CONVERGENCE RATE FOR WEIGHTED SUMS OF
ψ−MIXING RANDOM VARIABLES AND APPLICATIONS

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Abstract. The complete convergence result is obtained for weighted sums of ψ-mixing random variables without any conditions on mixing rate. As a special case, we can obtain the law of large numbers of Liu and Jin (J. Math. Ineq., 12, 2018). As applications, necessary and sufficient conditions are provided for the complete consistency of LS estimators in the errors-in-variables regression model with ψ-mixing errors.

1. Introduction

Convergence rate of the strong and weak law of large numbers is used prevalently in probability and statistics, and the complete convergence is a good tool to characterize the convergence rate. The concept of complete convergence was introduced by Hsu and Robbins (1947) as follows: a sequence $\{U_n, n \geqslant 1\}$ of random variables is said to converge completely to a constant u if

$$\sum_{n=1}^{\infty} P\{|U_n - u| > \varepsilon\} < \infty, \ \forall \varepsilon > 0.$$

By the Borel-Cantelli lemma, this implies that $U_n \to u$ a.s. The converse is true if $U_n, n \geqslant 1$, are independent. Hsu and Robbins (1947) proved that the sequence of arithmetic means of independent and identically distributed (i.i.d.) random variables converges completely to the expected value if the variance of the summands is finite. Erdös (1949) proved the converse.

The result of Hsu and Robbins (1947) has been generalized and extended by many authors. In the independent case, the celebrated results are due to Spitzer (1956) and Baum and Katz (1965) for partial sums, to Stout (1968) and Li et al. (1995) for weighted sums. In the negatively associated dependent case, Shao (2000) obtained the complete convergence for partial sums, and Sung (2012) and Wang et al. (2011) obtained the complete convergence for weighted sums. Chen et al. (2008) obtained the complete convergence for the moving average processes based on negatively associated random

variables. In the mixing case, Shao (1988) and Shao (1995) extended the result of Hsu and Robbins (1947) to the partial sums of φ-mixing and ρ-mixing random variables, respectively. Peligrad and Gut (1999) extended it to ρ^*-mixing random variables.

The limiting behavior for weighted sums of independent random variables plays an important role in probability theory and statistics, since many useful linear statistics, such as least squares estimators, nonparametric regression function estimators and jackknife estimates, are weighted sums. However, the independence assumption is not reasonable in many real applications of statistical problems. Hence, it is necessary to study the limiting properties for weighted sums of dependent random variables. When we study the limiting properties for weighted sums of mixing random variables such as φ-mixing and ρ-mixing, some conditions on mixing rate are often needed (see, for example, Yang (1995) and Huang et al. (2014)). However, it is not easy to check such mixing rate conditions.

We recall the concept of ψ-mixing random variables or random vectors.

DEFINITION 1.1. Define the ψ-mixing coefficient for a sequence of random variables or random vectors $\{X_n, n \geqslant 1\}$ as

$$\psi(n) = \sup_{m \geqslant 1} \sup_{A \in \mathscr{F}_1^m, B \in \mathscr{F}_{m+n}^\infty, P(A)P(B) \neq 0} \left| \frac{P(AB)}{P(A)P(B)} - 1 \right|,$$

where $\mathscr{F}_n^m = \sigma(X_i : n \leqslant i \leqslant m)$. Then $\{X_n, n \geqslant 1\}$ is said to be ψ-mixing or *-mixing if $\psi(n) \to 0$ as $n \to \infty$.

The concept of ψ-mixing was introduced by Blum et al. (1963), who obtained the Kolmogorov strong law of large numbers for identically distributed ψ-mixing random variables without any conditions on mixing rate. Under the condition $\sum_{n=1}^\infty \psi(n) < \infty$ on ψ-mixing rate, Yang (1995) obtained the moment inequality, exponent inequality and strong law for weighted sums, Wang et al. (2010) obtained the maximal inequality and gave some applications, and Xu and Tang (2013) discussed the strong law for Jamison's type weighted sums.

Recently, some limit theorems for ψ-mixing random variables have been established without any conditions on mixing rate. Hu et al. (2017) extended the strong law of large numbers of Blum et al. (1963) to weighted sums. Liu and Jin (2018) extended the Spitzer (1956) law of large numbers for i.i.d. random variables to ψ-mixing case.

In this paper, we obtain the complete convergence for weighted sums of ψ-mixing random variables without mixing rate conditions. The result of Liu and Jin (2018) can be obtained as a special case of our result. We also apply our result to the errors-in-variables (EV) regression models with ψ-mixing errors.

