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APPROXIMATION PROPERTIES OF GENERALIZED BLENDING TYPE LOTOTSKY–BERNSTEIN OPERATORS

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Abstract. In this paper, we introduce a family of blending type Bernstein operators $L_n^{\alpha,s}(f;x)$ which depends on two parameters, α and s. We prove a Korovkin type approximation theorem and obtain the rate of convergence of these operators. We also prove that these operators has monotonicity and convexity preserving properties for each α and s. So far, Lotosky matrices that generates blending type Bernstein operators were ignored. In this paper, we also introduce Lototsky matrices that generates these new family of blending type Bernstein operators.

1. Introduction

The classical Bernstein operators

$$B_n(f;x)=\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f\left(\frac{k}{n}\right), \tag{1}$$

where $f(x)$ is a continuous function on $[0,1]$, is one of the most important positive linear operators and have been investigated by different researchers in different directions (see [1, 3], [7, 10], [13], [14]). Moreover, a large number of generalizations of Bernstein polynomials and their properties are also considered by researchers. Very recently in [17], Srivastava et.al. introduced Stancu-type Bernstein operators based on Bĕzier bases and investigate their approximation properties.

It is well known that the Bernstein operators can be written in the form of

$$B_n(f,x)=\sum_{k=0}^{n}a_{nk}(x)f\left(\frac{k}{n}\right),$$

where $a_{nk}(x)$ is the Lototsky matrix obtained from the following equation

$$\prod_{k=1}^{n}(yx+1-x)=\sum_{k=0}^{n}a_{nk}(x)y^k. \tag{2}$$

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Paper JMI-16-50

In other words $a_{nk}(x)$ obtained from (2) is $\binom{n}{k}x^k(1-x)^{n-k}$.

For the more general case, the Lototsky matrix $a_{nk}(x)$ obtained from

$$\prod_{k=1}^{n}(yh(x)+1-h(x))=\sum_{k=0}^{n}a_{nk}(x)y^k,$$

where $0\leqslant h(x)\leqslant 1$, is used to define the operator

$$L_n(f,x)=\sum_{k=0}^{n}a_{nk}(x)f\left(\frac{k}{n}\right),$$

which is called Lototsky-Bernstein operators. Obviously, $L_n(f,x)$ is an extension of $B_n(f,x)$.

Recently, for a fixed real number $0\leqslant\alpha\leqslant 1$, the following blending type α-Bernstein operators are introduced and studied by Chen et al. [10],

$$B_{n,\alpha}(f;x)=\sum_{k=0}^{n}p_{n,k}^{(\alpha)}f\left(\frac{k}{n}\right) \tag{3}$$

where $p_{1,0}^{\alpha}(x)=1-x$, $p_{1,1}^{\alpha}(x)=x$ and,

$$p_{n,k}^{(\alpha)}(x)=\left[\binom{n-2}{k}(1-\alpha)x+\binom{n-2}{k-2}(1-\alpha)(1-x)\right.$$
$$\left.+\binom{n}{k}\alpha x(1-x)\right]x^{k-1}(1-x)^{n-k-1}$$

for $n\geqslant 2$ and $x\in[0,1]$.

So far, a large number of papers concerning blending type Bernstein operators are published. But in these papers researchers focus on only the operators and their approximation properties, the Lototsky matrices that generates these operators, which is another interesting problem, are not studied (for example [5], [7], [8], [9], [10], [12], [13], [16]).

One of the motivations of the present paper was to cover this lack by introducing Lototsky matrices that generates blending type Bernstein operators considered in [5], [7], [8], [9], [10], [12], [13], [16] and we did this in the present paper. The product given in (4) for $s=2$ gives Lototsky matrices that generates blending type Bernstein operators considered in these papers. It should be mentioned that, the product given in (4)(for $s=2$) which generates blending type Bernstein operators and product given in (2) that generates Bernstein operators are surprisingly quite different from each other.

The second motivation of the present paper was to use the generalized Lototsky matrices that can be obtained from product given in (4) to introduce more comprehensive generalization of both blending type Bernstein operators and standard Bernstein operators. This is also achieved (see operators $L_n^{\alpha,s}(f;x)$).

Finally, it should be mentioned that operators considered in some of previous papers does not preserves linear functions. For these papers, $L_n(e_i,x)=e_i(x)$, should be replace by $L_n(e_i,x) \to e_i(x)$. The same thing can be seen in some of above papers for bivariate case. Of course, conditions of Korovkin Theorem are still correct and same results can be obtained, but in any case some proofs need some modifications accordingly. To avoid this problem we define our operator peicewisely (see $L_n^{\alpha,s}(f;x)$).

2. Generalized Lototsky matrices and generalized blending type Bernstein operators

Let $0 \leqslant \alpha \leqslant 1$ be any real number and s be a positive integer, then consider the following generalization of (2),

$$\sum_{k=0}^{\infty} a_{nk}^{\alpha,s}(x)y^k = (1-\alpha)\left[y^s x \prod_{i=1}^{n-s}(yx+1-x) + (1-x)\prod_{i=1}^{n-s}(yx+1-x)\right] + \alpha \prod_{i=1}^{n}(yx+1-x), \tag{4}$$

where $0 \leqslant x \leqslant 1$,

$$\prod_{i=1}^{q} b = \begin{cases} b^q, & q \geqslant 1, \\ 1, & q = 0 \\ 0, & \text{otherwise.} \end{cases}$$

and

$$\binom{n}{k} = \begin{cases} \frac{n!}{k!(n-k)!}, & 0 \leqslant k \leqslant n, \\ 0, & \text{otherwise.} \end{cases}$$

It is obvious that for $\alpha = 1$, (4) reduces to (2). Now we can define the following α-Lototsky-Bernstein operators,

$$L_n^{\alpha,s}(f,x) = \sum_{k=0}^{n} b_{nk}^{\alpha,s}(x) f\left(\frac{k}{n}\right) \tag{5}$$

where

$$b_{n,k}^{\alpha,s}(x) = \begin{cases} \binom{n}{k}x^k(1-x)^{n-k}, & \text{if } n < s, \\ a_{n,k}^{\alpha,s}(x), & \text{if } n \geqslant s. \end{cases}$$

Obviously, $L_n^{\alpha,s}(f,x)$ is a positive linear operator and if $s = 1$, $L_n^{\alpha,1}(f,x)$, is the ordinary Bernstein operators, therefore the operators $L_n^{\alpha,s}(f,x)$ includes Bernstein operators as a special case. In the rest of the paper, to work with a non-trivial generalization of the Bernstein operators, we shall focus on the case $s \geqslant 2$.

