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SINGULAR VALUE INEQUALITIES FOR ACCRETIVE–DISSIPATIVE NORMAL OPERATORS

WASIM AUDEH

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Abstract. In this paper, we present singular value inequalities relevant to accretive-dissipative normal compact operators. In particular, we showed that if $X = A + iB$ and $Y = C + iD$ are accretive-dissipative normal compact operators, where $A \leqslant C$ and $B \leqslant D$. Then

$$s_j(X - Y) \leqslant \sqrt{2}s_j(X \oplus Y)$$

for $j = 1,2,\ldots$ Moreover, if $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ is accretive-dissipative normal compact operator. Then

$$\sqrt{2}s_j(Z) \leqslant s_j(X \oplus Y)$$

for $j = 1,2,\ldots$ We showed that these inequalities are equivalent. Also, we provide several singular value inequalities relevant to accretive- dissipative normal compact operators.

1. Introduction

Let $B(H)$ denote the space of all bounded linear operators on a complex separable Hilbert space H, and let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. The operator $A \in \mathbb{K}(\mathbb{H})$ is called normal operator if $A^*A = AA^*$. The absolute value of A is the positive operator $|A|$, where $|A| = (A^*A)^{1/2}$. The singular values of the compact operator A are the eigenvalues of the positive compact operator $|A|$, denoted by $s_1(A) \geqslant s_2(A) \geqslant \ldots$ and repeated according to multiplicity. Some basic properties for singular values of compact operators are listed below:

(a)

$$s_j(UAV) = s_j(A) = s_j(A^*) = s_j(|A|) = s_j(|A^*|) \tag{1}$$

for $j = 1,2,\ldots$ where U and V are unitary.

(b) $s_j(AA^*) = s_j(A^*A)$ for $j = 1,2,\ldots$

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(c)

$$s_j \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} = s_j \begin{bmatrix} 0 & B \\ A & 0 \end{bmatrix} \tag{2}$$

for $j = 1, 2, \ldots$, and they consist of those of A together with those of B, where $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ is defined on $\mathbb{H} \oplus \mathbb{H}$.

By applying Weyl's monotonicity principle (see, e.g., [10, p.63] or [13, p.26]), if $A, B \in \mathbb{K}(\mathbb{H})$ are positive and $A \leqslant B$, then

$$s_j(A) \leqslant s_j(B) \tag{3}$$

for $j = 1, 2, \ldots$ Moreover, for $A, B \in \mathbb{K}(\mathbb{H})$,

$$s_j(A) \leqslant s_j(B) \text{ if and only if } s_j(A \oplus A) \leqslant s_j(B \oplus B) \tag{4}$$

for $j = 1, 2, \ldots$ Every $T \in \mathbb{B}(\mathbb{H})$ can be written uniquely as $T = A + iB$, where $A = \frac{T+T^*}{2}$ and $B = \frac{T-T^*}{2i}$ are the real and imaginary parts of T, respectively, and they are self-adjoint operators. This is the cartesian decomposition of T. An operator $T \in \mathbb{B}(\mathbb{H})$ is called accretive-dissipative if in its cartesian decomposition, the operators A and B are positive.

Bhatia and Kittaneh in [11] have been proven that if $A, B \in \mathbb{K}(\mathbb{H})$, then

$$2s_j(AB^*) \leqslant s_j(A^*A + B^*B) \tag{5}$$

for $j = 1, 2, \ldots$ Zhan in [17] has been proven that if $A, B \in \mathbb{K}(\mathbb{H})$ are positive, then

$$s_j(A - B) \leqslant s_j(A \oplus B) \tag{6}$$

for $j = 1, 2, \ldots$ Tao has been proven in [16] that if $A, B, C \in \mathbb{K}(\mathbb{H})$ such that $\begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \geqslant 0$, then

$$2s_j(B) \leqslant s_j \begin{bmatrix} A & B \\ B^* & C \end{bmatrix} \tag{7}$$

for $j = 1, 2, \ldots$ Moreover, Bhatia and Kittaneh have been proven in [12] that if $A, B \in \mathbb{K}(\mathbb{H})$ such that A is self-adjoint, $B \geqslant 0$ and $\pm A \leqslant B$, then

$$s_j(A) \leqslant s_j(B \oplus B) \tag{8}$$

for $j = 1, 2, \ldots$

Many authors have conducted many studies to find effective inequalities on the subject of singular value inequalities, some of them succeed in that, Audeh and Kittaneh pointed out in [9] an equivalent form of inequality (7), which asserts that if $A, B, C \in \mathbb{K}(\mathbb{H})$, where A is self-adjoint, $B \geqslant 0$ and $\pm A \leqslant B$, then

$$2s_j(A) \leqslant s_j((B + A) \oplus (B - A)) \tag{9}$$

for $j = 1,2,\ldots$ It should be noted that inequalities in $[6,10]$ are proven for matrices, but we can ordinarily expand these inequalities to include all compact operators. While inequalities (6), (7), (8), and (9) are well-known for positive operators, we give related singular value inequalities for accretive-dissipative normal operators. For more general and comprehensive results related to singular value inequalities, we refer to [1], [2], [3], [4], [5], [6], [7], [8], [14] and [15] and the references therein.