We now state the main result. The proof of this result will be detailed in the next section. The applications of Theorem 1.1 will be shown in Section 3

THEOREM 1.1. Let $r \geqslant 1$, $p > 1$ and $q > 1$ with $1/p + 1/q = 1$. Let $\{X, X_n, n \geqslant 1\}$ be a sequence of identically distributed ψ-mixing random variables, and let $\{a_{nk}, n \geqslant 1, 1 \leqslant k \leqslant n\}$ be an array of constants satisfying

$$\sum_{k=1}^n |a_{nk}|^p = O(n). \tag{1.1}$$

If

$$EX = 0, \quad \begin{cases} E|X|^r < \infty, & \text{if } r < p, \\ E|X|^r \log(1+|X|) < \infty, & \text{if } r = p, \\ E|X|^{(r-1)q} < \infty, & \text{if } r > p, \end{cases} \tag{1.2}$$

then

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk} X_k \right| > \varepsilon n \right\} < \infty, \ \forall \, \varepsilon > 0. \tag{1.3}$$

Conversely, if (1.3) *holds for any array* $\{a_{nk}\}$ *satisfying* (1.1) *for some* $p > 1$, *then* $EX = 0$, $E|X|^{(r-1)q} < \infty$ *and* $E|X|^r < \infty$ *hold.*

REMARK 1.1. Recently, Wu and Wang (2021) proved Theorem 1.1 for m-extended negatively dependent random variables. They proved that the moment condition $EX|^r \log(1+|X|) < \infty$ is also necessary. However, their proof is not correct (the statement "It is easy to check that (1.2) is satisfied" on lines 7–8, page 17 is incorrect).

By Theorem 1.1, we immediately have the following corollary.

COROLLARY 1.1. *Let* $r \geqslant 1$. *Let* $\{X, X_n, n \geqslant 1\}$ *be a sequence of identically distributed* ψ*-mixing random variables. Then*

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} X_k \right| > \varepsilon n \right\} < \infty, \ \forall \, \varepsilon > 0 \tag{1.4}$$

holds if and only if $EX = 0$ *and* $E|X|^r < \infty$.

REMARK 1.2. Liu and Jin (2018) proved Corollary 1.1 when $r = 1$.

Throughout this paper, C always stands for a positive constant whose value is of no importance and may differ from one place to another, $I(A)$ denotes the indicator function of the event A.

2. Lemmas and proofs

To prove the main result, we need the following lemmas. The first one can refer to Lemma 2.1 in Liu and Jin (2018).

LEMMA 2.1. *Let* $\{Y_n, n \geqslant 1\}$ *be a sequence of random variables with the mixing coefficient* $\psi(\cdot)$. *Suppose that* $E|Y_n| < \infty$ *for each* $n \geqslant 1$. *Then*

$$|E(Y_{n+1}|\mathscr{G}) - EY_{n+1}| \leqslant \psi(1)E|Y_{n+1}| \ \text{a.s.}$$

for each σ *field* $\mathscr{G} \subset \sigma(Y_i : 1 \leqslant i \leqslant n)$, *and each* $n \geqslant 1$.

LEMMA 2.2. *Let* $\{Y_n, n \geqslant 1\}$ *be a sequence of* ψ*-mixing random variables with* $\sup_{n \geqslant 1} |Y_n| \leqslant M$ *a.s. for some constant* $M > 0$, *and let* $\{a_{nk}, n \geqslant 1, 1 \leqslant k \leqslant n\}$ *be an array of constants satisfying* (1.1) *for some* $p > 1$. *Then for any* $r \geqslant 1$,

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk} Y_k \right| > \varepsilon n \right\} < \infty, \ \forall \, \varepsilon > 0.$$

Proof. By the same argument as the proof of Lemma 2.3 in Hu et al. (2017), we have the desired result. $\square$

LEMMA 2.3. (see Theorem 2.11 in Hall and Heyde, 1980) *Let* $\{Y_k, \mathscr{F}_k, 1 \leqslant k \leqslant n\}$ *be a sequence of martingale differences and* $s > 0$. *Then there exists a constant C depending only on* s *such that*

$$E \left| \sum_{k=1}^{n} Y_k \right|^s \leqslant C \left\{ E \left(\sum_{k=1}^{n} E(Y_k^2 | \mathscr{F}_{k-1}) \right)^{s/2} + \sum_{k=1}^{n} E|Y_k|^s \right\}.$$