LEMMA 1. *For any* $0 \leqslant \alpha \leqslant 1$ *and* $s \geqslant 2$,

$$L_n^{\alpha,s}(f;x) = \begin{cases} B_n(f;x), \quad n < s, \\ \sum_{k=0}^{n}\left[(1-\alpha)\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k} + (1-\alpha)\binom{n-s}{k}x^k(1-x)^{n-s-k+1} \right. \\ \left. \quad + \alpha\binom{n}{k}x^k(1-x)^{n-k}\right] f(\frac{k}{n}), \quad n \geqslant s. \end{cases}$$

Proof. The case $n < s$ is obvious. Assume that $n \geqslant s$. From (4) we have,

$$\begin{aligned}
\sum_{k=0}^{\infty} a_{nk}^{\alpha,s}(x)y^k &= (1-\alpha)\Bigg[y^s x \sum_{k=0}^{n-s}\binom{n-s}{k} y^k x^k (1-x)^{n-s-k} \\
&\quad +(1-x)\sum_{k=0}^{n-s}\binom{n-s}{k} y^k x^k (1-x)^{n-s-k}\Bigg] + \alpha \sum_{k=0}^{n}\binom{n}{k} y^k x^k (1-x)^{n-k} \\
&= (1-\alpha)\Bigg[\sum_{k=0}^{n-s}\binom{n-s}{k} y^{k+s} x^{k+1} (1-x)^{n-s-k} \\
&\quad +\sum_{k=0}^{n-s}\binom{n-s}{k} y^k x^k (1-x)^{n-s-k+1}\Bigg] + \alpha \sum_{k=0}^{n}\binom{n}{k} y^k x^k (1-x)^{n-k} \\
&= (1-\alpha)\Bigg[\sum_{k=s}^{n}\binom{n-s}{k-s} y^k x^{k-s+1} (1-x)^{n-k} \\
&\quad +\sum_{k=0}^{n}\binom{n-s}{k} y^k x^k (1-x)^{n-s-k+1}\Bigg] + \alpha \sum_{k=0}^{n}\binom{n}{k} y^k x^k (1-x)^{n-k} \\
&= \sum_{k=0}^{n}\Bigg[(1-\alpha)\binom{n-s}{k-s} x^{k-s+1} (1-x)^{n-k} \\
&\quad +(1-\alpha)\binom{n-s}{k} x^k (1-x)^{n-s-k+1} + \alpha\binom{n}{k} x^k (1-x)^{n-k}\Bigg] y^k
\end{aligned}$$

which completes the proof. □

The following Lemma clarifies the relation between $L_n^{\alpha,s}(f,x)$ and the blending type Bernstein operators $B_{n,\alpha}(f,x)$ given in (3).

LEMMA 2. *For $s = 2$, $L_n^{\alpha,2}(f,x) = B_{n,\alpha}(f,x)$.*

Proof. It is easy to see that, for $s = 2$ we have,

$$b_{nk}^{\alpha,2}(x) = \begin{cases} \binom{n}{k} x^k (1-x)^{n-k}, & n < 2, \\ p_{n,k}^{(\alpha)}(x), & n \geqslant 2, \end{cases}$$

which completes the proof. □

COROLLARY 1. *Since $L_n^{\alpha,2}(f,x) = B_{n,\alpha}(f,x)$, the positive linear operators $L_n^{\alpha,s}(f,x)$ defined here, includes $B_{n,\alpha}(f,x)$ as a special case as well.*

The following theorem gives an alternative representation of the operator $L_n^{\alpha,s}(f;x)$.

THEOREM 1. *For any* $0 \leqslant \alpha \leqslant 1$ *and* $s \geqslant 2$,

$$L_n^{\alpha,s}(f;x) = \begin{cases} B_n(f;x), & n < s, \\ (1-\alpha)\sum_{k=0}^{n-s+1}\binom{n-s+1}{k}x^k(1-x)^{n-s-k+1}h_k + \alpha\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f_k, \\ \qquad n \geqslant s, \end{cases}$$

where

$$h_k = \left(1 - \frac{k}{n-s+1}\right)f_k + \frac{k}{n-s+1}f_{k+s-1},$$

and $f_k = f(\frac{k}{n})$.

Proof. Let $0 \leqslant \alpha \leqslant 1$ and let $s \geqslant 2$ be an arbitrary but fixed positive integer. By the definition of $b_{nk}^{\alpha,s}(x)$, $L_n^{\alpha,s}(f;x) = B_n(f,x)$ when $n < s$. Assume that $n \geqslant s$, then,

$$L_n^{\alpha,s}(f;x) = \sum_{k=0}^{n}\left[(1-\alpha)\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k} + (1-\alpha)\binom{n-s}{k}x^k(1-x)^{n-s-k+1} + \alpha\binom{n}{k}x^k(1-x)^{n-k}\right]f_k.$$

Say,

$$g_1 = \sum_{k=0}^{n}\binom{n-s}{k}x^k(1-x)^{n-s-k+1}f_k,$$

and

$$g_2 = \sum_{k=0}^{n}\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k}f_k.$$

Then,

$$g_1 = \sum_{k=0}^{n}\binom{n-s}{k}x^k(1-x)^{n-s-k+1}f_k = \sum_{k=0}^{n-s+1}\binom{n-s}{k}x^k(1-x)^{n-s-k+1}f_k$$

and

$$g_2 = \sum_{k=0}^{n}\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k}f_k = \sum_{k=s-1}^{n}\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k}f_k = \sum_{k=0}^{n-s+1}\binom{n-s}{k-1}x^k(1-x)^{n-s-k+1}f_{k+s-1}.$$

Therefore,

$$g_1+g_2=\sum_{k=0}^{n-s+1}\left[\binom{n-s}{k}f_k+\binom{n-s}{k-1}f_{k+s-1}\right]x^k(1-x)^{n-s-k+1}. \tag{6}$$

On the other hand, if we use the following equations

$$\binom{n-s}{k}=\binom{n-s+1}{k}\left(1-\frac{k}{n-s+1}\right)$$

and

$$\binom{n-s}{k-1}=\binom{n-s+1}{k}\frac{k}{n-s+1}$$

in (6) we get,

$$\begin{aligned} g_1+g_2 &= \sum_{k=0}^{n-s+1}\binom{n-s+1}{k}\left[\left(1-\frac{k}{n-s+1}\right)f_k+\frac{k}{n-s+1}f_{k+s-1}\right]x^k(1-x)^{n-s-k+1}. \\ &= \sum_{k=0}^{n-s+1}\binom{n-s+1}{k}h_k x^k(1-x)^{n-s-k+1}. \end{aligned}$$

Thus for $n\geqslant s$,

$$\begin{aligned} L_n^{\alpha,s}(f;x) &= \sum_{k=0}^{n} b_{nk}^{\alpha,s}(x)f_k \\ &= (1-\alpha)\sum_{k=0}^{n-s+1}\binom{n-s+1}{k}x^k(1-x)^{n-s-k+1}h_k \\ &\quad +\alpha\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}f_k. \quad \square \end{aligned}$$

