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IMPROVED JENSEN–TYPE INEQUALITIES VIA QUADRATIC INTERPOLATION AND APPLICATIONS

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Abstract. In the paper (J. Math. Inequal. 11 (2017), no. 2, 301–322.), Choi, Krnić and Pečarić used the linear interpolation to improve Jensen-type inequalities for convex functions. Their method also provides a unified approach with simpler proofs for many recent results related to Young-type and Heinz-type inequalities. In this paper, we propose new refinements of Jensen-type inequalities established by Choi, Krnić and Pečarić via the quadratic interpolation of convex functions. We also give Young-type and Heinz-type inequalities for both scalars and operator cases as an application.

1. Introduction

The classical Young inequality states that for all $a,b>0$ and $\nu\in[0,1]$, we have

$$(1-\nu)a+\nu b\geqslant a^{1-\nu}b^{\nu}; \tag{1}$$

moreover, the equality in (1) occurs if and only if $a=b$. The inequality (1) is also known in the literature as the weighted arithmetic-geometric mean inequality.

A striking refinement of (1) was established by Kittaneh and Manasrah [6] in 2010 as

$$(1-\nu)a+\nu b\geqslant a^{1-\nu}b^{\nu}+r_0(\nu)(\sqrt{a}-\sqrt{b})^2, \tag{2}$$

here $r_0(\nu)=\min\{\nu,1-\nu\}$. One year later, these two authors themself also gave in [7] a reversed version of (1) of the form

$$(1-\nu)a+\nu b\leqslant a^{1-\nu}b^{\nu}+R_0(\nu)(\sqrt{a}-\sqrt{b})^2, \tag{3}$$

where $R_0(\nu)=\max\{\nu,1-\nu\}$. The equality sign in the inequalities (2) and (3) also happen when $a=b$.

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The proof technique of (2) is to apply the inequality (1) to the difference between the weighted arithmetic mean $(1-\nu)a+\nu b$ and the quantity $r_0(\nu)(\sqrt{a}-\sqrt{b})^2$. Meanwhile, to prove (3), Kittaneh and Manasrah again used the inequality (2) and another inequality involving Heinz mean given by

$$H_\nu(a,b) := \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2} \geqslant \sqrt{ab}. \tag{4}$$

These proofs can not explain fully the source of quantities $r_0(\nu)(\sqrt{a}-\sqrt{b})^2$ and $R_0(\nu)(\sqrt{a}-\sqrt{b})^2$ in the inequalities (2) and (3), respectively. However, they will become clear when we apply the famous Jensen-type inequalities

$$\begin{aligned} r_0(\nu)\Big(\frac{f(0)+f(1)}{2}-f\Big(\frac{1}{2}\Big)\Big) &\leqslant (1-\nu)f(0)+\nu f(1)-f(\nu) \\ &\leqslant R_0(\nu)\Big(\frac{f(0)+f(1)}{2}-f\Big(\frac{1}{2}\Big)\Big) \end{aligned} \tag{5}$$

to the convex function $f(\nu)=a^{1-\nu}b^\nu$ defined on $[0,1]$. The inequalities in (5) is the refinement and reverse of Jensen's inequality proposed by Dragomir [3] in 2006. It is easy to see that

$$\varphi(\nu) := (1-\nu)f(0)+\nu f(1) - r_0(\nu)\Big(\frac{f(0)+f(1)}{2}-f\Big(\frac{1}{2}\Big)\Big)$$

is the linear interpolation of f at points $\nu = 0, \frac{1}{2}, 1$. In 2017, Choi, Krnić, Pečarić in [2] developed this idea via defining recursively the functions $r_n(\nu)$ as

$$\begin{aligned} r_0(\nu) &= \min\{\nu, 1-\nu\}, \\ r_n(\nu) &= \min\{2r_{n-1}(\nu), 1-2r_{n-1}(\nu)\}, \end{aligned} \tag{6}$$

for all $\nu \in [0,1]$, and using them to establish a refinement of Jensen's inequality as follows: If N is a non-negative integer and f is a convex function defined on $[0,1]$, then

$$(1-\nu)f(0)+\nu f(1) \geqslant f(\nu) + \sum_{n=0}^{N-1} r_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu), \tag{7}$$

here and in the future

$$\Delta_f(n,k) = f\Big(\frac{k-1}{2^n}\Big) + f\Big(\frac{k}{2^n}\Big) - 2f\Big(\frac{2k-1}{2^{n+1}}\Big), \tag{8}$$

and χ_I is the characteristic function of the interval I given by $\chi_I(\nu)=1$ if $\nu \in I$ and $\chi_I(\nu)=0$ otherwise. The inequality (7) is a new refinement of the left hand side of (5); moreover, using it, the authors provided a unified method with simpler proofs for many recent results involving Young's inequality, its refinements and reverses established by many researchers such as Kittaneh and Manasrah in [6, 7], Liao and Wu in [8], Sababheh and Choi in [9], Zhao and Wu in [10, 11].

An important observation is to establish the inequality (7), the authors built the function

$$\varphi_N(v) := (1-v)f(0) + vf(1) - \sum_{n=0}^{N-1} r_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v)$$

which is the linear interpolation of f at points $v = \frac{k}{2^N}, k = 0,1,\dots,2^N$; moreover, it has the property $\varphi_N(v) \geqslant f(v)$ for all $v \in [0,1]$. However, it is significant to emphasize that in very many situations, a quadratic interpolation will be better than a linear one when we employ the same interpolation nodes. Furthermore, it is enormously useful when we interpolate differentiable convex functions. Based on this significant observation, we utilize a quadratic interpolation to propose a new refinement of (7) in the present article. We also apply it to derive improved versions of Young-type and Heinz-type inequalities for both scalars and operator cases.

