Journal of
Mathematical
Inequalities
Volume 16, Number 2 (2022), 751–764
doi:10.7153/jmi-2022-16-53

SOME CHARACTERIZATIONS OF h–CONVEX FUNCTIONS

XIN JIN, BEIBEI JIN, JIANMIAO RUAN* AND XINSHENG MA

(*Communicated by T. Burić*)

Abstract. In this paper, we give some characterizations of h-convex functions, and some applications related to these functions are also obtained. According to these results, we can know better the relation between convex functions and h-convex functions.

1. Introduction and main results

Let I and J be intervals in $\mathbb{R}$, and $(0,1) \subseteq J$. In 2007, Varošanec [25] introduced the concept of h-convexity, which has received extensive attentions in recent years, see e.g. [2, 3, 6, 7, 9, 11, 12, 13, 16, 17, 18, 20, 21, 22, 23, 24, 26, 27], etc.

DEFINITION 1.1. Let $h : J \to [0,\infty)$ be a given function. We say that a nonnegative function $f : I \to [0,\infty)$ is h-convex or that f belongs to the class $SX(h,I)$, if for any $x,y \in I$ and $\alpha \in (0,1)$,

$$f(\alpha x + (1-\alpha)y) \leqslant h(\alpha)f(x) + h(1-\alpha)f(y). \tag{1.1}$$

If inequality (1.1) is reversed, then f is said to be h-concave and denoted by $f \in SV(h,I)$.

This notion generalizes the known classes of the usual convex functions, s-convex functions (in the second sense) [4], P-functions [19] and Godunova-Levin functions [8, 10, 14, 15], which are obtained by taking in (1.1) $h(\alpha) = \alpha$, $h(\alpha) = \alpha^s (s \in (0,1))$, $h(\alpha) = 1$ and and $h(\alpha) = 1/\alpha$ $(0 < \alpha \leqslant 1)$, respectively.

In 2015, Olbrýs [16] gave a characterization of h-convex functions under the condition $h(t) + h(1-t) = 1$, $t \in [0,1]$. In 2019, Delavar, Dragomir and de La Sen [5] showed a characterization of h-convex functions via Hermite-Hadamard inequality, Alomari [1] discussed the continuity of the functions and made geometric interpretation of them. This paper aims to give some other characterizations of them and provide some basic applications. In order to describe the exact setting of our focus, we first recall some notation.

Mathematics subject classification (2020): Primary 26A51; Secondary 26D07, 26D15.
Keywords and phrases: Convex function, h-convex function, equivalent characterizations.
The research was supported by the National Natural Science Foundation of China (No. 11771358) and the Teaching Reform Project of Zhejiang Province (No. jg20180252).
* Corresponding author.

DEFINITION 1.2. Let $h: J \to \mathbb{R}$. If

$$h(x)h(y) \leqslant h(xy) \tag{1.2}$$

for all $x,y \in J$, then h is said to be a supermultiplicative function. If inequality (1.2) is reversed, then h is said to be a submultiplicative function. If equality holds in (1.2), then h is said to be a multiplicative function.

DEFINITION 1.3. Let $h: J \to \mathbb{R}$. If for all $x,y \in J$,

$$h(x)+h(y) \leqslant h(x+y), \tag{1.3}$$

then h is said to be a superadditive function. If inequality (1.3) is reversed, we say that h is a subadditive function. If equality (1.3) holds, we say that h is an additive function.

In the sequel, unless otherwise specified, we assume that the function h satisfies $h(x) > 0$ $(x \neq 0)$. Now we are in a position to state our results, which are motivated by some classical properties of convex functions. Due to these conclusions, we can know better the relation between convex functions and h-convex functions.

THEOREM 1.1. *Let $h: J \to [0,\infty)$ be a continuous, supermultiplicative and superadditive function. If $f: I \to [0,\infty)$ is a continuous function, then the following four statements are equivalent*:

(i) *f is an h-convex function on I, i.e. $f \in SX(h,I)$*;
(ii) *If for all $x_1,x_2 \in I$, we have*

$$f\left(\frac{x_1+x_2}{2}\right) \leqslant h\left(\frac{1}{2}\right)(f(x_1)+f(x_2)); \tag{1.4}$$

(iii) *If for all $x_i \in I$, $i=1,2,\ldots,n$ $(n \geqslant 2)$, we have*

$$f\left(\frac{x_1+x_2+\cdots+x_n}{n}\right) \leqslant h\left(\frac{1}{n}\right)(f(x_1)+f(x_2)+\cdots+f(x_n)); \tag{1.5}$$

(iv) *If for all $x_i \in I$ and any $q_i > 0$, $i=1,2,\ldots,n$ $(n \geqslant 2)$, satisfying $q_1+q_2+\cdots+q_n = 1$, we have*

$$f(q_1x_1+q_2x_2+\cdots+q_nx_n) \leqslant h(q_1)f(x_1)+h(q_2)f(x_2)+\cdots+h(q_n)f(x_n). \tag{1.6}$$

REMARK. In fact, by the proof of theorem1.1 in Section 2, we can see that the assertions (i) and (iv) are equivalent provided h is a supermultiplicative function (see Theorem 19 in [25]), the assertions (iii) and (iv) are equivalent only if h satisfies the property of superaddition, the continuity of functions h and f are only used in proving the validity of (iv) from (iii).