LEMMA 3. *For any $0\leqslant\alpha\leqslant 1$ and s, the operators $L_n^{\alpha,s}(f;x)$, preserves linear polynomials, that is;*

$$L_n^{\alpha,s}(1;x)=1 \text{ and } L_n^{\alpha,s}(t;x)=x.$$

Proof. For the case $n<s$, $L_n^{\alpha,s}(1;x)=B_n(1;x)=1$ and $L_n^{\alpha,s}(t;x)=B_n(t;x)=x$. So we only need to prove both equations for the case $n\geqslant s$. If $f(x)=1$ then $f_k=h_k=1$. Therefore,

$$\begin{aligned} L_n^{\alpha,s}(1;x) &= (1-\alpha)\sum_{k=0}^{n-s+1}\binom{n-s+1}{k}x^k(1-x)^{n-s-k+1}+\alpha\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}. \\ &= (1-\alpha)B_{n-s+1}(1;x)+\alpha B_n(1;x)=1. \end{aligned}$$

On the other hand it is easy to see that, if $f(x)=x$ then,

$$h_k=\frac{k}{n-s+1}$$

and

$$\begin{aligned} L_n^{\alpha,s}(t;x) &= (1-\alpha)\sum_{k=0}^{n-s+1}\binom{n-s+1}{k}x^k(1-x)^{n-s-k+1}\frac{k}{n-s+1} \\ &\quad +\alpha\sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k}\frac{k}{n} \\ &= (1-\alpha)B_{n-s+1}(t;x)+\alpha B_n(t;x)=x. \quad \square \end{aligned}$$

3. Approximation properties of $L_n^{\alpha,s}(f;x)$

In this section, we shall study approximation properties of $L_n^{\alpha,s}(f;x)$.

THEOREM 2. *For any* $0 \leqslant \alpha \leqslant 1$ *and* $n \geqslant s \geqslant 2$,

$$\begin{aligned} L_n^{\alpha,s}(f;x) &= (1-\alpha)\sum_{r=0}^{n-s+1}\binom{n-s+1}{r}\left(\triangle^r f_0+\frac{r}{n-s+1}(\triangle^{r-1}f_s-\triangle^{r-1}f_1)\right)x^r \\ &\quad +\alpha\sum_{r=0}^{n}\binom{n}{r}x^r\triangle^r f_0. \end{aligned} \tag{7}$$

Proof. Recall that,

$$L_n^{\alpha,s}(f;x) = (1-\alpha)\sum_{k=0}^{n-s+1}\binom{n-s+1}{k}x^k(1-x)^{n-s-k+1}h_k+\alpha B_n(f;x),$$

where $0 \leqslant \alpha \leqslant 1$ and $n \geqslant s \geqslant 2$.

Now let,

$$\varphi(x) = \sum_{k=0}^{n-s+1}\binom{n-s+1}{k}x^k(1-x)^{n-s-k+1}h_k \tag{8}$$

then using the equation,

$$(1-x)^{n-s-k+1} = \sum_{j=0}^{n-s-k+1}\binom{n-s-k+1}{j}(-1)^j x^j$$

in (8) we have,

$$\begin{aligned} \varphi(x) &= \sum_{k=0}^{n-s+1}h_k\binom{n-s+1}{k}x^k\sum_{j=0}^{n-s-k+1}\binom{n-s-k+1}{j}(-1)^j x^j \\ &= \sum_{k=0}^{n-s+1}\sum_{j=0}^{n-s-k+1}h_k\binom{n-s+1}{k}\binom{n-s-k+1}{j}(-1)^j x^{k+j}. \end{aligned}$$

But,

$$\binom{n-s+1}{k}\binom{n-s-k+1}{j} = \binom{r}{k}\binom{n-s+1}{r}$$

where $r=k+j$. Thus,

$$\begin{aligned}\varphi(x) &= \sum_{r=0}^{n-s+1}\binom{n-s+1}{r}x^r\sum_{k=0}^{r}\binom{r}{k}(-1)^{r-k}h_k\\ &= \sum_{r=0}^{n-s+1}\binom{n-s+1}{r}x^r\triangle^r h_0. \end{aligned} \tag{9}$$

On the other hand it is known that,

$$B_n(f,x) = \sum_{r=0}^{n}\binom{n}{r}x^r\triangle^r f_0. \tag{10}$$

Using (9) and (10) in the definition of $L_n^{\alpha,s}(f;x)$, we have,

$$L_n^{\alpha,s}(f;x) = (1-\alpha)\sum_{r=0}^{n-s+1}\binom{n-s+1}{r}x^r\triangle^r h_0+\alpha\sum_{r=0}^{n}\binom{n}{r}x^r\triangle^r f_0.$$

Finally, by using mathematical induction and the Leibniz rule

$$\triangle^n(g_if_i) = \sum_{k=0}^{n}\binom{n}{k}(\triangle^k f_i)(\triangle^{n-k}g_{i+k})$$

for $n=1$, we can prove that,

$$\triangle^r h_i = \left(1-\frac{i}{n-s+1}\right)\triangle^r f_i+\frac{i}{n-s+1}\triangle^r f_{i+s-1}+\frac{r}{n-s+1}(\triangle^{r-1}f_{i+s}-\triangle^{r-1}f_{i+1})$$

which gives

$$\triangle^r h_0 = \triangle^r f_0+\frac{r}{n-s+1}(\triangle^{r-1}f_s-\triangle^{r-1}f_1)$$

for $i=0$. This completes the proof. □

Recall that, we have the following relation between derivatives and differences;

$$\triangle^k f_i = \frac{1}{n^k}f^{(k)}(\xi_i),\ \ \text{for some } \xi_i\in\left(\frac{i}{n},\frac{i+k}{n}\right).$$

If $f(x)=x^m$ then,

$$\triangle^k f_0 = 0 \text{ for } k>m$$

and

$$\triangle^m f_i = \frac{1}{n^m}f^{(m)}(\xi_i) = \frac{m!}{n^m}.$$

Moreover,

$$\triangle^k f_1 = 0\ \ \text{for } k>m.$$

LEMMA 4. *For any* $0 \leqslant \alpha \leqslant 1$ *and* $n \geqslant s \geqslant 2$,

$$L_n^{\alpha,s}(t^2;x) = x^2 + \frac{x(1-x)\left[n+(1-\alpha)s(s-1)\right]}{n^2} \tag{11}$$

$$\begin{aligned} L_n^{\alpha,s}(t^3;x) &= x^3 + x^2(1-x)\left[\frac{3n-2}{n^2} + (1-\alpha)\frac{s(s-1)(3n-2s-2)}{n^3}\right] \\ &\quad + x(1-x)\left[\frac{1}{n^2} + (1-\alpha)\frac{s(s-1)(s+1)}{n^3}\right] \end{aligned} \tag{12}$$

$$\begin{aligned} L_n^{\alpha,s}(t^4;x) &= x^4 + x^3(1-x)\left[\frac{6n^2-11n+6}{n^3}\right. \\ &\quad \left. +(1-\alpha)\frac{s(1-s)\left[3(s+1)(s+2)+2n(3n-4s-7)\right]}{n^4}\right] \\ &\quad + x^2(1-x)\left[\frac{7(n-1)}{n^3} + (1-\alpha)\frac{s(s-1)[(n-s)(4s+10)-7]}{n^4}\right] \\ &\quad + x(1-x^2)\left[\frac{1}{n^3} + (1-\alpha)\frac{s(s-1)(s^2+s+1)}{n^4}\right]. \end{aligned} \tag{13}$$