The paper is organized as follows. In Section 2, we propose a quadratic interpolation of twice differentiable and convex functions. Applying this interpolation, we gain improved Jensen-type inequalities and its reverses which is a refinement and reverse of (7). The main results in this section is given in Theorem 2.3 below. In Section 3, we provide applications of Theorem 2.3 to refine the most general forms of recent Young-type and Heinz-type inequalities. In the last Section, we give operator versions of the obtained inequalities in Section 3.

2. Improved Jensen-type inequalities related to convex and piecewise convex functions

The main goal of this section is to give new refinements of Jensen-type inequalities and their reverses. Our method is to utilize a quadratic interpolation for convex and piecewise convex functions. Throughout the paper, we will also use the functions r_n given in (6); moreover, they can be expressed as multipart functions in the following.

LEMMA 2.1. ([2, Lemma 1]) *Let $0 \leqslant v \leqslant 1$ and n be a non-negative integer. If $\frac{k-1}{2^n} \leqslant v \leqslant \frac{k}{2^n}$ for $1 \leqslant k \leqslant 2^n$, then*

$$r_n(v) = \begin{cases} 2^n v - k + 1, & \frac{k-1}{2^n} \leqslant v \leqslant \frac{2k-1}{2^{n+1}}, \\ k - 2^n v, & \frac{2k-1}{2^{n+1}} < v \leqslant \frac{k}{2^n}. \end{cases} \tag{9}$$

Notice that the functions r_n are continuous and linear on intervals $\left(\frac{k-1}{2^{n+1}},\frac{k}{2^{n+1}}\right)$ for $1 \leqslant k \leqslant 2^{n+1}$. We can now define the quadratic functions $\mathfrak{r}_n$ via the functions r_n and get a quadratic interpolation in the following.

LEMMA 2.2. *Let f be a function defined on $[0,1]$. For a positive integer N, we define ψ_N by*

$$\psi_N(v) = (1-v)f(0) + vf(1) - \sum_{n=0}^{N-1} \mathfrak{r}_n(v) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{\left(\frac{k-1}{2^n},\frac{k}{2^n}\right)}(v), \tag{10}$$

where the quadratic functions $\mathfrak{r}_n$ *are given by*

$$\mathfrak{r}_n(\nu) = r_n(\nu) + \alpha r_{N-1}(\nu)(1/2 - r_{N-1}(\nu)) \tag{11}$$

with the arbitrary constant $\alpha \neq 0$. *Then,* ψ_N *is the quadratic interpolation of* f *at points* $\nu = k/2^N$ *for* $k = 0, 1, \ldots, 2^N$.

Proof. By Lemma 2.1, we have $r_n(k/2^n) = 0$ for $0 \leqslant k \leqslant 2^n$. Thus, the interval of the characteristic function may contain boundary points. Following the proof of [2, Lemma 2] and the representation (11) of the quadratic functions $\mathfrak{r}_n$, we can rewrite ψ_N as

$$\begin{aligned}
\psi_N(\nu) &= \varphi_N(\nu) - \alpha \sum_{n=0}^{N-1} r_{N-1}(\nu)(1/2 - r_{N-1}(\nu)) \sum_{k=1}^{2^n} \Delta_f(n,k) \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}(\nu) \\
&= (k - 2^N \nu) f\Big(\frac{k-1}{2^N}\Big) + (2^N \nu - k + 1) f\Big(\frac{k}{2^N}\Big) \\
&\quad - \alpha r_{N-1}(\nu)(1/2 - r_{N-1}(\nu)) \sum_{n=0}^{N-1} \sum_{k=1}^{2^n} \Delta_f(n,k) \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}(\nu)
\end{aligned} \tag{12}$$

for $\frac{k-1}{2^N} \leqslant \nu \leqslant \frac{k}{2^N}$ and $1 \leqslant k \leqslant 2^N$. Obviously, since $r_{N-1}(\frac{k}{2^{N-1}}) = 0$ for all $k = 0, 1, \ldots, 2^{N-1}$, we deduce that if $k = 2m$, then $r_{N-1}(\frac{k}{2^N}) = r_{N-1}(\frac{m}{2^{N-1}}) = 0$; and, if $k = 2m - 1$, then $r_{N-1}(\frac{k}{2^N}) = r_{N-1}(\frac{2m-1}{2^N}) = 1/2$. These two facts, combined with the equality (12), implies that $\psi_N(\frac{k}{2^N}) = f(\frac{k}{2^N})$ for $k = 0, 1, \ldots, 2^N$, which show that ψ_N is the quadratic interpolation of f at points $\nu = \frac{k}{2^N}$ for $k = 0, 1, \ldots, 2^N$. $\square$

THEOREM 2.3. *Let* N *be a positive integer. If* f *is a twice differentiable convex function defined on* $[0,1]$ *satisfying that* $0 < m \leqslant f''(\nu) \leqslant M < \infty$, *then*

$$(1-\nu) f(0) + \nu f(1) \geqslant f(\nu) + \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k) \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}(\nu), \tag{13}$$

and

$$\begin{aligned}
(1-\nu) f(0) + \nu f(1) &\leqslant f(0) + f(1) - f(1-\nu) \\
&\quad - \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n, 2^n - k + 1) \chi_{(\frac{k-1}{2^n}, \frac{k}{2^n})}(\nu),
\end{aligned} \tag{14}$$

where the quadratic functions $\mathfrak{r}_n$ *are defined as in* (11) *with* $\alpha = \frac{6m}{(4^N - 1)M}$.

Proof. Observe first that $r_n(\nu) \in [0, \frac{1}{2}]$ for all n and $\nu \in [0,1]$. Hence, for each positive integer N, we have $\varphi_N(\nu) \geqslant \psi_N(\nu)$ because $\mathfrak{r}_n(\nu) \geqslant r_n(\nu)$.