2. Main results

To reach our findings, we need the following lemmas. The first lemma is a consequence of the min-max principle (see, e.g., [10, p. 75], or [13, p. 27]).

LEMMA 1. *Let $A,B,X \in \mathbb{B}(\mathbb{H})$ such that X is compact. Then*

$$s_j(AXB) \leqslant \|A\|\,\|B\|\,s_j(X) \tag{10}$$

for $j = 1,2,\ldots$

LEMMA 2. *Let $A,B \in \mathbb{K}(\mathbb{H})$ and let $X = A+iB$ be accretive-dissipative normal operator. Then*

$$\frac{1}{\sqrt{2}}s_j(A+B) \leqslant s_j(X) \leqslant s_j(A+B) \tag{11}$$

for $j = 1,2,\ldots$

Proof. $|A+B|^2 \leqslant 2(|A|^2+|B|^2)$. Hence

$$\begin{aligned} s_j(A+B) &= s_j(|A+B|) \leqslant \sqrt{2}s_j\left(\sqrt{|A|^2+|B|^2}\right) \\ &\qquad \text{(by inequalities (1) and (3))}, \\ &= \sqrt{2}s_j(|X|) = \sqrt{2}s_j(X) \text{ (since } A,\,B \geqslant 0) \end{aligned}$$

which proves the first inequality. Moreover, since X is normal, we have $|X| = \sqrt{X^*X} = \sqrt{A^2+B^2} \leqslant |A|+|B|$. This implies that

$$\begin{aligned} s_j(X) = s_j(|X|) \leqslant s_j(|A|+|B|) = s_j(A+B), \\ \text{(by inequalities (1) and (3))}, \end{aligned}$$

which proves the second inequality. $\square$

By making use of lemma 2 incites, we here by present the following result. Throughout this paper, when will discuss accretive-dissipative normal operators $X = A+iB$ and $Y = C+iD$, we assume that $C \geqslant A$ and $D \geqslant B$.

THEOREM 1. *Let $X,Y \in \mathbb{K}(\mathbb{H})$ be accretive-dissipative normal operators where $X = A+iB$ and $Y = C+iD$. Then*

$$s_j(Y-X) \leqslant \sqrt{2}s_j(Y \oplus X) \tag{12}$$

for $j = 1,2,\ldots$

Proof. Since $X = A + iB$ and $Y = C + iD$ are the cartesian decompositions of the accretive-dissipative normal operators X and Y respectively, it follows that A, B, C and D are positive operators. Now

$$\begin{aligned} s_j(Y - X) &= s_j((C + iD) - (A + iB)) \\ &= s_j((C - A) + i(D - B)) \\ &\leqslant s_j((C - A) + (D - B)) \text{ (by inequality (11)} \\ &= s_j((C + D) - (A + B)) \\ &\leqslant s_j((C + D) \oplus (A + B)) \text{ (by inequality (6))} \\ &\leqslant \sqrt{2} s_j((C + iD) \oplus (A + iB)) \text{ (by inequality (11)} \\ &= \sqrt{2} s_j(Y \oplus X), \end{aligned}$$

which is the conclusion of inequality (12). $\square$

EXAMPLE 1. Let $X = \begin{bmatrix} 1 & 0 \\ 0 & 2+i \end{bmatrix}$ and $Y = \begin{bmatrix} 2+2i & 0 \\ 0 & 4+4i \end{bmatrix}$ be accretive-dissipative normal operators. Then

$$s_j(Y - X) = \sqrt{13}, \sqrt{5}, 0, 0$$

and

$$\sqrt{2} s_j(Y \oplus X) = 8, 4, \sqrt{10}, \sqrt{2}.$$

The following theorem is a generalization of inequality (12).

THEOREM 2. *Let $X, Y \in \mathbb{K}(\mathbb{H})$ be accretive-dissipative normal operators and let $A, B \in \mathbb{B}(\mathbb{H})$. Then*

$$s_j(A(Y - X)B) \leqslant \sqrt{2}\, \|A\|\, \|B\|\, s_j(Y \oplus X) \tag{13}$$

for $j = 1, 2, \ldots$

Proof.

$$\begin{aligned} s_j(A(Y - X)B) &\leqslant \|A\|\, \|B\|\, s_j(Y - X), \text{ (by inequality (10))} \\ &\leqslant \sqrt{2}\, \|A\|\, \|B\|\, s_j(Y \oplus X), \text{ (by inequality (12)).} \quad \square \end{aligned}$$

By making use of Theorem 1, we here by present the following theorem.