Proof. From equation (7) we have,

$$\begin{aligned} L_n^{\alpha,s}(t^2;x) &= (1-\alpha)\left[(n-s+1)\left(\triangle f_0 + \frac{1}{n-s+1}(f_s - f_1)\right)x\right. \\ &\quad \left. + \frac{(n-s+1)(n-s)}{2}\left(\triangle^2 f_0 + \frac{2}{n-s+1}(\triangle f_s - \triangle f_1\right)x^2\right] \\ &\quad + \alpha\left[n\triangle f_0 x + \frac{n(n-1)}{2}\triangle^2 f_0 x^2\right] \\ &= (1-\alpha)\left[\left(\frac{s^2-s+n}{n^2}\right)x + x^2 + \left(\frac{-s^2+s-n}{n^2}\right)x^2\right] + \alpha\left[\frac{x}{n} + x^2 - \frac{x^2}{n}\right] \\ &= x^2 + \frac{x(1-x)\left[n+(1-\alpha)s(s-1)\right]}{n^2} \end{aligned}$$

which proves (11). Equations (12) and (13), can be proved in a parallel way. □

THEOREM 3. *Let* $f(x)$ *be any continuous function on the interval* $[0,1]$. *Then for any positive integer* s, *and for any* $0 \leqslant \alpha \leqslant 1$, $L_n^{\alpha,s}(f;x)$ *converges uniformly to* $f(x)$ *on* $[0,1]$.

Proof. As a consequence of the Korovkin theorem [4, 15] it is enough to show that, $L_n^{\alpha,s}(e_i;x)$ converges uniformly to $e_i(x)$, where $e_i(x) = x^i$, $i = 0,1,2$. By Lemma 3, we have, $L_n^{\alpha,s}(1;x) = 1$ and $L_n^{\alpha,s}(x;x) = x$. On the other hand by Lemma 4 and the definition of the operator $L_n^{\alpha,s}(f;x)$ we have,

$$L_n^{\alpha,s}(t^2;x) = \begin{cases} x^2 + \frac{x(1-x)}{n}, & n < s, \\ x^2 + \frac{x(1-x)\left[n+(1-\alpha)s(s-1)\right]}{n^2}, & n \geqslant s. \end{cases}$$

Obviously, for both cases $L_n^{\alpha,s}(t^2;x)$ converges uniformly to $f(x)=x^2$, and this completes the proof. □

LEMMA 5. *Let*

$$S_m=\sum_{k=0}^{n}(k-nx)^m a_{n,k}^{\alpha,s}(x).$$

Then for $n\geqslant s$ we have

$$\begin{aligned}
S_0(x) &= 1\\
S_1(x) &= 0\\
S_2(x) &= x(1-x)\big[n+(1-\alpha)s(s-1)\big]\\
S_3(x) &= x(1-x)(1-2x)\big[n+(1-\alpha)s(s-1)(s+1)\big]\\
S_4(x) &= 3x^2(1-x)^2\Bigg[n(n-2)+(1-\alpha)s(s-1)[2n-(s+1)(s+2)]\Bigg]\\
&\quad +x(1-x)\Bigg[n+(1-\alpha)s(s-1)(s^2+s+1)\Bigg].
\end{aligned}$$

Proof. It is obvious that $S_0(x)=1$ and $S_1(x)=0$.

$$\begin{aligned}
S_2(x) &= \sum_{k=0}^{n}(k-nx)^2 a_{n,k}^{\alpha,s}(x)\\
&= \sum_{k=0}^{n}(k^2-2nkx+x^2)a_{n,k}^{\alpha,s}(x)\\
&= n^2\sum_{k=0}^{n}\left(\frac{k^2}{n^2}-2x\frac{k}{n}+x^2\right)a_{n,k}^{\alpha,s}(x)\\
&= n^2\sum_{k=0}^{n}\frac{k^2}{n^2}a_{n,k}^{\alpha,s}(x)-2xn^2\sum_{k=0}^{n}\frac{k}{n}a_{n,k}^{\alpha,s}(x)+x^2n^2\sum_{k=0}^{n}a_{n,k}^{\alpha,s}(x)\\
&= n^2x^2+\big(x(1-x)\big[n+(1-\alpha)s(s-1)\big]\big)-2n^2x^2+n^2x^2\\
&= x(1-x)\big[n+(1-\alpha)s(s-1)\big].
\end{aligned}$$

Equations for $S_3(x)$ and $S_4(x)$ can be obtained in a parallel way. □

LEMMA 6. *For any $0\leqslant\alpha\leqslant 1$, and positive integer s there exists a constant C independent of n such that,*

$$\sum_{|\frac{k}{n}-x|\geqslant n^{-\frac{1}{8}}} b_{n,k}^{\alpha,s}(x)\leqslant\frac{C}{n^{\frac{3}{2}}}, \tag{14}$$

for all $x\in[0,1]$.

Proof. Given a positive integer s. For the case $n < s$, $b_{n,k}^{\alpha,s}(x) = \binom{n}{k}x^k(1-x)^{n-k}$ and (14) holds. Assume that $n \geqslant s$. Since $S_4(x)$ is a second order polynomial in n. There exists a constant C such that $S_4(x) \leqslant Cn^2$ for all $x \in [0,1]$. On the other hand, $|\frac{k}{n} - x| \geqslant n^{-\frac{1}{8}}$ implies that $\frac{(k-nx)^4}{n^{\frac{7}{2}}} \geqslant 1$. Thus,

$$\sum_{|\frac{k}{n}-x|\geqslant n^{-\frac{1}{8}}} a_{n,k}^{\alpha,s}(x) \leqslant \frac{1}{n^{\frac{7}{2}}} \sum_{k=0}^{n} (k-nx)^4 a_{n,k}^{\alpha,s}(x) \leqslant \frac{1}{n^{\frac{7}{2}}} S_4(x) \leqslant \frac{1}{n^{\frac{7}{2}}} Cn^2 \leqslant \frac{C}{n^{\frac{3}{2}}}. \qquad \square$$

Now we can prove the following Voronovskaja type theorem [18] for $L_n^{\alpha,s}(f;x)$.

THEOREM 4. *Let $f(x)$ be a bounded function on $[0,1]$, if $f''(x)$ exists at a point $x \in [0,1]$ then*

$$\lim_{n\to\infty} n\left[L_n^{\alpha,s}(f;x) - f(x)\right] = \frac{1}{2}x(1-x)f''(x) \tag{15}$$

where $0 \leqslant \alpha \leqslant 1$ and s is a positive integer.