We will show $\psi_N(\nu) \geqslant f(\nu)$ for all $\nu \in [0,1]$. To see this, let us consider the function

$$g(\nu) := \psi_N(\nu) - f(\nu), \quad \forall \nu \in [0,1].$$

By Lemma 2.2, it is easy to see that $g(\frac{k-1}{2^N}) = g(\frac{k}{2^N}) = 0$ for $1 \leqslant k \leqslant 2^N$. Hence, it suffices to prove that g is concave on intervals $(\frac{k-1}{2^N}, \frac{k}{2^N})$ for $1 \leqslant k \leqslant 2^N$. Indeed, by Jensen's inequality, we have

$$0 \leqslant \Delta_f(n,k) \leqslant 2^{-2n-2}M. \tag{15}$$

On the other hand, from Lemma 2.1, it is easy to see that the quadratic functions $\mathfrak{r}_n$ are twice differentiable on intervals $(\frac{k-1}{2^N}, \frac{k}{2^N})$. Therefore, we obtain, for all $\nu \in (\frac{k-1}{2^N}, \frac{k}{2^N})$,

$$\begin{aligned} g''(\nu) &= \sum_{n=0}^{N-1} 2\alpha(r'_{N-1}(\nu))^2 \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) - f''(\nu) \\ &= 2\alpha \sum_{n=0}^{N-1} 2^{2N-2} \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) - f''(\nu) \\ &\leqslant \frac{\alpha M}{8} \sum_{n=0}^{N-1} 2^{2(N-n)} - f''(\nu) \\ &= \frac{\alpha M}{6}(4^N - 1) - f''(\nu) \\ &\leqslant \frac{\alpha M}{6}(4^N - 1) - m = 0, \end{aligned}$$

which proves that g is concave on $(\frac{k-1}{2^N}, \frac{k}{2^N})$ for $1 \leqslant k \leqslant 2^N$. This fact leads to $g(\nu) \geqslant 0$ on intervals $[\frac{k-1}{2^N}, \frac{k}{2^N}]$ for $1 \leqslant k \leqslant 2^N$, namely, $\psi_N(\nu) \geqslant f(\nu)$ for $\nu \in [0,1]$, which is equivalent to the desired inequality (13).

Since the functions r_n are symmetric about $\nu = 1/2$, replacing ν by $1-\nu$ in (13), we get

$$\nu f(0) + (1-\nu)f(1) \geqslant f(1-\nu) + \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(1-\nu).$$

Hence,

$$\begin{aligned} (1-\nu)f(0) + \nu f(1) &= f(0) + f(1) - [\nu f(0) + (1-\nu)f(1)] \\ &\leqslant f(0) + f(1) - f(1-\nu) - \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} \Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(1-\nu). \end{aligned}$$

Clearly, $1-\nu \in (\frac{k-1}{2^n}, \frac{k}{2^n})$ if and only if $\nu \in (1-\frac{k}{2^n}, 1-\frac{k-1}{2^n})$. Thus, replacing k by $2^n - k + 1$ in the inner summation of the above inequality, we obtain the desired inequality (14). This finishes the proof of the theorem. $\square$

REMARK 2.4. (i) The inequalities (13) and (14) in Theorem 2.3 refine the main results in [2, Theorem 3].

(ii) From the proof of Theorem 2.3, we deduce that the inequalities (13) and (14) are still valid for twice differentiable convex functions f on intervals of the form

$[\frac{\ell-1}{2^{N+1}}, \frac{\ell}{2^{N+1}}], 1 \leqslant \ell \leqslant 2^{N+1}$ satisfying that $0 < m \leqslant f''(v) \leqslant M < \infty$ on those intervals, and $\Delta_f(N,k) \geqslant 0$ for $1 \leqslant k \leqslant 2^N$.

(iii) When $N = 1$, the upper bound $M/4$ in (15) can be replaced by $\Delta_f(0,1)$ if it is different from zero. In that case we obtain the following inequalities

$$\begin{aligned}(1-v)f(0)+vf(1)-f(v) &\geqslant r_0(v)\Delta_f(0,1)+\frac{1}{2}mr_0(v)(1/2-r_0(v))\\ &\geqslant \big(r_0(v)+2\frac{m}{M}r_0(v)(1/2-r_0(v))\big)\Delta_f(0,1).\end{aligned}$$

Notice that the first inequality above can also follow from the theory of strongly convex functions. However, when $N \geqslant 2$, applying (7) to this theory gives us the weaker results. More presicely, for $N \geqslant 2$, we have a series of inequalities

$$\begin{aligned}(1-v)f(0)+vf(1) &\geqslant f(v)+\sum_{n=0}^{N-1}\mathfrak{r}_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\\ &\geqslant f(v)+\sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\\ &\geqslant f(v)+\sum_{n=0}^{N-1}r_n(v)\sum_{k=1}^{2^n}\Delta_f(n,k)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(v)\\ &\quad+\Big(\frac{v(1-v)}{2}-\sum_{n=0}^{N-1}\frac{r_n(v)}{2^{n+2}}\Big)m,\end{aligned}$$

because, following [2], the inequality

$$\frac{v(1-v)}{2} \leqslant \sum_{n=0}^{N-1}\frac{r_n(v)}{2^{n+2}}$$

holds for all $v \in [0,1]$.

3. Improved versions of Young-type and Heinz-type inequalities

In this section, we use Theorem 2.3 to establish improved versions of Young-type and Heinz-type inequalities. These results are refinements of recently established inequalities by many mathematicians, for instance, Kittaneh and Manasrah in [6, 7], Liao and Wu in [8], Sababheh and Choi in [9], Zhao and Wu in [10, 11].

The following theorem gives general refinements and reverses of Young-type inequalities.