THEOREM 1.2. *Let $h: J \to [0,\infty)$ be a supermultiplicative and superadditive function. Suppose that $f \in SX(h,I)$. Then for all $x_1,x_2,x_3 \in I$ with $x_1 < x_2 < x_3$ and $x_3-x_1, x_3-x_2, x_2-x_1 \in J$, we have*:

(i)
$$\frac{f(x_2)-f(x_1)}{h(x_2-x_1)} \leqslant \frac{f(x_3)-f(x_1)}{h(x_3-x_1)}; \tag{1.7}$$

(ii)
$$\frac{f(x_3)-f(x_1)}{h(x_3-x_1)} \leqslant \frac{f(x_3)-f(x_2)}{h(x_3-x_2)}; \tag{1.8}$$

(iii)
$$\frac{f(x_2)-f(x_1)}{h(x_2-x_1)} \leqslant \frac{f(x_3)-f(x_2)}{h(x_3-x_2)}. \tag{1.9}$$

As a consequence of Theorem 1.2, we have the following result.

COROLLARY 1.3. *Let h and f be as in Theorem 1.2. Then*

$$\frac{f(x_2)-f(x_1)}{h(x_2-x_1)} \leqslant \frac{f(x_3)-f(x_1)}{h(x_3-x_1)} \leqslant \frac{f(x_3)-f(x_2)}{h(x_3-x_2)}$$

holds for all $x_1,x_2,x_3 \in I$ with $x_1 < x_2 < x_3$ and $x_3-x_1, x_3-x_2, x_2-x_1 \in J$.

THEOREM 1.4. *Let $h : J \to [0,\infty)$ be a submultiplicative and subadditive function. Suppose that $f : I \to \mathbb{R}$ be a nonnegative function. If one of the inequalities (1.7), (1.8) or (1.9) holds, then $f \in SX(h,I)$.*

Combining Theorem 1.2 and Theorem 1.4, we conclude the following representation for h-convex functions.

THEOREM 1.5. *Let $h : J \to [0,\infty)$ be a multiplicative and additive function. Suppose that $f : I \to \mathbb{R}$ be a nonnegative function. Then $f \in SX(h,I)$ if and only if one of the inequalities (1.7), (1.8) or (1.9) holds,*

The rest of this paper is organized as follows. In Section 2, we will prove the above theorems. In section 3, we will provide some applications of h-convex functions related continuity, differentiability, integrability and so on.

2. Proof of theorems

Proof of Theorem 1.1. In order to prove the theorem, we will take the following strategy: (i) $\Leftrightarrow$ (iv), (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv). Since (iv) $\Rightarrow$ (i) is clear and (i) $\Rightarrow$ (iv) is a special case of Theorem 19 in [25], it remains to show that (ii) $\Leftrightarrow$ (iii) and (iii) $\Leftrightarrow$ (iv).

(ii)$\Leftrightarrow$(iii): It is obvious that (iii) $\Rightarrow$ (ii). Now we will show that (ii) $\Rightarrow$ (iii) by taking two steps as follows: ① Use the forward induction to establish the inequality (1.5) holds for a sub-sequence of the natural numbers $n = 2^k$, $k \in \mathbb{N}$. ② Use the backward induction to prove (1.5) holds for $n = k$ if it is true for $n = k+1$.

① *Step* 1: If $k=1$ i.e. $n=2$, inequality (1.5) is derived directly by (1.4). Now we prove (1.5) holds for $k=2$ i.e. $n=4$. Let $x_1,x_2,x_3,x_4 \in I$. It follows from (1.4) that

$$\begin{aligned} f\left(\frac{x_1+x_2+x_3+x_4}{4}\right) &= f\left(\frac{\frac{x_1+x_2}{2}+\frac{x_3+x_4}{2}}{2}\right) \\ &\leqslant h\left(\frac{1}{2}\right)\left[f\left(\frac{x_1+x_2}{2}\right)+f\left(\frac{x_3+x_4}{2}\right)\right] \\ &\leqslant h^2\left(\frac{1}{2}\right)\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)\right] \\ &\leqslant h\left(\frac{1}{4}\right)\left[f(x_1)+f(x_2)+f(x_3)+f(x_4)\right], \end{aligned}$$

where the last inequality is obtained by the supermultiplicativity of h. This means that (1.5) holds for $n=4$. For any natural number k, we repeat similar process k times as above and obtain that

$$\begin{aligned} f\left(\frac{x_1+x_2+\cdots+x_{2^k}}{2^k}\right) &\leqslant h^k\left(\frac{1}{2}\right)\left[f(x_1)+f(x_2)+\cdots+f(x_{2^k})\right] \\ &\leqslant h\left(\frac{1}{2^k}\right)\left[f(x_1)+f(x_2)+\cdots+f(x_{2^k})\right], \end{aligned}$$

which implies that (1.5) holds for $n=2^k,\ k\in\mathbb{N}$.

