu-Weitzenböck type for tetrahedron:

$$a^2+b^2+c^2+a'^2+b'^2+c'^2 \geqslant 12\sqrt[3]{9V^2}. \tag{31}$$

2) If the tetrahedron $ABCD$ is orthocentric (a tetrahedron where all three pairs of opposite edges are perpendicular), then we have:

$$a^2+a'^2=b^2+b'^2=c^2+c'^2,$$

so, the above inequality becomes:

$$a^2+a'^2 \geqslant 4\sqrt[3]{9V^2}.$$

3) From equality (19), we deduce the inequlity:

$$R^2 \geqslant \frac{1}{16}\left(a^2+b^2+c^2+a'^2+b'^2+c'^2\right) \tag{32}$$

Combining inequalities (31) and (32), we obtain:

$$R^2 \geqslant \frac{3}{4}\sqrt[3]{9V^2},$$

which is equivalent to

$$R \geqslant \frac{1}{2}\sqrt[3]{9\sqrt{3}V}. \tag{33}$$

Let S be the surface area of the tetrahedron. If we apply the Ionescu-Weitzenböck (I-W) inequality for each face of the tetrahedron, we deduce the inequality:

$$a^2+b^2+c^2+a'^2+b'^2+c'^2 \geqslant 2\sqrt{3}S. \tag{34}$$

which, using the inequality (29) prove the following inequality:

$$R^2 \geqslant \frac{\sqrt{3}}{8}S.$$

Equality holds if the tetrahedron is regular. Using Grace-Danielsson's inequality [12]

$$OI^2+4r^2 \leqslant (R-r)^2$$

and inequality also inequality (19), we can deduce now the following inequality:

$$S_AS_BAB^2+S_AS_CAC^2+S_AS_DAD^2+S_BS_CBC^2+S_BS_DBD^2+S_CS_DCD^2 \geqslant r(2R+3r)S^2. \tag{35}$$

Next, we will present some new results starting from the very well known inequality (F-H).

THEOREM 6. *In any triangle ABC, with usual notations, the following inequality holds:*

$$\frac{a^3+b^3+c^3}{s}+6Rr \geqslant Q(a,b,c)+4\sqrt{3}\Delta. \tag{36}$$

Proof. Starting with the Finsler-Hadwiger inequality:

$$a^2+b^2+c^2 \geqslant Q(a,b,c)+4\sqrt{3}\Delta.$$

Multiplying with $a+b+c=2s$, the both terms of inequality, one obtains succesivelly:

$$a^3+b^3+c^3+a^2(b+c)+b^2(a+c)+c^2(a+b) \geqslant (Q(a,b,c)+4\sqrt{3}\Delta)2s.$$

Now, using the Schur inequality, we have

$$2\left(a^3+b^3+c^3\right)+3abc \geqslant a^3+b^3+c^3+a^2(b+c)+b^2(a+c)+c^2(a+b)$$

and replacing in the above inequality, we get:

$$2\left(a^3+b^3+c^3\right)+3abc \geqslant (Q(a,b,c)+4\sqrt{3}\Delta)2s$$

and from this, we get the conclusion of the theorem. □

In [11], we found the following equality:

$$a^2+b^2+c^2 = Q(a,b,c)+4\Delta\sum\tan\frac{A}{2}. \tag{37}$$

But, by calculation we have

$$\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\frac{4R+r}{s},$$

where in the triangle ABC, we have a,b,c the side lengths, s the semiperimeter, R the circumradius, r the inradius. Therefore, we deduce the equality

$$a^2+b^2+c^2 = Q(a,b,c)+4\Delta\frac{4R+r}{s},$$

which is equivalent to

$$\frac{a^2+b^2+c^2-Q(a,b,c)}{4\Delta}=\frac{4R+r}{s}. \tag{38}$$

We remark that the Finsler-Hadwiger inequality is equivalent to the geometric inequality

$$4R+r \geqslant s\sqrt{3}. \tag{39}$$

Using the Garfunkel-Bankhoff inequality (see [16]),

$$\sum\tan^2\frac{A}{2} \geqslant 2-8\prod\sin\frac{A}{2},$$

we have

$$\left(\sum\tan\frac{A}{2}\right)^2 = 2+\sum\tan^2\frac{A}{2} \geqslant 4-8\prod\sin\frac{A}{2}.$$

But, it is known that $\prod\sin\frac{A}{2}=\frac{r}{4R}$. Consequently, we deduce the following inequality:

$$\sum\tan\frac{A}{2} \geqslant \sqrt{4-\frac{2r}{R}},$$

and using relation (37), we obtain

$$\frac{a^2+b^2+c^2-Q(a,b,c)}{4\Delta} \geqslant \sqrt{4-\frac{2r}{R}}, \tag{40}$$

which is an inequality given by Kouba in [4]. This inequality proved an inequality of Lupu and Pohoaţă [5]:

$$\frac{a^2+b^2+c^2-Q(a,b,c)}{4\Delta} \geqslant \sqrt{3+\frac{4(R-2r)}{4R+r}}, \tag{41}$$

because $4-\frac{2r}{R}\geqslant 3+\frac{4(R-2r)}{4R+r}$.

In [2], Crelle gave the following Theorem: let V and R be the volume and the circumradius of a tetrehedron $ABCD$ respectively. Then the quantities aa', bb', cc' are side-lengths of a triangle whose area Δ' is given by the formula $\Delta'=6RV$. This triangle is called *Crelle triangle* associated to the tetrahedron $ABCD$. In the Crelle triangle, if we have s' the semiperimeter, R' the circumradius and r' the inradius, then we obtain $s'=\frac{1}{2}(aa'+bb'+cc')$, $R'=\frac{aa'bb'cc'}{24RV}$ and $r'=\frac{12RV}{aa'+bb'+cc'}$. Consequently, inequality (39) becomes:

$$abca'b'c'(aa'+bb'+cc')+288R^2V^2\geqslant 12\sqrt{3}RV(aa'+bb'+cc')^2. \tag{41}$$

If we apply inequality (36) in the Crelle triangle, the we obtain the following inequality:

$$2\left((aa')^3+(bb')^3+(cc')^3\right)+3abca'b'c'\geqslant 24\sqrt{3}RV(aa'+bb'+cc'). \tag{42}$$

From inequality (40), we deduce

$$\frac{(aa')^2+(bb')^2+(cc')^2-Q(aa',bb',cc')}{48RV}\geqslant\sqrt{4-\frac{(24RV)^2}{abca'b'c'(aa'+bb'+cc')}}. \tag{43}$$

If we apply in the Crelle triangle a series of inequalities that are true in a triangle, then we obtain a series of inequalities valid in a tetrahedron.

7. Conclusion

In this paper we have investigated some new metric relations in triangle and tetrahedron using two different approach. On the one hand the classical vector theory proofs and on the other hand we have proved the same results using the Wachpress's coordinates. The results obtained in this paper, reflect that even we used two different approaches we can get the validity of the theorems presented in this paper. In the last section of the paper we present also some interesting results regarding some applications in inequalities theory for tetrahedron using the previous results obtained in the paper.

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