THEOREM 3.1. *Let $0 < m \leqslant a,b \leqslant M < \infty$, $0 \leqslant v \leqslant 1$ and N be a positive integer. For each non-negative integer n, we denote by*

$$\mathfrak{r}_n(v) = r_n(v) + \frac{6m}{(4^N-1)M}r_{N-1}(v)(1/2-r_{N-1}(v))$$

and by $\mathfrak{R}_0(\nu) = 1 - \mathfrak{r}_0(\nu)$. *Then, we have*

$$
\begin{aligned}
(1-\nu)a+\nu b &\geqslant a^{1-\nu}b^{\nu} + \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} \mathfrak{g}_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\
&= a^{1-\nu}b^{\nu} + \mathfrak{r}_0(\nu)(\sqrt{a}-\sqrt{b})^2 + \sum_{n=1}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} \mathfrak{g}_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)
\end{aligned} \tag{16}
$$

and

$$
\begin{aligned}
(1-\nu)a+\nu b &\leqslant a+b-a^{\nu}b^{1-\nu} - \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} (\nu)\mathfrak{g}_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\
&= 2\sqrt{ab} - a^{\nu}b^{1-\nu} + \mathfrak{R}_0(\nu)(\sqrt{a}-\sqrt{b})^2 \\
&\quad - \sum_{n=1}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} (\nu)\mathfrak{g}_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\
&\leqslant a^{1-\nu}b^{\nu} + \mathfrak{R}_0(\nu)(\sqrt{a}-\sqrt{b})^2 \\
&\quad - \sum_{n=1}^{N-1} \mathfrak{r}_n(\nu) \sum_{k=1}^{2^n} (\nu)\mathfrak{g}_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),
\end{aligned} \tag{17}
$$

where $\mathfrak{g}_{n,k}(a,b) = \Delta_f(n,k)$ *with* $f(\nu) = a^{1-\nu}b^{\nu}$, *namely,*

$$
\begin{aligned}
\mathfrak{g}_{n,k}(a,b) &= a^{1-(k-1)/2^n}b^{(k-1)/2^n} + a^{1-k/2^n}b^{k/2^n} - 2a^{1-(2k-1)/2^{n+1}}b^{(2k-1)/2^{n+1}} \\
&= \left(\sqrt{a^{1-(k-1)/2^n}b^{(k-1)/2^n}} - \sqrt{a^{1-k/2^n}b^{k/2^n}}\right)^2.
\end{aligned}
$$

Proof. Since $f(\nu) = a^{1-\nu}b^{\nu}$ is a twice differentiable and convex function on $[0,1]$ with its second derivative $f''(\nu) = (\ln\frac{b}{a})^2 a^{1-\nu}b^{\nu}$ and $a,b \in [m,M]$, it follows that

$$
m\left(\ln\frac{b}{a}\right)^2 \leqslant f''(\nu) \leqslant M\left(\ln\frac{b}{a}\right)^2,
$$

which implies the constant $\alpha = \frac{6m}{(4^N-1)M}$. Thus, the functions $\mathfrak{r}_n$ and $\mathfrak{R}_0$ can be defined as written above. Now, the inequality (16) follows directly from (13) and the first inequality in (17) is deduced from (14). The second inequality in (17) is inferred from the geometric-arithmetic mean inequality $2\sqrt{ab} \leqslant a^{1-\nu}b^{\nu} + a^{\nu}b^{1-\nu}$. $\square$

REMARK 3.2. (i) The inequalities (16) and (17) provide new refinements of the main results in [1, 9].

(ii) By taking $N = 1$ in (16) and (17), we obtain the following inequalities, which are refinements of famous inequalities established by Manasrah and Kittaneh in [6] and [7],

$$
(1-\nu)a+\nu b \geqslant a^{1-\nu}b^{\nu} + \mathfrak{r}_0(\nu)(\sqrt{a}-\sqrt{b})^2
$$

and

$$(1-\nu)a+\nu b \leqslant a^{1-\nu}b^{\nu}+\mathfrak{R}_0(\nu)(\sqrt{a}-\sqrt{b})^2,$$

where $\mathfrak{r}_0(\nu)=r_0(\nu)+2\frac{m}{M}r_0(\nu)(1/2-r_0(\nu))$ and $\mathfrak{R}_0(\nu)=1-\mathfrak{r}_0(\nu)$.

(iii) By choosing $N=2$ in (16) and (17), we get new refinements of the results in [11]:

$$\begin{aligned}(1-\nu)a+\nu b \geqslant\, & a^{1-\nu}b^{\nu}+\mathfrak{r}_0(\nu)\big(\sqrt{a}-\sqrt{b}\big)^2\\ &+\mathfrak{r}_1(\nu)\Big[\big(\sqrt{a}-\sqrt[4]{ab}\big)^2\chi_{(0,\frac{1}{2})}(\nu)+\big(\sqrt[4]{ab}-\sqrt{b}\big)^2\chi_{(\frac{1}{2},1)}(\nu)\Big],\end{aligned}$$

and

$$\begin{aligned}(1-\nu)a+\nu b \leqslant\, & a^{1-\nu}b^{\nu}+\mathfrak{R}_0(\nu)(\sqrt{a}-\sqrt{b})^2\\ &-\mathfrak{r}_1(\nu)\Big((\sqrt{b}-\sqrt[4]{ab})^2\chi_{(0,\frac{1}{2})}(\nu)+(\sqrt{a}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\Big),\end{aligned}$$

where

$$\begin{aligned}\mathfrak{r}_0(\nu)&=r_0(\nu)+\frac{2m}{5M}r_1(\nu)(1/2-r_1(\nu)),\\ \mathfrak{r}_1(\nu)&=r_1(\nu)+\frac{2m}{5M}r_1(\nu)(1/2-r_1(\nu)),\\ \mathfrak{R}_0(\nu)&=1-\mathfrak{r}_0(\nu).\end{aligned}$$

The following theorem provides complete refinements and reverses of Young-type inequalities involving the Kantorovich constants.