② *Step* 2: Let $A=\frac{x_1+x_2+\cdots+x_k}{k}$. Then $x_1+x_2+\cdots+x_k=kA$ and

$$A=\frac{x_1+x_2+\cdots+x_k+A}{k+1}.$$

If (1.5) holds for $n=k+1$, then

$$\begin{aligned} f(A) &= f\left(\frac{x_1+x_2+\cdots+x_k+A}{k+1}\right) \\ &\leqslant h\left(\frac{1}{k+1}\right)\left[f(x_1)+f(x_2)+\cdots+f(x_k)+f(A)\right], \end{aligned}$$

which tells us that

$$\left[1-h\left(\frac{1}{k+1}\right)\right]f(A)\leqslant h\left(\frac{1}{k+1}\right)\left[f(x_1)+f(x_2)+\cdots+f(x_k)\right].$$

Multiplying by $h\left(\frac{1}{k}\right)$ on both sides of the proceeding inequality yields that

$$\begin{aligned} &\left[h\left(\frac{1}{k}\right)-h\left(\frac{1}{k}\right)h\left(\frac{1}{k+1}\right)\right]f(A) \\ &\leqslant h\left(\frac{1}{k}\right)h\left(\frac{1}{k+1}\right)\left[f(x_1)+f(x_2)+\cdots+f(x_k)\right]. \end{aligned} \tag{2.1}$$

Noting that $h: J \to [0,\infty)$ is a supermultiplicative and superadditive function, we have

$$h\left(\frac{1}{k^2+k}\right) \geqslant h\left(\frac{1}{k}\right)h\left(\frac{1}{k+1}\right), h\left(\frac{1}{k}\right) \geqslant h\left(\frac{1}{k^2+k}\right)+h\left(\frac{1}{k+1}\right),$$

and

$$\begin{aligned} h\left(\frac{1}{k}\right)-h\left(\frac{1}{k}\right)h\left(\frac{1}{k+1}\right) &\geqslant h\left(\frac{1}{k^2+k}\right)+h\left(\frac{1}{k+1}\right)-h\left(\frac{1}{k}\right)h\left(\frac{1}{k+1}\right) \\ &\geqslant h\left(\frac{1}{k+1}\right) \geqslant 0. \end{aligned} \tag{2.2}$$

Therefore, we infer from (2.1) and (2.2) that

$$f(A) \leqslant h\left(\frac{1}{k}\right)\left[f(x_1)+f(x_2)+\cdots+f(x_k)\right].$$

This means that (1.5) holds for $n=k$.

Thus we complete the proof of the assertion by step 1 and step 2.

(iii)$\Leftrightarrow$(iv): Since (iv) $\Rightarrow$ (iii) is easily obtained by taking $q_i=\frac{1}{n}$, $i=1,2,\ldots,n$ in (1.6), it remains to verify that (iii) $\Rightarrow$ (iv).

① Firstly, we consider the case of that all q_i are rational numbers. Let $q_i=\frac{a_i}{m}$, where a_i, $i=1,2,\ldots,n$ are positive integers and $a_1+a_2+\cdots+a_n=m$. Then

$$f(q_1x_1+q_2x_2+\cdots+q_nx_n)=f\left(\underbrace{\frac{x_1}{m}+\frac{x_1}{m}+\cdots+\frac{x_1}{m}}_{a_1}+\cdots+\underbrace{\frac{x_n}{m}+\frac{x_n}{m}+\cdots+\frac{x_n}{m}}_{a_n}\right).$$

It follows from (1.5) that

$$\begin{aligned} &f(q_1x_1+q_2x_2+\cdots+q_nx_n) \\ &\leqslant \underbrace{h\left(\frac{1}{m}\right)f(x_1)+\cdots+h\left(\frac{1}{m}\right)f(x_1)}_{a_1}+\cdots+\underbrace{h\left(\frac{1}{m}\right)f(x_n)+\cdots+h\left(\frac{1}{m}\right)f(x_n)}_{a_n}. \end{aligned}$$

Since h possesses the property of superaddition, we obtain

$$\begin{aligned} f(q_1x_1+q_2x_2+\cdots+q_nx_n) &\leqslant h\left(\frac{a_1}{m}\right)f(x_1)+h\left(\frac{a_2}{m}\right)f(x_2)+\cdots+h\left(\frac{a_n}{m}\right)f(x_n) \\ &= h(q_1)f(x_1)+h(q_2)f(x_2)+\cdots+h(q_n)f(x_n). \end{aligned}$$

② Secondly, we consider the case of that there exists some q_i are irrational numbers. Without loss of generality, we assume that the first numbers q_k $(0<k\leqslant n)$ are irrational, then there are rational sequences $\{q_{i,l}\}$ such that

$$\lim_{l\to\infty} q_{i,l}=q_i \ (i=1,2,\ldots,k,\ 0<k\leqslant n). \tag{2.3}$$

According to the argument in ①, we have

$$\begin{aligned}&f\left(q_{1,l}x_1+q_{2,l}x_2+\cdots+q_{k,l}x_k+q_{k+1}x_{k+1}+\cdots+q_nx_n\right)\\ \leqslant\ & h\left(q_{1,l}\right)f\left(x_1\right)+\cdots+h\left(q_{k,l}\right)f\left(x_k\right)+h\left(q_{k+1}\right)f\left(x_{k+1}\right)+\cdots+h\left(q_{n,l}\right)f\left(x_n\right).\end{aligned}$$