THEOREM 3. *Let $X, Y, Z \in \mathbb{K}(\mathbb{H})$ such that $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ is accretive-dissipative normal block matix. Then*

$$\sqrt{2} s_j(Z) \leqslant s_j \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \tag{14}$$

for $j = 1, 2, \ldots$

Proof. Let $A = \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ be accretive-dissipative normal block matix, note that $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ is unitarily equivalent to $\begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix}$. To show this, note that $\begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix} = UAU^*$, where $U = \begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$. It follows that $\begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix}$ is accretive-dissipative normal block matix. Apply the result of Theorem 1, we have

$$s_j\left(\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} - \begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix}\right) \leqslant \sqrt{2}s_j\left(\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \oplus \begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix}\right).$$

This implies that

$$2s_j\left(\begin{bmatrix} 0 & Z \\ Z^* & 0 \end{bmatrix}\right) \leqslant \sqrt{2}s_j\left(\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \oplus \begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix}\right),$$

which is equivalent to saying that

$$\sqrt{2}s_j\,(Z \oplus Z^*) \leqslant s_j\left(\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \oplus \begin{bmatrix} X & -Z \\ -Z^* & Y \end{bmatrix}\right), \text{(by inequality (2))},$$

which implies that

$$\sqrt{2}s_j(Z) \leqslant s_j\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \text{ (by inequality (4)).} \quad \square$$

The following theorem is a generalization of inequality (3).

THEOREM 4. *Let $X,Y,Z \in \mathbb{K}(\mathbb{H})$ such that $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ is accretive-dissipative normal block matix, $A,B \in \mathbb{B}(\mathbb{H})$. Then*

$$\sqrt{2}s_j(AZB) \leqslant \|A\|\,\|B\|\,s_j\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \tag{15}$$

for $j = 1,2,\ldots$

Proof.

$$\begin{aligned}\sqrt{2}s_j(AZB) &\leqslant \sqrt{2}\,\|A\|\,\|B\|\,s_j(Z) \text{ (by using inequality (10))} \\ &\leqslant \|A\|\,\|B\|\,s_j\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix} \text{ (by inequality (14))},\end{aligned}$$

which is precisely inequality (15). $\square$

At this stage of our discussion, we provide the following singular value inequality for accretive-dissipative normal operators.

THEOREM 5. *Let $X,Y \in \mathbb{K}(\mathbb{H})$ where $X = A+iB$ and $Y = C+iD$ be accretive-dissipative normal operators. Then*

$$\sqrt{2}s_j(X) \leqslant s_j((Y+X) \oplus (Y-X)) \tag{16}$$

for $j = 1,2,\ldots$

Proof.

$$\begin{aligned} s_j \begin{bmatrix} Y & X \\ X & Y \end{bmatrix} &= s_j \begin{bmatrix} C+iD & A+iB \\ A+iB & C+iD \end{bmatrix} \\ &= s_j \left(\begin{bmatrix} C & A \\ A & C \end{bmatrix} + i \begin{bmatrix} D & B \\ B & D \end{bmatrix} \right) \\ &\leqslant s_j \left(\begin{bmatrix} C & A \\ A & C \end{bmatrix} + \begin{bmatrix} D & B \\ B & D \end{bmatrix} \right), \text{(by inequality (11))} \\ &= s_j \left(\begin{bmatrix} C+D & A+B \\ A+B & C+D \end{bmatrix} \right). \end{aligned}$$

It follows that

$$\begin{aligned} 2s_j(X) &= 2s_j(A+iB) \\ &\leqslant 2s_j(A+B), \text{(by inequality (11))} \\ &\leqslant s_j \left(\begin{bmatrix} C+D & A+B \\ A+B & C+D \end{bmatrix} \right), \text{(by inequality (7))} \\ &= s_j \left(\begin{bmatrix} C & A \\ A & C \end{bmatrix} + \begin{bmatrix} D & B \\ B & D \end{bmatrix} \right) \\ &\leqslant \sqrt{2}s_j \left(\begin{bmatrix} C & A \\ A & C \end{bmatrix} + i \begin{bmatrix} D & B \\ B & D \end{bmatrix} \right), \text{(by inequality (11))} \\ &= \sqrt{2}s_j \left(\begin{bmatrix} C+iD & A+iB \\ A+iB & C+iD \end{bmatrix} \right) \\ &= \sqrt{2}s_j \left(\begin{bmatrix} Y & X \\ X & Y \end{bmatrix} \right). \end{aligned}$$

for $j = 1,2,\ldots$ Our inequality has thus been substantiated. □

EXAMPLE 2. Let $X = \begin{bmatrix} 2i & 0 \\ 0 & 3+i \end{bmatrix}$ and $Y = \begin{bmatrix} 1+4i & 0 \\ 0 & 4+3i \end{bmatrix}$ be accretive-dissipative normal operators. Then

$$\sqrt{2}s_j(X) = 2\sqrt{5}, 2\sqrt{2}, 0, 0$$

and

$$s_j((Y+X) \oplus (Y-X)) = \sqrt{65}, \sqrt{37}, \sqrt{5}, \sqrt{5}.$$

The following theorem is a generalization of inequality (16).