THEOREM 3.3. *Let $0<m\leqslant a,b\leqslant M<\infty$, $0\leqslant\nu\leqslant 1$ and N be a positive integer. For each non-negative integer n, we denote by*

$$s_n(\nu)=r_n(\nu)+\frac{6m}{(4^N-1)MK_N(a,b)^{1/2}}r_{N-1}(\nu)(1/2-r_{N-1}(\nu))$$

and by $S_0(\nu)=1-s_0(\nu)$, where $K_N(a,b)$ is the Kantorovich constant given by

$$K_N(a,b)=\frac{\big(a^{1/2^N}+b^{1/2^N}\big)^2}{4(ab)^{1/2^N}}.$$

Then, we have

$$\begin{aligned}(1-\nu)a+\nu b &\geqslant K_N(a,b)^{r_N(\nu)}a^{1-\nu}b^{\nu}+\sum_{n=0}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\\ &=K_N(a,b)^{r_N(\nu)}a^{1-\nu}b^{\nu}+s_0(\nu)(\sqrt{a}-\sqrt{b})^2\\ &\quad+\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(a,b)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\end{aligned}\tag{18}$$

and

$$
\begin{aligned}
(1-\nu)a+\nu b &\leqslant a+b-K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu} \\
&\quad -\sum_{n=0}^{N-1} s_n(\nu)\sum_{k=1}^{2^n}(\nu)\mathfrak{g}_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\
&= 2\sqrt{ab}-K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}+S_0(\nu)(\sqrt{a}-\sqrt{b})^2 \\
&\quad -\sum_{n=1}^{N-1} s_n(\nu)\sum_{k=1}^{2^n}(\nu)\mathfrak{g}_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu) \\
&\leqslant K_N(a,b)^{-r_N(\nu)}a^{1-\nu}b^{\nu}+S_0(\nu)(\sqrt{a}-\sqrt{b})^2 \\
&\quad -\sum_{n=1}^{N-1} s_n(\nu)\sum_{k=1}^{2^n}(\nu)\mathfrak{g}_{n,k}(b,a)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),
\end{aligned}
\tag{19}
$$

where $\mathfrak{g}_{n,k}$ *is defined as in Theorem 3.1.*

Proof. Let $f(\nu)=K_N(a,b)^{r_N(\nu)}a^{1-\nu}b^{\nu}$ be a function defined on $[0,1]$. Observe first that by Lemma 2.1, the function f is of the form $\gamma\delta^{\nu}$ on each interval $I_\ell=[\frac{\ell-1}{2^{N+1}},\frac{\ell}{2^{N+1}}]$ for some $\delta,\gamma>0$ and $1\leqslant \ell\leqslant 2^{N+1}$. Although f is not convex on $[0,1]$, it is convex and twice differentiable on each interval I_ℓ. Moreover, there exist positive constants C_ℓ such that the second derivative f'' of f satisfies

$$
mC_\ell^2\leqslant f''(\nu)\leqslant K_N(a,b)^{1/2}C_\ell^2M
$$

for all $\nu\in I_\ell$ and $1\leqslant \ell\leqslant 2^{N+1}$. Hence, we can take the constant

$$
\alpha=\frac{6m}{(4^N-1)MK_N(a,b)^{1/2}}
$$

and obtain the quadratic functions s_n and S_0 as above.

On the other hand, since $r_N(\frac{k}{2^N})=0$ for $k=0,1,\dots,2^N$, it is easy to compute and get

$$
\Delta_f(n,k)=\begin{cases}\mathfrak{g}_{n,k}(a,b), & 0\leqslant n<N,\\ 0, & n=N,\end{cases}
$$

which shows $\Delta_f(N,k)\geqslant 0$ for $1\leqslant k\leqslant 2^N$. Thus, by Remark 2.4, we can apply Theorem 2.3 to the function f to gain the inequality (18) and the first inequality in (19). The second inequality in (19) follows directly from the arithmetic-geometric mean inequality

$$
2\sqrt{ab}\leqslant K_N(a,b)^{r_N(\nu)}a^{\nu}b^{1-\nu}+K_N(a,b)^{-r_N(\nu)}a^{1-\nu}b^{\nu},
$$

this also finishes the proof of the theorem. $\square$

REMARK 3.4. (i) For $N=1$ and $N=2$, the inequalities in (18) and (19) respectively refine the results which proposed by Wu and Zhao in [10]

$$
(1-\nu)a+\nu b\geqslant K_1(a,b)^{r_1(\nu)}a^{1-\nu}b^{\nu}+s_0(\nu)(\sqrt{a}-\sqrt{b})^2,
$$

$$(1-\nu)a+\nu b \leqslant K_1(a,b)^{-r_1(\nu)}a^{1-\nu}b^{\nu}+S_0(\nu)(\sqrt{a}-\sqrt{b})^2,$$

where

$$s_0(\nu) = r_0(\nu)+\frac{2m}{MK_1(a,b)^{1/2}}r_0(\nu)(1/2-r_0(\nu)),$$
$$S_0(\nu) = 1-s_0(\nu),$$

and by Liao and Wu in [8]:

$$\begin{aligned}(1-\nu)a+\nu b \geqslant{}& K_2(a,b)^{r_2(\nu)}a^{1-\nu}b^{\nu}+\mathbf{s}_0(\nu)(\sqrt{a}-\sqrt{b})^2\\ &+\mathbf{s}_1(\nu)\Big[(\sqrt[4]{ab}-\sqrt{a})^2\chi_{(0,\frac{1}{2})}(\nu)+(\sqrt{b}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\Big],\\ (1-\nu)a+\nu b \leqslant{}& K_2(a,b)^{-r_2(\nu)}a^{1-\nu}b^{\nu}+\mathbf{S}_0(\nu)(\sqrt{a}-\sqrt{b})^2\\ &-\mathbf{s}_1(\nu)\Big[(\sqrt[4]{ab}-\sqrt{b})^2\chi_{(0,\frac{1}{2})}(\nu)+(\sqrt{a}-\sqrt[4]{ab})^2\chi_{(\frac{1}{2},1)}(\nu)\Big],\end{aligned}$$

where

$$\mathbf{s}_0(\nu) = r_0(\nu)+\frac{2m}{5MK_2(a,b)^{1/2}}r_1(\nu)(1/2-r_1(\nu)),$$
$$\mathbf{s}_1(\nu) = r_n(\nu)+\frac{2m}{5MK_1(a,b)^{1/2}}r_1(\nu)(1/2-r_1(\nu)),$$
$$\mathbf{S}_0(\nu) = 1-\mathbf{s}_0(\nu).$$

Next, we will discuss the Heinz mean $H_\nu(a,b)$ in parameter ν of two positive numbers a,b which we have just mentioned in (4). The Heinz mean interpolates the geometric mean and the arithmetic mean, namely,

$$\sqrt{ab} \leqslant H_\nu(a,b) \leqslant \frac{a+b}{2}. \tag{20}$$

The right-hand side of this inequality was refined by Kittaneh and Krnić (see [5]) to

$$H_\nu(a,b) \leqslant (1-2r_0(\nu))\frac{a+b}{2}+2r_0(\nu)\sqrt{ab}. \tag{21}$$

Now, by virtue of Theorem 2.3, we can give a general improved form of this result as follows.