Letting $l\to\infty$, the continuity of the functions f and h tells us that

$$f\left(q_1x_1+q_2x_2+\cdots+q_nx_n\right)\leqslant h\left(q_1\right)f\left(x_1\right)+h\left(q_2\right)f\left(x_2\right)+\cdots+h\left(q_n\right)f\left(x_n\right),$$

which finishes the proof of Theorem 1.1. □

Proof of Theorem 1.2. (i) Since $x_1,\ x_2,\ x_3\in I$ and $x_1<x_2<x_3$, we have

$$\frac{x_3-x_2}{x_3-x_1},\ \frac{x_2-x_1}{x_3-x_1}\in(0,1)\subset J$$

and

$$\frac{x_3-x_2}{x_3-x_1}+\frac{x_2-x_1}{x_3-x_1}=1. \tag{2.4}$$

In view of the supermultiplicativity of h,

$$h\left(x_3-x_2\right)\geqslant h\left(\frac{x_3-x_2}{x_3-x_1}\right)h\left(x_3-x_1\right), \tag{2.5}$$

$$h\left(x_2-x_1\right)\geqslant h\left(\frac{x_2-x_1}{x_3-x_1}\right)h\left(x_3-x_1\right). \tag{2.6}$$

Let $\alpha=\frac{x_3-x_2}{x_3-x_1}$. Then (2.4) implies that

$$x_2=\alpha x_1+(1-\alpha)x_3.$$

It follows from (2.5), (2.6) and the definition of the h-convex function that

$$\begin{aligned}f(x_2)&\leqslant h\left(\frac{x_3-x_2}{x_3-x_1}\right)f\left(x_1\right)+h\left(\frac{x_2-x_1}{x_3-x_1}\right)f\left(x_3\right)\\ &\leqslant\frac{h\left(x_3-x_2\right)}{h\left(x_3-x_1\right)}f\left(x_1\right)+\frac{h\left(x_2-x_1\right)}{h\left(x_3-x_1\right)}f\left(x_3\right),\end{aligned}$$

which yields that

$$\frac{f(x_2)-f(x_1)}{h(x_2-x_1)}\leqslant\frac{1}{h(x_3-x_1)}\left(\frac{h(x_3-x_2)-h(x_3-x_1)}{h(x_2-x_1)}f(x_1)+f(x_3)\right). \tag{2.7}$$

Recalling that h is a superadditive function, we have

$$h\left(x_3-x_1\right)\geqslant h\left(x_3-x_2\right)+h\left(x_2-x_1\right).$$

This means that

$$\frac{h(x_3-x_2)-h(x_3-x_1)}{h(x_2-x_1)}\leqslant-1. \tag{2.8}$$

Since $f \geqslant 0$, (2.7) and (2.8) tell us that

$$\frac{f(x_2)-f(x_1)}{h(x_2-x_1)} \leqslant \frac{f(x_3)-f(x_1)}{h(x_3-x_1)}.$$

This completes the proof of (i).

The inequalities (1.8) and (1.9) can be obtained by a similar argument as above, we leave the details to readers. □

Proof of Theorem 1.4. Without loss of generality, we will prove that $f \in SX(h,I)$ if inequality (1.7) holds. The proof of the other two is only notation difference and it does not require new ideas.

If (1.7) holds, then for any $x_1, x_2, x_3 \in I$, with $x_1 < x_2 < x_3$ and $x_3 - x_1$, $x_3 - x_2$, $x_2 - x_1 \in J$,

$$\frac{f(x_2)-f(x_1)}{h(x_2-x_1)} \leqslant \frac{f(x_3)-f(x_1)}{h(x_3-x_1)}.$$

This implies that

$$f(x_2) \leqslant \frac{h(x_2-x_1)}{h(x_3-x_1)} f(x_3) + \frac{h(x_3-x_1)-h(x_2-x_1)}{h(x_3-x_1)} f(x_1).$$

Since h possesses the property of subaddition, the proceeding inequality yields that

$$f(x_2) \leqslant \frac{h(x_2-x_1)}{h(x_3-x_1)} f(x_3) + \frac{h(x_3-x_2)}{h(x_3-x_1)} f(x_1). \tag{2.9}$$

On the other hand, the submultiplicativity of h shows that

$$h(x_3-x_2) = h\left[\frac{x_3-x_2}{x_3-x_1}(x_3-x_1)\right] \leqslant h\left(\frac{x_3-x_2}{x_3-x_1}\right) h(x_3-x_1),$$

i.e.

$$\frac{h(x_3-x_2)}{h(x_3-x_1)} \leqslant h\left(\frac{x_3-x_2}{x_3-x_1}\right). \tag{2.10}$$

Similarly,

$$\frac{h(x_2-x_1)}{h(x_3-x_1)} \leqslant h\left(\frac{x_2-x_1}{x_3-x_1}\right). \tag{2.11}$$

Combining (2.9), (2.10) and (2.11), we conclude that

$$f(x_2) \leqslant h\left(\frac{x_2-x_1}{x_3-x_1}\right) f(x_3) + h\left(\frac{x_3-x_2}{x_3-x_1}\right) f(x_1). \tag{2.12}$$