THEOREM 6. *Let* $X, Y \in \mathbb{K}(\mathbb{H})$ *be accretive-dissipative normal operators and let* $A, B \in \mathbb{B}(\mathbb{H})$. *Then*

$$\sqrt{2} s_j(AXB) \leqslant \|A\| \, \|B\| \, s_j((Y+X) \oplus (Y-X)) \tag{17}$$

for $j = 1, 2, \ldots$

Proof.

$$\begin{aligned}\sqrt{2} s_j(AXB) &\leqslant \sqrt{2} \|A\| \, \|B\| \, s_j(X), \text{ (by using inequality (10))} \\ &\leqslant \|A\| \, \|B\| \, s_j((Y+X) \oplus (Y-X)), \text{ (by inequality (16)).}\end{aligned}$$

Inequality (17) has thus been substantiated. □

In the next theorem, we prove that inequalities (12), (14) and (16) are equivalent.

THEOREM 7. *The following statements are equivalent.*

(1) Let $X, Y \in K(H)$ *be accretive-dissipative normal operators. Then*

$$s_j(Y - X) \leqslant \sqrt{2} s_j(Y \oplus X)$$

for $j = 1, 2, \ldots$

(2) Let $X, Y, Z \in \mathbb{K}(\mathbb{H})$ *be such that* $\begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$ *is accretive-dissipative normal block matix. Then*

$$\sqrt{2} s_j(Z) \leqslant s_j \begin{bmatrix} X & Z \\ Z^* & Y \end{bmatrix}$$

for $j = 1, 2, \ldots$

(3) Let $X, Y \in \mathbb{K}(\mathbb{H})$ *be accretive-dissipative normal operators. Then*

$$\sqrt{2} s_j(X) \leqslant s_j((Y+X) \oplus (Y-X))$$

for $j = 1, 2, \ldots$

Proof. $(1) \to (2)$ This implication follows from the proof of Theorem 3.

$(2) \to (1)$ To prove inequality (12), let X and Y be accretive-dissipative normal operators, which implies that $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$ is also accretive-dissipative normal operator matrix. Define $D = \begin{bmatrix} \frac{X+Y}{2} & \frac{X-Y}{2} \\ \frac{X-Y}{2} & \frac{X+Y}{2} \end{bmatrix}$. Then D is unitarily equivalent to $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}$, note that $\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} = UDU^*$, where $U = \frac{1}{\sqrt{2}} \begin{bmatrix} I & -I \\ -I & I \end{bmatrix}$. Now, applying inequality (14) to the accretive-dissipative normal operator matrix D, we have

$$\begin{aligned}
\sqrt{2} s_j \left(\frac{X-Y}{2} \right) &\leqslant s_j \left(\begin{bmatrix} \frac{X+Y}{2} & \frac{X-Y}{2} \\ \frac{X-Y}{2} & \frac{X+Y}{2} \end{bmatrix} \right) \\
&= s_j \left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right) \\
&= s_j (X \oplus Y),
\end{aligned}$$

which is inequality (14). Thus inequalities (12) and (14) are equivalent.

$(1) \rightarrow (3)$ Let $Y-X$ and $Y+X$ substitute X and Y respectively in inequality (12). Then inequality (16) is satisfied.

$(3) \rightarrow (1)$ Let $Y-X$ and $Y+X$ substitute X and Y respectively in inequality (16). Then inequality (12) is satisfied. Thus inequalities (12) and (16) are equivalent. Theorem 7 has thus been substantiated. $\square$

THEOREM 8. *Let $X, Y \in \mathbb{K}(\mathbb{H})$ be accretive-dissipative normal operators, where $X = A + iB$ and $Y = C + iD$. Then*

$$s_j(X) \leqslant \sqrt{2} s_j(Y) \tag{18}$$

for $j = 1, 2, \ldots$

Proof.

$$\begin{aligned}
s_j(X) &= s_j(A+iB) \\
&\leqslant s_j(A+B), \text{(by inequality (11))} \\
&\leqslant s_j(C+D), \text{(by inequality (3)} \\
&\leqslant \sqrt{2} s_j(C+iD) \text{ (by inequality (11)} \\
&= \sqrt{2} s_j(Y). \quad \square
\end{aligned}$$

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Wasim Audeh
Department of Mathematics
University of Petra
Amman, Jordan
e-mail: waudeh@uop.edu.jo