THEOREM 3.5. *Let $0<a,b<\infty$, $0\leqslant \nu \leqslant 1$ and N be a positive integer. For each non-negative integer n, we denote by*

$$h_n(\nu) = r_n(\nu)+\frac{12\sqrt{ab}}{(4^N-1)(a+b)}r_{N-1}(\nu)(1/2-r_{N-1}(\nu)).$$

Then, we have

$$\begin{aligned}\frac{a+b}{2}-\sum_{n=0}^{N-1}h_n(\nu)\sum_{k=1}^{2^n}\big[H_{\frac{k-1}{2^n}}(a,b)+H_{\frac{k}{2^n}}(a,b)-2H_{\frac{2k-1}{2^{n+1}}}(a,b)\big]\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\\ \geqslant H_\nu(a,b).\end{aligned} \tag{22}$$

Proof. Let $f(\nu) = H_\nu(a,b) = (a^{1-\nu}b^\nu + a^\nu b^{1-\nu})/2$ be a twice differentiable and convex function on $[0,1]$; moreover, its second derivative is

$$f''(\nu) = \frac{a^{1-\nu}b^\nu + a^\nu b^{1-\nu}}{2}\left(\ln\frac{b}{a}\right)^2.$$

Clearly,

$$\sqrt{ab}\left(\ln\frac{b}{a}\right)^2 \leqslant f''(\nu) \leqslant \left(\ln\frac{b}{a}\right)^2\frac{a+b}{2},$$

which leads to the constant

$$\alpha = \frac{12\sqrt{ab}}{(4^N-1)(a+b)}.$$

Hence, applying Theorem 2.3 to f, we obtain the desired inequality (22). □

REMARK 3.6. For $N=1$, we have a series of inequalities

$$\begin{aligned} H_\nu(a,b) &\leqslant (1-2h_0(\nu))\frac{a+b}{2} + 2h_0(\nu)\sqrt{ab} \\ &\leqslant (1-2r_0(\nu))\frac{a+b}{2} + 2r_0(\nu)\sqrt{ab} \leqslant \frac{a+b}{2}, \end{aligned}$$

where

$$h_0(\nu) = r_0(\nu) + \frac{4\sqrt{ab}}{a+b}r_0(\nu)(1/2 - r_0(\nu)).$$

4. Operator versions of Young-type and Heinz-type inequalities

Our main goal in this section is to use scalar versions of Young-type and Heinz-type inequalities from the previous section to establish their operator forms.

Throughout this section, invertible positive operators on a complex Hilbert space H will be denoted by uppercase letters and I stands for the identity operator on H. We also use the following notations:

(i) $A \geqslant 0$ $(A > 0)$ if A is a positive (invertible positive) operator;

(ii) $A \geqslant B$ $(A > B)$ if $A - B$ is a positive (invertible positive) operator.

For $A, B > 0$ and $0 \leqslant \nu \leqslant 1$, the ν-weighted arithmetic and geometric means of A and B are defined respectively by

$$\begin{aligned} A\nabla_\nu B &= (1-\nu)A + \nu B, \\ A\sharp_\nu B &= A^{1/2}(A^{-1/2}BA^{-1/2})^\nu A^{1/2}. \end{aligned}$$

We also write $A\nabla B$ and $A\sharp B$ instead of $A\nabla_{\frac{1}{2}}B$ and $A\sharp_{\frac{1}{2}}B$, respectively.

The main idea for showing operator inequalities corresponding to their scalar versions is to use the operator monotonicity of continuous functions in the following.

LEMMA 4.1. ([11]) *Let X be an arbitrary self-adjoint operator. If f and g are continuous real-valued functions on the spectrum $\mathrm{Sp}(X)$ satisfying that $f(t) \geqslant g(t)$ for all $t \in \mathrm{Sp}(X)$, we then have an operator inequality $f(X) \geqslant g(X)$.*

Now, we will discuss operator forms of Young-type inequalities. First, the operator version of (1) says that for two invertible positive operators A,B and $0 \leqslant \nu \leqslant 1$, we have

$$A\nabla_\nu B \geqslant A\sharp_\nu B.$$

This inequality was improved by Dragomir [4] to

$$\frac{1}{2}\nu(1-\nu)A\sharp_{f_{\min}}B \leqslant A\nabla_\nu B - A\sharp_\nu B \leqslant \frac{1}{2}\nu(1-\nu)A\sharp_{f_{\max}}B,$$

where $A,B > 0$, $0 \leqslant \nu \leqslant 1$, and

$$f_{\min}(x) = \min\{1,x\}(\ln x)^2,$$
$$f_{\max}(x) = \max\{1,x\}(\ln x)^2,$$

for $x > 0$. The more general versions of these results were showed in [1] (see also [2] for the matrix case) of the following forms

$$A\nabla_\nu B \geqslant A\sharp_\nu B + 2r_0(\nu)(A\nabla B - A\sharp B) + \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n} \mathfrak{g}_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)$$

and

$$A\nabla_\nu B \leqslant A\sharp_\nu B + 2R_0(\nu)(A\nabla B - A\sharp B) - \sum_{n=0}^{N-1} r_n(\nu)\sum_{k=1}^{2^n} \mathfrak{g}_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),$$

where $A,B > 0$, $0 \leqslant \nu \leqslant 1, N$ is a positive integer, and

$$\mathfrak{g}_{n,k}(A,B) = A\sharp_{\frac{k-1}{2^n}}B + A\sharp_{\frac{k}{2^n}}B - 2A\sharp_{\frac{2k-1}{2^{n+1}}}B, \tag{23}$$

for all $1 \leqslant k \leqslant 2^n$. By virtue of Theorem 3.1, these results can be extended as follows.