Let $x, y \in I$ and $\alpha \in (0,1)$. If $x < y$, we define x_1, x_2, x_3 as following

$$x_1 = x,\ x_2 = \alpha x + (1-\alpha) y,\ x_3 = y,$$

then $\frac{x_3-x_2}{x_3-x_1} = \alpha$ and $\frac{x_2-x_1}{x_3-x_1} = 1-\alpha$. Therefore (2.12) gives that

$$f(\alpha x + (1-\alpha y)) \leqslant h(\alpha) f(x) + h(1-\alpha) f(y). \tag{2.13}$$

If $y < x$, we prove (2.13) in a similar way. If $x = y$, then from subadditivity and submultiplicativity of function h we have

$$1 \leqslant h(1) = h(\alpha + (1-\alpha)) \leqslant h(\alpha) + h(1-\alpha),$$

which implies that

$$f(\alpha x + (1-\alpha)y) = f(x)\,(h(\alpha) + h(1-\alpha)) = h(\alpha)f(x) + h(1-\alpha)f(y),$$

i.e. (2.13) holds. Thus we finish the proof of Theorem 1.4. □

3. Some applications of h-convex functions

In this section, we will give some interesting properties for nonnegative h-convex functions.

THEOREM 3.1. *Let $h : J \to [0,\infty)$ be a supermultiplicative and superadditive function with $\lim_{t\to 0^+} \frac{h(t)}{t} = A > 0$. Suppose that $f \in SX(h,I)$. Then:*

(i) *For any interior point x_0 in I, the one-sided derivatives $f'_-(x_0)$, $f'_+(x_0)$ exist with $f'_-(x_0) \leqslant f'_+(x_0)$, and f'_-, f'_+ are both increasing on any open subinterval of I.*

(ii) *The functions f'_-, f'_+ are Riemann integrable on any open interval $K \subset I$. Furthermore, for any $a,b \in K$, we have*

$$\int_a^b f'_-(t)dt = f(b) - f(a) = \int_a^b f'_+(t)dt. \tag{3.1}$$

Proof. (i) For any interior point x_0 of I, there is an interval $x_0 \in (c,d) \subset [c,d] \subset I$ satisfying $0 < d - c < 1$. Define

$$F(t) = \begin{cases} \dfrac{f(x_0+t) - f(x_0)}{h(t)}, & 0 < t < d - x_0, \\ \dfrac{f(x_0) - f(x_0+t)}{h(-t)}, & c - x_0 < t < 0. \end{cases}$$

Then we infer from Corollary 1.3 that the function F is increasing on the intervals $(c-x_0, 0) \bigcup (0, d-x_0) \subset J$. According to the definition of F and using Corollary 1.3 again, we can check that the set $\{F(t) : 0 < t < d - x_0\}$ has lower-bound and $\{F(t) : c - x_0 < t < 0\}$ has upper-bound. Thus, the Monotone Convergence Criterion for Real Numbers tells us that the one-sided limits $\lim_{t\to 0^+} F(t)$ and $\lim_{t\to 0^-} F(t)$ exist, and

$$\lim_{t\to 0^+} F(t) = \lim_{t\to 0^+} \frac{f(x_0+t) - f(x_0)}{h(t)} = f'_+(x_0) \lim_{t\to 0^+} \frac{t}{h(t)} = \frac{f'_+(x_0)}{A}, \tag{3.2}$$

$$\lim_{t\to 0^-} F(t) = \lim_{t\to 0^-} \frac{f(x_0) - f(x_0+t)}{h(-t)} = f'_-(x_0) \lim_{t\to 0^-} \frac{-t}{h(-t)} = \frac{f'_-(x_0)}{A}. \tag{3.3}$$

In view of the nonnegativity of h, we have $A > 0$. Thus, (3.2), (3.3) and Corollary 1.3 yield that

$$f'_-(x_0) \leqslant f'_+(x_0).$$

By virtue of the above discussion and Corollary 1.3 again, it is not difficult to check that f'_-, f'_+ are both increasing on any open interval of I.

(ii) It is clear that f'_-, f'_+ are Riemann integrable on K by the monotonicity properties of the two functions. Now we pay more attention to proving (3.1) and just consider the right-hand side equality, since the left-hand side is obtained by a similar argument.

Suppose that $a < b$. Let $\{x_i\}_{i=0}^n$ be a partition of $[a,b]$, i.e. $a = x_0 < x_1 < \ldots < x_n = b$. Then

$$f(b) - f(a) = \sum_{i=1}^{n} [f(x_i) - f(x_{i-1})].$$

It follows from Corollary 1.3 and (3.2) that

$$f(x_i) - f(x_{i-1}) \geqslant \frac{f'_+(x_{i-1})}{A} h(x_i - x_{i-1}) = f'_+(x_{i-1})(x_i - x_{i-1}) \frac{h(x_i - x_{i-1})}{(x_i - x_{i-1})A},$$

which means that

$$f(b) - f(a) \geqslant \sum_{i=1}^{n} f'_+(x_{i-1})(x_i - x_{i-1}) \frac{h(x_i - x_{i-1})}{(x_i - x_{i-1})A}. \tag{3.4}$$