Proof. Given a positive integer s. If $n < s$, $L_n^{\alpha,s}(f;x) = B_n(f;x)$ and (15) holds. Assume that $n \geqslant s$, by the Taylor's formula we have

$$f(t) = f(x) + (t-x)f'(x) + \frac{1}{2}(t-x)^2 f''(x) + p(t)(t-x)^2,$$

where $\lim_{t\to x} p(t) = 0$. Taking $t = \frac{k}{n}$ where $k \leqslant n$ gives,

$$f(\frac{k}{n}) = f(x) + \left(\frac{k}{n} - x\right)f'(x) + \frac{1}{2}\left(\frac{k}{n} - x\right)^2 f''(x) + p\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2. \tag{16}$$

Equation (16) gives that,

$$\begin{aligned} n[L_n^{\alpha,s}(f;x) - f(x)] &= \frac{1}{2n}S_2(x)f''(x) + n\sum_{k=0}^{n} p\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2 a_{n,k}^{\alpha,s}(x) \\ &= x(1-x)\left[\frac{1}{2} + \frac{(1-\alpha)s(s-1)}{2n}\right] f''(x) + nP_n^{\alpha,s}(x), \end{aligned}$$

for any $0 \leqslant \alpha \leqslant 1$ and s, where

$$P_n^{\alpha,s}(x) = \sum_{k=0}^{n} p\left(\frac{k}{n}\right)\left(\frac{k}{n} - x\right)^2 a_{n,k}^{\alpha,s}(x).$$

On the other hand,

$$|P_n^{\alpha,s}(x)| \leqslant \sum_{|\frac{k}{n}-x|<n^{-\frac{1}{8}}} \left|p\left(\frac{k}{n}\right)\right|\left(\frac{k}{n} - x\right)^2 a_{n,k}^{\alpha,s}(x) + \sum_{|\frac{k}{n}-x|\geqslant n^{-\frac{1}{8}}} \left|p\left(\frac{k}{n}\right)\right|\left(\frac{k}{n} - x\right)^2 a_{n,k}^{\alpha,s}(x).$$

Given $\varepsilon > 0$, we can find n, large enough such that $|\frac{k}{n} - x| < n^{-\frac{1}{8}}$ implies $p(\frac{k}{n}) < \varepsilon$. Therefore,

$$\begin{aligned} |P_n^{\alpha,s}(x)| &\leqslant \frac{\varepsilon}{n^2} S_2(x) + K \sum_{|\frac{k}{n}-x| \geqslant n^{-\frac{1}{8}}} |a_{n,k}^{\alpha,s}(x)| \\ &\leqslant \varepsilon x(1-x)\left[\frac{1}{2n} + \frac{(1-\alpha)s(s-1)}{2n^2}\right] + \frac{KC}{n^{\frac{3}{2}}}, \end{aligned}$$

where $K = sup_{0 \leqslant t \leqslant 1} p(t)(t-x)^2$. In other words,

$$n|P_n^{\alpha,s}(x)| \leqslant \varepsilon x(1-x)\left[\frac{1}{2} + \frac{(1-\alpha)s(s-1)}{2n}\right] + \frac{KC}{n^{\frac{1}{2}}}.$$

Since ε is arbitrary (15) follows. □

Recall that the modulus of continuity of a function $f(x)$ which is defined on an interval $[a,b]$ is defined by,

$$\omega(f;\delta) = sup_{a \leqslant x,y \leqslant b,\ |x-y|<\delta} \left| f(x) - f(y) \right|$$

where $\delta > 0$. It is also well-known that, $\omega(\delta)$ satisfies following properties.

(i) If $0 < \delta \leqslant \gamma$, then $\omega(f;\delta) \leqslant \omega(f;\gamma)$.

(ii) $f(x)$ is uniformly continuous on [a,b] if and only if $\lim_{\delta \to 0} \omega(f;\delta) = 0$.

(iii) If $\mu > 0$, then $\omega(f;\mu\delta) \leqslant (1+\mu)\omega(f;\delta)$.

THEOREM 5. *Let f be a bounded function on $[0,1]$ and let $0 \leqslant \alpha \leqslant 1$ be any real number than,*

$$\|L_n^{\alpha,s}(f;x) - f(x)\|_\infty \leqslant \frac{3}{2}\omega\Big(f, \frac{\sqrt{n+(1-\alpha)s(s-1)}}{n}\Big)$$

where $\|\cdot\|_\infty$ is the supremum norm and $n \geqslant s$.

Proof. For any $\alpha \in [0,1]$, and $n \geqslant s$ we have,

$$\begin{aligned} &|L_n^{\alpha,s}(f;x) - f(x)| \\ &= \Big| \sum_{k=0}^{n} a_{nk}^{\alpha,s} f\Big(\frac{k}{n}\Big) - f(x) \Big| \\ &\leqslant \sum_{k=0}^{n} a_{nk}^{\alpha,s} \Big| f\Big(\frac{k}{n}\Big) - f(x) \Big| \\ &\leqslant \sum_{k=0}^{n} \omega\Big(f; \Big|x - \frac{k}{n}\Big|\Big) a_{nk}^{\alpha,s} \\ &\leqslant \sum_{k=0}^{n} \omega\left(f; \frac{\sqrt{n+(1-\alpha)s(s-1)}}{n} \Big|x - \frac{k}{n}\Big| \frac{n}{\sqrt{n+(1-\alpha)s(s-1)}}\right) a_{nk}^{\alpha,s} \end{aligned}$$

$$\begin{aligned}
&\leqslant \sum_{k=0}^{n}\left(1+\frac{n}{\sqrt{n+(1-\alpha)s(s-1)}}\left|x-\frac{k}{n}\right|\right) \\
&\quad\times\omega\left(f;\frac{\sqrt{n+(1-\alpha)s(s-1)}}{n}\right)a_{nk}^{\alpha,s} \\
&\leqslant \omega\left(f;\frac{\sqrt{n+(1-\alpha)s(s-1)}}{n}\right) \\
&\quad\times\sum_{k=0}^{n}\left(1+\frac{n}{\sqrt{n+(1-\alpha)s(s-1)}}\left|x-\frac{k}{n}\right|\right)a_{nk}^{\alpha,s} \\
&\leqslant \omega\left(f;\frac{\sqrt{n+(1-\alpha)s(s-1)}}{n}\right) \\
&\quad\times\left(1+\frac{n}{\sqrt{n+(1-\alpha)s(s-1)}}\sum_{k=0}^{n}\left|x-\frac{k}{n}\right|a_{nk}^{\alpha,s}\right). \qquad (17)
\end{aligned}$$

On the other hand as a consequence of Schwarz's inequality we have,

$$\begin{aligned}
\sum_{k=0}^{n}|x-\frac{k}{n}|a_{nk}^{\alpha,s} &= \sum_{k=0}^{n}\left|x-\frac{k}{n}\right|\sqrt{a_{nk}^{\alpha,s}}\sqrt{a_{nk}^{\alpha,s}} \\
&\leqslant \left[\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2a_{nk}^{\alpha,s}\right]^{\frac{1}{2}}\left[\sum_{k=0}^{n}a_{nk}^{\alpha,s}\right]^{\frac{1}{2}} \\
&\leqslant \left[\sum_{k=0}^{n}\left(x-\frac{k}{n}\right)^2a_{nk}^{\alpha,s}\right]^{\frac{1}{2}}.
\end{aligned}$$