THEOREM 4.2. *Let $A,B > 0$ be such that $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset [m,M]$ for some positive numbers m,M. Then, for $0 \leqslant \nu \leqslant 1$, a positive integer N and $\mathfrak{r}_n, \mathfrak{R}_0$ are defined as in Theorem 3.1, we have*

$$A\nabla_\nu B \geqslant A\sharp_\nu B + 2\mathfrak{r}_0(\nu)(A\nabla B - A\sharp B) + \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu)\sum_{k=1}^{2^n} \mathfrak{g}_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)$$

and

$$A\nabla_\nu B \leqslant A\sharp_\nu B + 2\mathfrak{R}_0(\nu)(A\nabla B - A\sharp B) - \sum_{n=0}^{N-1} \mathfrak{r}_n(\nu)\sum_{k=1}^{2^n} \mathfrak{g}_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),$$

where $\mathfrak{g}_{n,k}(A,B)$ are given in (23).

Proof. From the inequalities (16) and (17) in Theorem 3.1, we have, for all $x \in [\frac{m}{M}, \frac{M}{m}]$,

$$1-\nu+\nu x \geqslant x^{\nu}+2\mathfrak{r}_0(\nu)\Big(\frac{1+x}{2}-\sqrt{x}\Big)+\sum_{n=1}^{N-1}\mathfrak{r}_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(1,x)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)$$

and

$$1-\nu+\nu x \leqslant x^{\nu}+2\mathfrak{R}_0(\nu)\Big(\frac{1+x}{2}-\sqrt{x}\Big)-\sum_{n=1}^{N-1}\mathfrak{r}_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(1,x)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),$$

where

$$\mathfrak{g}_{n,k}(1,x)=x^{(k-1)/2^n}+x^{k/2^n}-2x^{(2k-1)/2^{n+1}}.$$

Thus, by Lemma 4.1, for any positive operator X with its spectrum in $[\frac{m}{M}, \frac{M}{m}]$, we have

$$\begin{aligned}(1-\nu)I+\nu X \geqslant X^{\nu}&+2\mathfrak{r}_0(\nu)\Big(\frac{1+X}{2}-X^{1/2}\Big)\\&+\sum_{n=1}^{N-1}\mathfrak{r}_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(I,X)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\end{aligned}$$

and

$$\begin{aligned}(1-\nu)I+\nu X \leqslant X^{\nu}&+2\mathfrak{R}_0(\nu)\Big(\frac{I+X}{2}-X^{1/2}\Big)\\&-\sum_{n=1}^{N-1}\mathfrak{r}_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(I,X)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),\end{aligned}$$

where

$$\mathfrak{g}_{n,k}(I,X)=X^{(k-1)/2^n}+X^{k/2^n}-2X^{(2k-1)/2^{n+1}}.$$

Since $\mathrm{Sp}(A), \mathrm{Sp}(B)$ are in $[m,M]$, we obtain the spectrum $\mathrm{Sp}(A^{-1/2}BA^{-1/2})$ of $A^{-1/2}BA^{-1/2}$ in the interval $[\frac{m}{M}, \frac{M}{m}]$. Therefore, replacing X by $A^{-1/2}BA^{-1/2}$ in the above inequalities and multiplying the obtained inequalities by $A^{1/2}$ both-sidely, we get the desired inequalities. □

In view of Theorem 3.3, we can also propose improved versions of Young-type inequalities involving the Kantorovich constant. Recall (see [2, 4]) that for any non-negative integer N, the Kantorovich constant of order N is given by

$$K_N(2,t)=\frac{(1+t^{1/2^N})^2}{4t^{1/2^N}}, \quad t>0.$$

The function $K_N(2,\cdot)$ is decreasing on $(0,1)$ and increasing on $(1,\infty)$; moreover, it has the property

$$K_N(2,t)=K_N(2,1/t), \quad \forall t\in(0,\infty).$$

Now, we are ready to state the result as follows.

THEOREM 4.3. *Let $A,B>0$ satisfy one of the following conditions:*

(i) $0<mI\leqslant A\leqslant \gamma I<\Gamma I\leqslant B\leqslant MI$,

(ii) $0<mI\leqslant B\leqslant \gamma I<\Gamma I\leqslant A\leqslant MI$,

where $0<m,M,\gamma,\Gamma<\infty$ are scalars. With the notations S_0,s_n and $\mathfrak{g}_{n,k}$ being as in Theorem 3.3 and in (23) respectively, we have

$$A\nabla_\nu B\geqslant K_N(2,h)^{r_N(\nu)}A\sharp_\nu B+2s_0(\nu)\Big(A\nabla B-A\sharp B\Big)\\ +\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)$$

and

$$A\nabla_\nu B\leqslant K_N(2,h)^{-r_N(\nu)}A\sharp_\nu B+2S_0(\nu)\Big(A\nabla B-A\sharp B\Big)\\ -\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(A,B)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),$$

where

$$K_N(2,h)=\frac{\big(\gamma^{1/2^N}+\Gamma^{1/2^N}\big)^2}{4(\gamma\Gamma)^{1/2^N}}\quad \textit{and}\quad K_N(2,h')=\frac{\big(m^{1/2^N}+M^{1/2^N}\big)^2}{4(mM)^{1/2^N}}.$$