Similarly,

$$f(b) - f(a) \leqslant \sum_{i=1}^{n} f'_+(x_i)(x_i - x_{i-1}) \frac{h(x_i - x_{i-1})}{(x_i - x_{i-1})A}. \tag{3.5}$$

Denote $\delta = \max_{i=1,\ldots,n}\{x_i - x_{i-1}\}$. Since f'_+ is integrable on K, letting $\delta \to 0$ and noting that $\lim_{t\to 0^+} \frac{h(t)}{t} = A$, (3.4) and (3.5) tell us that

$$\int_a^b f'_+(t)dt \leqslant f(b) - f(a) \leqslant \int_a^b f'_+(t)dt,$$

that is

$$\int_a^b f'_+(t)dt = f(b) - f(a).$$

Thus we finish the proof of Theorem 3.1. $\square$

As a consequence of Theorem 3.1, we have the following results related the continuity and differentiability of h-convex functions.

THEOREM 3.2. *Let h and f be as in Theorem 3.1. Then:*
(i) *f is continuous on any interior point in I.*
(ii) *f is differentiable almost everywhere on I.*

Proof. Since (i) is derived directly by Theorem 3.1 (i), it remains to show that statement (ii) is valid. Without loss of generality, we may assume that I is an open interval in $\mathbb{R}$. Thus, in order to prove the theorem, it sufficient to prove that f is differentiable almost everywhere on any closed interval $[a,b]$ of I. Due to (3.1),

$$\int_a^b \left[f'_+(t) - f'_-(t)\right] dt = 0. \tag{3.6}$$

On the other hand, it is well known that every Riemann integrable function is also Lebesgue integrable on $[a,b]$, which means that $f'_+ - f'_-$ is Lebesgue integrable on $[a,b]$ by Theorem 3.1 (ii). Recalling that $f'_-(t) \leqslant f'_+(t), t \in [a,b]$ by Theorem 3.1 (i), (3.6) and the above argument yield that

$$f'_+(t) - f'_-(t) = 0,\ a.e.\ \ t \in [a,b],$$

which implies that f is differentiable a.e. on $[a,b]$. Thus we finish the proof of Theorem 3.2. $\square$

THEOREM 3.3. *Let h and f be as in Theorem 3.1. Then for any interior point x_0 in I, there is a constant $\alpha \geqslant 0$ satisfies*

$$f(x) \geqslant -\alpha h(x_0 - x) + f(x_0),\ \textit{for all } x < x_0,\ x \in I \textit{ and } x_0 - x \in J, \tag{3.7}$$

$$f(x) \geqslant \alpha h(x - x_0) + f(x_0),\ \textit{for all } x > x_0,\ x \in I \textit{ and } x - x_0 \in J. \tag{3.8}$$

Proof. Let x_0 be an interior point in I. For any $x < x_0$, $x \in I$ and $x_0 - x \in J$, it follows from (3.3) and Theorem 1.2 that

$$\frac{f(x_0) - f(x)}{h(x_0 - x)} \leqslant \frac{f'_-(x_0)}{A}. \tag{3.9}$$

Thus for any positive constant $\alpha \geqslant \frac{f'_-(x_0)}{A}$, (3.9) shows that

$$f(x) \geqslant -\frac{f'_-(x_0)}{A} h(x_0 - x) + f(x_0) \geqslant -\alpha h(x_0 - x) + f(x_0). \tag{3.10}$$

Similarly, we take $\alpha \leqslant \frac{f'_+(x_0)}{A}$, then for any $x > x_0$, $x \in I$ and $x - x_0 \in J$, (3.2) and Theorem 1.2 imply that

$$f(x) \geqslant \frac{f'_+(x_0)}{A} h(x - x_0) + f(x_0) \geqslant \alpha h(x - x_0) + f(x_0). \tag{3.11}$$

Noting that $f'_-(x_0) \leqslant f'_+(x_0)$, we finish the proof of Theorem 3.3 by (3.10) and (3.11). $\square$

As a consequence of Theorem 1.5 and Theorem 3.3, we arrive at the following equivalent characterization for h-convex functions.

THEOREM 3.4. *Let $h: J \to [0,\infty)$ be a multiplicative and additive function with $\lim_{t\to 0^+} \frac{h(t)}{t} = A > 0$. Suppose that $f: I \to \mathbb{R}$ be a nonnegative function. Then $f \in SX(h,I)$ if and only if for any interior point x_0 in I, there is a constant $\alpha \geqslant 0$ such that inequalities (3.7) and (3.8) are both valid.*

THEOREM 3.5. *Let $h: J \to [0,\infty)$ be bounded on $(0,1)$ and $f \in SX(h,I)$. Then f is bounded on any closed subinterval of I.*

Proof. Let $[a,b] \subset I$ be a closed subinterval of I and $M = \max\{f(a), f(b)\}$. By the assumption of h, there exists a positive constant C satisfying $0 \leqslant h(x) \leqslant C$, $x \in (0,1)$.

Firstly, we prove that f is upper-bounded on $[a,b]$. In fact, for any $x \in (a,b) \subset I$, there is $\alpha \in (0,1)$ such that $x = \alpha a + (1-\alpha)b$. Therefore,

$$f(x) \leqslant h(\alpha)f(a) + h(1-\alpha)f(b) \leqslant 2CM,$$

which means that f is a upper-bounded function on $[a,b]$.