For $s \geqslant 2$ we get,

$$\begin{aligned}
\sum_{k=0}^{n}(x-\frac{k}{n})^2a_{nk}^{\alpha,s} &= \frac{x(1-x)\big(n+(1-\alpha)s(s-1)}{n^2} \\
&\leqslant \frac{n+(1-\alpha)s(s-1)}{4n^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
|L_n^{\alpha,s}(f;x)-f(x)| &\leqslant \omega\left(f;\frac{\sqrt{n+(1-\alpha)s(s-1)}}{n}\right) \\
&\quad\times\left(1+\frac{n}{\sqrt{n+(1-\alpha)s(s-1)}}\frac{\sqrt{n+(1-\alpha)s(s-1)}}{2n}\right) \\
&= \frac{3}{2}\omega\left(f,\frac{\sqrt{n+(1-\alpha)s(s-1)}}{n}\right). \quad \square
\end{aligned}$$

4. Shape preserving properties of $L_n^{\alpha,s}(f;x)$

This section is devoted to the shape preserving properties of $L_n^{\alpha,s}(f;x)$. Particularly, we shall prove that $L_n^{\alpha,s}(f;x)$ preserves monotonicity and convexity properties.

THEOREM 6. *Let f be a continuous function on the interval $[0,1]$. If f is monotonically increasing (or monotonically decreasing) on $[0,1]$, then for any positive integer s, and for any $0 \leqslant \alpha \leqslant 1$, $L_n^{\alpha,s} f$ is also monotonically increasing (or monotonically decreasing) on $[0,1]$.*

Proof. If $s=1$ or $n<s$, $L_n^{\alpha,s}(f;x)=B_n(f,x)$ so it has monotonicity and convexity properties. Now we can assume that $n \geqslant s \geqslant 2$. From Lemma 1,

$$L_n^{\alpha,s}(f;x) = (1-\alpha)\sum_{k=0}^{n}\left[\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k} + \binom{n-s}{k}x^{k}(1-x)^{n-s-k+1}\right]f_k + \alpha B_n(f,x).$$

Say

$$T=\sum_{k=0}^{n}\left[\binom{n-s}{k-s}x^{k-s+1}(1-x)^{n-k}+\binom{n-s}{k}x^{k}(1-x)^{n-s-k+1}\right]f_k$$

then,

$$\begin{aligned} T' =& \sum_{k=0}^{n}\binom{n-s}{k-s}(k-s+1)x^{k-s}(1-x)^{n-k}f_k \\ &-\sum_{k=0}^{n}\binom{n-s}{k-s}(n-k)x^{k-s+1}(1-x)^{n-k-1}f_k \\ &+\sum_{k=0}^{n}\binom{n-s}{k}kx^{k-1}(1-x)^{n-s-k+1}f_k \\ &-\sum_{k=0}^{n}\binom{n-s}{k}(n-k-s+1)x^{k}(1-x)^{n-s-k}f_k. \end{aligned}$$

The first and third summations are zero for $k=0$ and second and fourth summations are zero for $k=n$. So if we modify the summations and then if we replace k by $k+1$ in first and third summations we get,

$$\begin{aligned} T' =& \sum_{k=0}^{n-1}\binom{n-s}{k-s+1}(k-s+2)x^{k-s+1}(1-x)^{n-k-1}f_{k+1} \\ &-\sum_{k=0}^{n-1}\binom{n-s}{k-s}(n-k)x^{k-s+1}(1-x)^{n-k-1}f_k \end{aligned}$$

$$+\sum_{k=0}^{n-1}\binom{n-s}{k+1}(k+1)x^k(1-x)^{n-s-k}f_{k+1}$$
$$-\sum_{k=0}^{n-1}\binom{n-s}{k}(n-k-s+1)x^k(1-x)^{n-s-k}f_k. \tag{18}$$

Using the equations

$$\binom{n-s}{k-s}(n-k)=\binom{n-s}{k-s+1}(k-s+1)$$

and

$$\binom{n-s}{k+1}(k+1)=\binom{n-s}{k}(n-s-k)$$

in (18) we have,

$$T'=\sum_{k=0}^{n-1}\binom{n-s}{k-s+1}(k-s+1)x^{k-s+1}(1-x)^{n-k-1}\triangle f_k$$
$$+\sum_{k=0}^{n-1}\binom{n-s}{k}(n-s-k)x^k(1-x)^{n-s-k}\triangle f_k$$
$$+\sum_{k=0}^{n-1}\binom{n-s}{k-s+1}x^{k-s+1}(1-x)^{n-k-1}f_{k+1}$$
$$-\sum_{k=0}^{n-1}\binom{n-s}{k}x^k(1-x)^{n-s-k}f_k.$$

Since $\binom{n-s}{k-s+1}=0$ when $k<s-1$ and $\binom{n-s}{k}=0$ when $k>n-s$, we get

$$T'=\sum_{k=0}^{n-1}\binom{n-s}{k-s+1}(k-s+1)x^{k-s+1}(1-x)^{n-k-1}\triangle f_k$$
$$+\sum_{k=0}^{n-1}\binom{n-s}{k}(n-s-k)x^k(1-x)^{n-s-k}\triangle f_k$$
$$+\sum_{k=s-1}^{n-1}\binom{n-s}{k-s+1}x^{k-s+1}(1-x)^{n-k-1}f_{k+1}$$
$$-\sum_{k=0}^{n-s}\binom{n-s}{k}x^k(1-x)^{n-s-k}f_k.$$

Replace k by $k+s-1$ in the third summation we get,

$$T' = \sum_{k=0}^{n-1} \binom{n-s}{k-s+1} (k-s+1) x^{k-s+1} (1-x)^{n-k-1} \triangle f_k + \sum_{k=0}^{n-1} \binom{n-s}{k} (n-s-k) x^k (1-x)^{n-s-k} \triangle f_k + \sum_{k=0}^{n-s} \binom{n-s}{k} x^k (1-x)^{n-s-k} f_{k+s} - \sum_{k=0}^{n-s} \binom{n-s}{k} x^k (1-x)^{n-s-k} f_k,$$

or

$$T' = \sum_{k=0}^{n-1} \binom{n-s}{k-s+1} (k-s+1) x^{k-s+1} (1-x)^{n-k-1} \triangle f_k + \sum_{k=0}^{n-1} \binom{n-s}{k} (n-s-k) x^k (1-x)^{n-s-k} \triangle f_k + \sum_{k=0}^{n-s} \binom{n-s}{k} x^k (1-x)^{n-s-k} (f_{k+s} - f_k). \tag{19}$$