Proof. Observe first that $\frac{m}{M}\leqslant\frac{\gamma}{\Gamma}<1<\frac{\Gamma}{\gamma}\leqslant\frac{M}{m}$. We now consider the case when the operators A,B satisfy the condition (i). Utilizing the inequality (18), we have, for all $x\in[\frac{\Gamma}{\gamma},\frac{M}{m}]\subset[\frac{m}{M},\frac{M}{m}]$,

$$\begin{aligned}(1-\nu)+\nu x&\geqslant K_N(2,x)^{r_N(\nu)}x^\nu+s_0(\nu)(1+x-2\sqrt{x})\\&\quad+\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(1,x)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\\&\geqslant\Big(\min_{h\leqslant x\leqslant h'}K_N(2,x)^{r_N(\nu)}\Big)x^\nu+s_0(\nu)(1+x-2\sqrt{x})\\&\quad+\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(1,x)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)\\&=K_N(2,h)^{r_N(\nu)}x^\nu+s_0(\nu)(1+x-2\sqrt{x})\\&\quad+\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(1,x)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu),\end{aligned}\tag{24}$$

where $h=\frac{\Gamma}{\gamma}$ and $h'=\frac{M}{m}$. By Lemma 4.1, for any invertible positive operator X with its spectrum in $[h,h']$, we obtain

$$(1-\nu)I+\nu X\geqslant K_N(2,h)^{r_N(\nu)}X^\nu+2s_0(\nu)\Big(\frac{I+X}{2}-X^{1/2}\Big)\\ +\sum_{n=1}^{N-1}s_n(\nu)\sum_{k=1}^{2^n}\mathfrak{g}_{n,k}(I,X)\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu).$$

It follows from the condition (i) that the spectrum $\mathrm{Sp}(A^{-1/2}BA^{-1/2})$ of the operator $A^{-1/2}BA^{-1/2}$ is in $[\frac{\Gamma}{\gamma},\frac{M}{m}]$. Hence, replacing X by $A^{-1/2}BA^{-1/2}$ in the above inequality and computing as in the proof of Theorem 4.2, we will gain the first inequality in the theorem.

If the operators A,B satisfy the condition (ii), the inequality (24) still holds for all $x\in[\frac{m}{M},\frac{\gamma}{\Gamma}]\subset[\frac{m}{M},\frac{M}{m}]$ instead of $[\frac{\Gamma}{\gamma},\frac{M}{m}]$. The spectrum $\mathrm{Sp}(A^{-1/2}BA^{-1/2})$ of $A^{-1/2}BA^{-1/2}$ in this case belongs in the interval $[\frac{m}{M},\frac{\gamma}{\Gamma}]$. Since the remaining steps in the proof of this case are similar to the previous case, we omit the details.

The other inequality is proved similarly to the first one, and so we do not again present the details here. This also finishes the proof of the theorem. □

The operator versions of Heinz-type inequalities (20) and (21) are of the forms

$$A\sharp B\leqslant H_\nu(A,B)\leqslant A\nabla B$$

and

$$A\nabla B-H_\nu(A,B)\geqslant 2r_0(\nu)(A\nabla B-A\sharp B),$$

where $A,B>0$, $0\leqslant\nu\leqslant 1$ and

$$H_\nu(A,B)=\frac{A\sharp_\nu B+A\sharp_{1-\nu}B}{2}, \tag{25}$$

see [5] for the details. Notice that the second inequality above refines the right-hand side of the previous inequality. Due to Theorem 3.5, we can give the most general form of this refinement in the last theorem of the paper as follows.

THEOREM 4.4. *Let $0\leqslant\nu\leqslant 1$ and N be a positive integer. Let $A,B>0$ have the spectra $\mathrm{Sp}(A),\mathrm{Sp}(B)\subset[a,b]$ for some $0<a<b<\infty$. Then, we have*

$$\begin{aligned}A\nabla B-\sum_{n=0}^{N-1}h_n(\nu)\sum_{k=1}^{2^n}\Big[H_{\frac{k-1}{2^n}}(A,B)+H_{\frac{k}{2^n}}(A,B)-2H_{\frac{2k-1}{2^{n+1}}}(A,B)\Big]\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)&\\ \geqslant H_\nu(A,B),&\end{aligned}$$

where h_n are defined as in Theorem 3.5 and $H_\nu(A,B)$ is given in (25).

Proof. From the inequality (22) in Theorem 3.5, we have

$$\begin{aligned}\frac{1+x}{2}-\sum_{n=0}^{N-1}h_n(\nu)\sum_{k=1}^{2^n}\Big[H_{\frac{k-1}{2^n}}(1,x)+H_{\frac{k}{2^n}}(1,x)-2H_{\frac{2k-1}{2^{n+1}}}(1,x)\Big]\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)&\\ \geqslant H_\nu(1,x),&\end{aligned}$$

where $x\in[\frac{a}{b},\frac{b}{a}]$. By Lemma 4.1, for any arbitrary invertible positive operator X with $\mathrm{Sp}(X)\subset[\frac{a}{b},\frac{b}{a}]$, we deduce

$$\begin{aligned}\frac{I+X}{2}-\sum_{n=0}^{N-1}h_n(\nu)\sum_{k=1}^{2^n}\Big[H_{\frac{k-1}{2^n}}(I,X)+H_{\frac{k}{2^n}}(I,X)-2H_{\frac{2k-1}{2^{n+1}}}(I,X)\Big]\chi_{(\frac{k-1}{2^n},\frac{k}{2^n})}(\nu)&\\ \geqslant H_\nu(I,X).&\end{aligned}$$

On the other hand, since $\mathrm{Sp}(A), \mathrm{Sp}(B) \subset [a,b]$, we find that $\mathrm{Sp}(A^{-1/2}BA^{-1/2})$ of $A^{-1/2}BA^{-1/2}$ is in $[\frac{a}{b}, \frac{b}{a}]$. Therefore, replacing X by $A^{-1/2}BA^{-1/2}$ in the above inequality and multiplying the obtained inequality by $A^{1/2}$ both-sidely, we will gain the desired inequality. $\square$

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