Next, we show that f is lower-bounded on $[a,b]$. For any $x \in (a,b)$, there is some $t_x \in (-(b-a)/2, (b-a)/2)$ such that

$$x = \frac{a+b}{2} + t_x.$$

According to the definition of h-convex functions, we have

$$\begin{aligned} f\left(\frac{a+b}{2}\right) &= f\left(\frac{1}{2}\left[\left(\frac{a+b}{2}+t_x\right)+\left(\frac{a+b}{2}-t_x\right)\right]\right) \\ &\leqslant h\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}+t_x\right) + h\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}-t_x\right). \end{aligned}$$

That is

$$f\left(\frac{a+b}{2}\right) \leqslant h\left(\frac{1}{2}\right) f(x) + h\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}-t_x\right),$$

which implies that

$$f(x) \geqslant \max\left\{0, \frac{f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right) f\left(\frac{a+b}{2}-t_x\right)}{h\left(\frac{1}{2}\right)}\right\} \geqslant \max\left\{0, \frac{f\left(\frac{a+b}{2}\right) - h\left(\frac{1}{2}\right) M}{h\left(\frac{1}{2}\right)}\right\}$$

holds for all $x \in (a,b)$. Thus we finish the proof of Theorem 3.5. □

THEOREM 3.6. *Let $h: (0,1) \to [0,\infty)$ be a superadditive and continuous function. If $f \in SX(h,(a,b))$ is a bounded function on (a,b), then the limits $\lim_{x\to b^-} f(x)$ and $\lim_{x\to a^+} f(x)$ exist.*

Proof. We only prove $\lim_{x\to b^-} f(x)$ exits, the existence of $\lim_{x\to a^+} f(x)$ can be obtained by a similar discussion. For any $x \in (c,b)$, we choose some fixed points $x_0, x_1 \in (d,b)$ with $x_0 < x_1 < x$, where $c = \max\left\{\frac{a+b}{2}, b-\frac{1}{4}\right\}$, $d = \max\left\{\frac{a+b}{2}, b-\frac{1}{2}\right\}$. Obviously, $0 < b-c \leqslant b-d < 1$. Corollary 1.3 means that the function

$$G(x) := \frac{f(x)-f(x_0)}{h(x-x_0)}$$

is increasing on (c,b). Since f is a bounded function on (a,b) and h is a superadditive function on $[0,1]$,

$$G(x) \leqslant \frac{M-f(x_0)}{h(x_1-x_0)} \tag{3.12}$$

holds for $x \in (c,b)$, where $M = \max\{f(x) : x \in (a,b)\}$. Then (3.12) means that G is a bounded function on the interval (c,b). Therefore, the Monotone Convergence Theorem implies that the one-sided limit $\lim_{x\to b^-} G(x)$ exists. By the definition of G and the continuity of h,

$$\lim_{x\to b^-} f(x) = \lim_{x\to b^-} \left[h(x-x_0)G(x)+f(x_0)\right] = h(b-x_0)\lim_{x\to b^-} G(x)+f(x_0),$$

which completes the proof of Theorem 3.6. $\square$

THEOREM 3.7. *Let h and f be as in Theorem 3.1. Suppose that $g : K \to I$ is a Riemann integrable function, where K is an interval in $\mathbb{R}$. Then for any $[a,b] \subset K$,*

$$f\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \leqslant \frac{A}{b-a}\int_a^b f(g(t))dt.$$

Proof. Since I is an interval , by Theorem 3.2, we may assume that f is continuous on I. Therefore, by the integrability of g and the continuity of f, the composition of $f(g(t)), t \in [a,b]$ is also integrable. Let $t_i = \frac{i(b-a)}{n}$ $(i = 0,1,\dots,n)$ be a partition of $[a,b]$ and denote $g_i = g(t_i)$. In view of Theorem 1.1,

$$f\left(\frac{1}{n}\sum_{i=1}^{n} g_i\right) \leqslant h\left(\frac{1}{n}\right)\sum_{i=1}^{n} f(g_i) = nh\left(\frac{1}{n}\right)\frac{1}{n}\sum_{i=1}^{n} f(g_i).$$

Letting $n\to\infty$, the continuity of f and the integrability of $f(g(\cdot))$ on $[a,b]$ imply that

$$f\left(\frac{1}{b-a}\int_a^b g(t)dt\right) \leqslant \frac{A}{b-a}\int_a^b f(g(t))dt,$$

which completes the proof of Theorem 3.7. $\square$

Thanks to the anonymous reviewer's reminder, we know that Theorem 3.7 was firstly established in [28] with slightly modified assumptions.

Acknowledgements. The authors would like to express their deep thanks to the referees for many helpful comments and suggestions.

REFERENCES

[1] M. ALOMARI, *A note on h-convex functions*, e-J. Anal. Appl. Math., **2019** (1) (2019), 55–67.

[2] P. BURAI AND A. HÁZY, *On approximately h-convex functions*, J. Convex Anal., **18** (2) (2011), 1–9.

[3] M. BOMBARDELLI AND S. VAROŠANEC, *Properties of h-convex functions related to the Hermite-Hadamard-Fejér inequalities*, Comput. Math. Appl., **58** (2009), 1869–1877.