Therefore, using (19) and the fact that

$$B_n'(f,x) = \sum_{k=0}^{n-1} \binom{n}{k} (n-k) x^k (1-x)^{n-k-1} \triangle f_k$$

we get,

$$(L_n^{\alpha,s})'(f;x) = (1-\alpha) \left[\sum_{k=0}^{n-1} \binom{n-s}{k-s+1} (k-s+1) x^{k-s+1} (1-x)^{n-k-1} \triangle f_k + \sum_{k=0}^{n-1} \binom{n-s}{k} (n-s-k) x^k (1-x)^{n-s-k} \triangle f_k + \sum_{k=0}^{n-s} \binom{n-s}{k} x^k (1-x)^{n-s-k} (f_{k+s} - f_k) \right] + \alpha \sum_{k=0}^{n-1} \binom{n}{k} (n-k) x^k (1-x)^{n-k-1} \triangle f_k. \tag{20}$$

Now, if $f(x)$ is monotonically increasing then $\triangle f_k \geqslant 0$, $f_{k+s} - f_k \geqslant 0$ and $(L_n^{\alpha,s})'(f;x) \geqslant 0$, or if $f(x)$ is monotonically decreasing then $\triangle f_k \leqslant 0$, $f_{k+s} - f_k \leqslant 0$ and $(L_n^{\alpha,s})'(f;x) \leqslant 0$. □

THEOREM 7. *Let $f(x)$ be a continuous function on the interval $[0,1]$. If $f(x)$ is convex on $[0,1]$, then for any positive integer s, and for any $0 \leqslant \alpha \leqslant 1$, $L_n^{\alpha,s}(f;x)$ is also convex on $[0,1]$.*

Proof. If $s=1$ or $n<s$, $L_n^{\alpha,s}(f;x)=B_n(f,x)$ so it is convex. Now we can assume that $n\geqslant s\geqslant 2$. The derivative of (18) gives that,

$$\begin{aligned}
T'' = &\sum_{k=0}^{n-1}\binom{n-s}{k-s+1}(k-s+1)(k-s+2)x^{k-s}(1-x)^{n-k-1}f_{k+1}\\
&-\sum_{k=0}^{n-1}\binom{n-s}{k-s+1}(k-s+2)(n-k-1)x^{k-s+1}(1-x)^{n-k-2}f_{k+1}\\
&-\sum_{k=0}^{n-1}\binom{n-s}{k-s}(n-k)(k-s+1)x^{k-s}(1-x)^{n-k-1}f_k\\
&+\sum_{k=0}^{n-1}\binom{n-s}{k-s}(n-k)(n-k-1)x^{k-s+1}(1-x)^{n-k-2}f_k\\
&+\sum_{k=0}^{n-1}\binom{n-s}{k+1}k(k+1)x^{k-1}(1-x)^{n-s-k}f_{k+1}\\
&-\sum_{k=0}^{n-1}\binom{n-s}{k+1}(n-s-k)(k+1)x^{k}(1-x)^{n-s-k-1}f_{k+1}\\
&-\sum_{k=0}^{n-1}\binom{n-s}{k}(n-s-k+1)kx^{k-1}(1-x)^{n-s-k}f_k\\
&+\sum_{k=0}^{n-1}\binom{n-s}{k}(n-s-k+1)(n-s-k)x^{k}(1-x)^{n-s-k-1}f_k.
\end{aligned}$$

First, third, fifth and seventh summations are zero for $k=0$ and other summations are zero for $k=n-1$. Modify end-points of each summation accordingly and replace k by $k+1$ in first, third, fifth and seventh summations we have,

$$\begin{aligned}
T'' = &\sum_{k=0}^{n-2}\binom{n-s}{k-s+2}(k-s+2)(k-s+3)x^{k-s+1}(1-x)^{n-k-2}f_{k+2}\\
&-2\sum_{k=0}^{n-2}\binom{n-s}{k-s+1}(k-s+2)(n-k-1)x^{k-s+1}(1-x)^{n-k-2}f_{k+1}\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k-s}(n-k)(n-k-1)x^{k-s+1}(1-x)^{n-k-2}f_k\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k+2}(k+1)(k+2)x^{k}(1-x)^{n-s-k-1}f_{k+2}\\
&-2\sum_{k=0}^{n-2}\binom{n-s}{k+1}(n-s-k)(k+1)x^{k}(1-x)^{n-s-k-1}f_{k+1}\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k}(n-s-k+1)(n-s-k)x^{k}(1-x)^{n-s-k-1}f_k.
\end{aligned} \tag{21}$$

Using following equations

$$\binom{n-s}{k-s+2}(k-s+2)=\binom{n-s}{k-s+1}(n-k-1),$$

$$\binom{n-s}{k-s}(n-k)=\binom{n-s}{k-s+1}(k-s+1),$$

$$\binom{n-s}{k+2}(k+2)=\binom{n-s}{k+1}(n-s-k-1),$$

and

$$\binom{n-s}{k}(n-s-k)=\binom{n-s}{k+1}(k+1)$$

in (21) we get,

$$\begin{aligned}
T'' = &\sum_{k=0}^{n-2}\binom{n-s}{k-s+1}(n-k-1)(k-s+3)x^{k-s+1}(1-x)^{n-k-2}f_{k+2}\\
&-2\sum_{k=0}^{n-2}\binom{n-s}{k-s+1}(k-s+2)(n-k-1)x^{k-s+1}(1-x)^{n-k-2}f_{k+1}\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k-s+1}(n-k-1)(k-s+1)x^{k-s+1}(1-x)^{n-k-2}f_{k}\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k+1}(k+1)(n-s-k-1)x^{k}(1-x)^{n-s-k-1}f_{k+2}\\
&-2\sum_{k=0}^{n-2}\binom{n-s}{k+1}(n-s-k)(k+1)x^{k}(1-x)^{n-s-k-1}f_{k+1}\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k+1}(k+1)(n-s-k+1)x^{k}(1-x)^{n-s-k-1}f_{k}
\end{aligned}$$

$$\begin{aligned}
T'' = &\sum_{k=0}^{n-2}\binom{n-s}{k-s+1}(n-k-1)(k-s+1)x^{k-s+1}(1-x)^{n-k-2}\,\triangle^2 f_k\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k-s+1}(n-k-1)x^{k-s+1}(1-x)^{n-k-2}(2f_{k+2}-2f_{k+1})\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k+1}(k+1)(n-s-k-1)x^{k}(1-x)^{n-s-k-1}\,\triangle^2 f_k\\
&+\sum_{k=0}^{n-2}\binom{n-s}{k+1}(k+1)x^{k}(1-x)^{n-s-k-1}(-2f_{k+1}+2f_k).
\end{aligned}$$