[4] W. BRECKNER, *Stetigkeitsaussagen für eine Klasse verallgemeinerter Konvexer funktionen in topologischen linearen Raumen*, Publ. Inst. Math., **23** (1978), 13–20.

[5] M. DELAVAR, S. DRAGOMIR AND M. DE LA SEN, *A note on characterization of h-convex functions via Hermite-Hadamard type inequality*, Probl. Anal. Issues Anal., **8** (26), (2019), 28–36.

[6] S. DRAGOMIR, *Inequalities of Jensen type for h-convex functions on linear spaces*, Math. Moravica, **19** (1) (2015), 107–121.

[7] S. DRAGOMIR, *Inequalities of Hermite-Hadamard type for h-convex functions on linear spaces*, Proyecciones J. Math., **34** (4) (2015), 323–341.

[8] S. DRAGOMIR, J. PEČARIĆ AND L. PERSSON, *Some inequalities of Hadamard type*, Soochow J. Math., **21** (1995), 335–341.

[9] M. FENG, J. RUAN AND X. MA, *Hermite-Hadamard inequalities for multidimentional strongly h-convex functions*, Math. Inequal. Appl., **24** (4) (2021), 897–911.

[10] E. GODUNOVA AND V. LEVIN, *Neravenstva dlja funkcii širokogo klassa, soderžaščego vypuklye, monotonnye i nekotorye drugie vidy funkcii*, Vyčislitel. Mat. i. Mat. Fiz. Mežvuzov. Sb. Nauč. Trudov, MGPI, Moskva, (1985), 138–142.

[11] A. HÁZY, *Bernstein-Doetsch type results for h-convex functions*, Math. Inequal. Appl., **14** (3) (2011), 499–508.

[12] M. LATIF, *On some inequalities for h-convex functions*, Int. J. Math. Anal., **4** (30) (2010), 1473–1482.

[13] M. MATŁOKA, *On Hadamard's inequality for h-convex function on a disk*, Appl. Math. Comput., **235** (2014), 118–123.

[14] D. MITRINOVIĆ AND J. PEČARIĆ, *Note on a class of functions of Godunova and Levin*, C. R. Math. Rep. Acad. Sci. Can., **12** (1990), 33–36.

[15] D. MITRINOVIĆ, J. PEČARIĆ AND A. FINK, *Classical and new inequalities in analysis*, Kluwer Academic, Dordrecht, 1993.

[16] A. OLBRYŚ, *Representation theorems for h-convexity*, J. Math. Anal. Appl., **426** (2015), 986–994.

[17] M. ÖZDEMIR, M. GÜRBÜZ, AND A. AKDEMIR, *Inequalities for h-convex functions via further properties*, RGMIA Research Report Collection, **14**, (22) (2011).

[18] M. ÖZDEMIR, M. TUNC AND M. GÜRBÜZ, *Definitions of h-logaritmic, h-geometric and h-multi convex functions and some inequalities related to them*, arXiv:1211.2570v1, (2012), 1–8.

[19] C. PEARCE AND A. RUBINOV, *P-functions, quasi-convex functions and Hadamard-type inequalities*, J. Math. Anal. Appl., **240** (1999), 92–104.

[20] M. SARIKAYA, A. SAGLAM AND H. YILDRIM, *On some Hadamard-type inequalities for h-convex functions*, J. Math. Inequal., **2** (3) (2008), 335–341.

[21] M. TUNC, *On some integral inequalities via h-convexity*, Miskolc Math. Notes, **14** (3) (2013), 1041–1057.

[22] M. TUNC, *Ostrowski-type inequalities via h-convex functions with applications to special means*, J. Inequal. Appl., **326** (2013), 1–10.

[23] M. TUNC AND H. YILDRIM, *New estimations for h-convex functions via further properties*, arXiv:1202.2207v1, (2012), 1–11.

[24] M. TUNC, H. YILDRIM AND A. EKINCI, *On some inequalities of Simpson's type via h-convex functions*, Hacettepe J. Math. Stat., **42** (4) (2013), 309–317.

[25] S. VAROŠANEC, *On h-convexity*, J. Math. Anal. Appl., **326** (2007), 303–311.

[26] T. WANG, M. FENG, J. RUAN AND B. SHAO, *Hermite-Hadamard inequalities for co-ordinated $\log - h$-convex functions*, J. Math. Inequal., **15** (1) (2021), 31–46.

[27] X. WANG, J. RUAN AND X. MA, *On the Hermite-Hadamard inequalities for h-convex functions on balls and ellipsoids*, Filomat, **33** (18) (2019), 5817–5886.
[28] D. ZHAO, T. AN, G. YE AND W. LIU, *New Jensen and Hermite–Hadamard type inequalities for h-convex interval-valued functions*, J. Inequal. Appl., **302** (2018), 1–14.

(Received August 1, 2021)

Xin Jin
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: 1031141781@qq.com

Beibei Jin
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: 804484864@qq.com

Jianmiao Ruan
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: rjmath@163.com

Xinsheng Ma
Department of Mathematics
Zhejiang International Studies University
Hangzhou 310014, China
e-mail: xsma@zisu.edu.cn