The first two summations are zero for $k < s-1$ and last two summations are zero for $k > n-s-1$, so we have

$$\begin{aligned}T'' = &\sum_{k=s-1}^{n-2}\binom{n-s}{k-s+1}(n-k-1)(k-s+1)x^{k-s+1}(1-x)^{n-k-2}\triangle^2 f_k\\&+\sum_{k=s-1}^{n-2}\binom{n-s}{k-s+1}(n-k-1)x^{k-s+1}(1-x)^{n-k-2}(2f_{k+2}-2f_{k+1})\\&+\sum_{k=0}^{n-s-1}\binom{n-s}{k+1}(k+1)(n-s-k-1)x^k(1-x)^{n-s-k-1}\triangle^2 f_k\\&+\sum_{k=0}^{n-s-1}\binom{n-s}{k+1}(k+1)x^k(1-x)^{n-s-k-1}(-2f_{k+1}+2f_k).\end{aligned}\tag{22}$$

$$\begin{aligned}T'' = &\sum_{k=0}^{n-s-1}\binom{n-s}{k}k(n-k-s)x^k(1-x)^{n-k-s-1}\triangle^2 f_{k+s-1}\\&+\sum_{k=0}^{n-s-1}\binom{n-s}{k+1}(k+1)(n-s-k-1)x^k(1-x)^{n-s-k-1}\triangle^2 f_k\\&+2\sum_{k=0}^{n-s-1}\binom{n-s}{k}(n-k-s)x^k(1-x)^{n-k-s-1}\\&\times(f_{k+s+1}-f_{k+s}-f_{k+1}+f_k).\end{aligned}\tag{23}$$

By using mathematical induction one can prove that

$$f_{k+s+1}-f_{k+s}-f_{k+1}+f_k=\sum_{m=0}^{s-1}\triangle^2 f_{k+m}.\tag{24}$$

Therefore, using (24) in (23) we get,

$$\begin{aligned}T'' = &\sum_{k=0}^{n-s-1}\binom{n-s}{k}(n-k-s)x^k(1-x)^{n-k-s-1}(k)\triangle^2 f_{k+s-1}\\&+\sum_{k=0}^{n-s-1}\binom{n-s}{k+1}(k+1)x^k(1-x)^{n-s-k-1}(n-s-k-1)\triangle^2 f_k\\&+2\sum_{k=0}^{n-s-1}\binom{n-s}{k}(n-k-s)x^k(1-x)^{n-k-s-1}\sum_{m=0}^{s-1}\triangle^2 f_{k+m}.\end{aligned}$$

If f is convex then

$$\triangle^2 f_k \geqslant 0,$$

which means $T'' \geqslant 0$ and $B_n''(f;x) \geqslant 0$ or equivalently,

$$(L_n^{\alpha,s})''(f;x) = (1-\alpha)T'' + \alpha B_n''(f;x) \geqslant 0,$$

which completes the proof. □

5. Concluding remarks

In this paper, we introduced the generalized blending type Bernstein operators $L_n^{\alpha,s}$ which depends on two parameters α and s. So far, many research papers are published concerning blending type Bernstein operators but the Lototsky matrices that generates these blending type Bernstein operators were not investigated. In this paper, we also introduced the Lototsky matrix that generates our blending type Bernstein operators. It should be noted that Lototsky matrix that generates α-Bernstein operators (3) can be obtained from (4) for $s=2$.

The generalized blending type Bernstein operators has the following properties:

(i) For $\alpha=1$ or $s=1$, $L_n^{\alpha,s}(f;x)$ reduces to Bernstein operators.

(ii) For $s=2$, $L_n^{\alpha,s}(f;x)$ reduces to α-Bernstein operators (3) given by Chen et al. [10].

(iii) For any $0\leqslant\alpha\leqslant 1$ and positive integer s, $L_n^{\alpha,s}(f;x)$ reproduce the linear functions.

(iv) If $f(x)$ is a continuous function on $[0,1]$, then for any real number $\alpha\in[0,1]$ and for any s, $L_n^{\alpha,s}(f;x)$ converges uniformly to $f(x)$.

(v) For any $0\leqslant\alpha\leqslant 1$ and positive integer s, $L_n^{\alpha,s}(f;x)$ has monotonicity and convexity properties.

(vi) The Lototsky matrices, introduced in (4), generates $L_n^{\alpha,s}(f;x)$. The particular case of these Lototsky matrices for $s=2$ generates α-Bernstein operators given by Chen et al. [10] .

(vii) An upper bound for the approximation error of these operators are obtained in terms of modulus of continuity.

(viii) In Figure 1, the approximation of $L_n^{\alpha,s}(f;x)$ to f, for $f(x)=x(x-1)(2x-1)$ is shown for $\alpha=0.5$, $s=5$ and $n=25,50,200$.

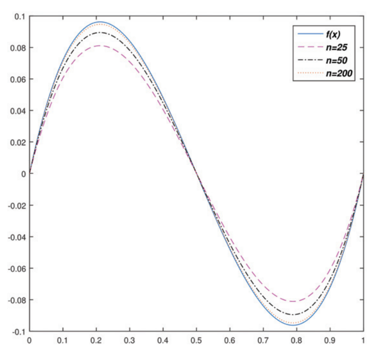

Figure 1: *Approximation of* $L_n^{\alpha,s}(f;x)$ *to* $f(x)=x(x-1)(2x-1)$ *for* $s=5$, $\alpha=0.5$ *and* $n=25,50,200$

(ix) Figure 2, shows the approximation of $L_n^{\alpha,s}(f;x)$ to f, for $f(x) = x(x-1)(2x-1)$ for fixed n and α. It can be seen that smaller s values gives better approximation to f.

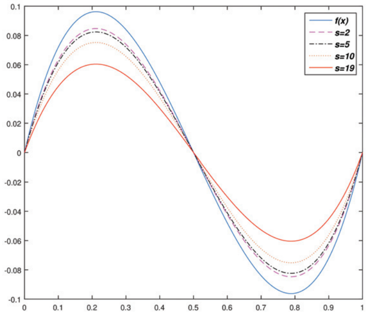

Figure 2: *Approximation of* $L_n^{\alpha,s}(f;x)$ *to* $f(x) = x(x-1)(2x-1)$ *for* $n = 25$*,* $\alpha = 2/3$ *and* $s = 2, 5, 10, 19$

(x) Figure 3, shows the approximation of $L_n^{\alpha,s}(f;x)$ to f, for $f(x) = x(x-1)(2x-1)$ for fixed n and s. It can be seen that large values of α gives better approximation to f.

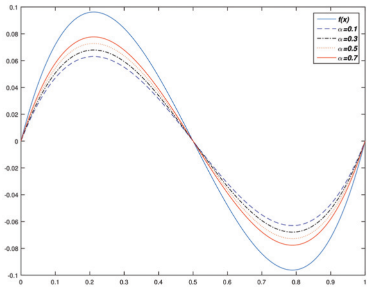

Figure 3: *Approximation of* $L_n^{\alpha,s}(f;x)$ *to* $f(x) = x(x-1)(2x-1)$ *for* $s = 9$*,* $n = 25$ *and* $\alpha = 0.1, 0.3, 0.5, 0.7$

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