Proof of Theorem 1.1. Sufficiency. We can assume that $0 \leqslant \psi(1) < \infty$ by the subsequence method, since $\psi(n) \to 0$ as $n \to \infty$, where $\psi(\cdot)$ is the ψ-mixing coefficient of $\{X_n, n \geqslant 1\}$. By (1.1), without loss of generality, we can assume that for all $n \geqslant 1$,

$$\sum_{k=1}^{n} |a_{nk}|^p \leqslant n,$$

which implies that

$$\sum_{k=1}^{n} |a_{nk}|^t \leqslant n \tag{2.1}$$

for all $t \in (0, p)$ by the Hölder inequality. For any fixed $\varepsilon > 0$, there exists a positive number $M = M(\varepsilon)$ such that

$$E|X|I(|X| > M) < \varepsilon/(8 + 8\psi(1)), \tag{2.2}$$

since $E|X| < \infty$. Note that by $EX = 0$,

$$\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk} X_k \right| > \varepsilon n \right\}$$

$$\subset \left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk} (X_k I(|X_k| \leqslant M) - E X_k I(|X_k| \leqslant M)) \right| > \varepsilon n/2 \right\}$$

$$\cup \left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk} (X_k I(|X_k| > M) - E X_k I(|X_k| > M)) \right| > \varepsilon n/2 \right\}, \tag{2.3}$$

and by (2.1) and (2.2),

$$\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk} (X_k I(|X_k| > M) - E X_k I(|X_k| > M)) \right| > \varepsilon n/2 \right\}$$

$$\subset \left\{ \sum_{k=1}^{n} |a_{nk} X_k| I(|X_k| > M) > (\varepsilon/2 - E|X|I(|X| > M))n \right\}$$

$$\subset \left\{ \sum_{k=1}^{n} |a_{nk}X_k|I(|X_k| > M) > 3\varepsilon n/8 \right\}$$

$$\subset \cup_{k=1}^{n}\{|a_{nk}X_k| > n\} \cup \left\{ \sum_{k=1}^{n} |a_{nk}X_k'|I(|a_{nk}X_k'| \leqslant n) > 3\varepsilon n/8 \right\}, \quad (2.4)$$

where $X_k' = X_kI(|X_k| > M)$. Since $|X_nI(|X_n| \leqslant M)| \leqslant M$ for all $n \geqslant 1$, we have by Lemma 2.2 that

$$\sum_{n=1}^{\infty} n^{r-2}P \left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} a_{nk}(X_kI(|X_k| \leqslant M) - EX_kI(|X_k| \leqslant M)) \right| > \varepsilon n/2 \right\} < \infty. \quad (2.5)$$

By Lemma 2.2 in Chen and Sung (2018),

$$\sum_{n=1}^{\infty} n^{r-2}P(\cup_{k=1}^{n}\{|a_{nk}X_k| > n\}) \leqslant \begin{cases} CE|X|^r, & \text{if } r < p, \\ CE|X|^r \log(1+|X|), & \text{if } r = p, \\ CE|X|^{(r-1)q}, & \text{if } r > p \end{cases}$$
$$< \infty. \quad (2.6)$$

Therefore, to prove (1.3), it suffices by (2.3)–(2.6) to show that

$$\sum_{n=1}^{\infty} n^{r-2}P \left\{ \sum_{k=1}^{n} |a_{nk}X_k'|I(|a_{nk}X_k'| \leqslant n) > 3\varepsilon n/8 \right\} < \infty. \quad (2.7)$$

Set $X_{nk} = |a_{nk}X_k'|I(|a_{nk}X_k'| \leqslant n)$, $\mathscr{F}_{n,k} = \sigma(X_{ni} : 1 \leqslant i \leqslant k)$ for $1 \leqslant k \leqslant n$, and $\mathscr{F}_{n,0} = \{\emptyset, \Omega\}$. By Lemma 2.1, (2.1) and (2.2),

$$\left| \sum_{k=1}^{n} E(X_{nk}|\mathscr{F}_{n,k-1}) \right| \leqslant \sum_{k=1}^{n} (|E(X_{nk}|\mathscr{F}_{n,k-1}) - EX_{nk}| + EX_{nk})$$

$$\leqslant (1 + \psi(1)) \sum_{k=1}^{n} EX_{nk} \text{ a.s.}$$

$$\leqslant (1 + \psi(1)) \sum_{k=1}^{n} |a_{nk}|E|X_k|I(|X_k| > M) \text{ a.s.}$$

$$= (1 + \psi(1)) \left(\sum_{k=1}^{n} |a_{nk}| \right) E|X|I(|X| > M) \leqslant \varepsilon n/8 \text{ a.s.}$$

Thus, to prove (2.7), it suffices to prove that

$$\sum_{n=1}^{\infty} n^{r-2}P \left\{ \left| \sum_{k=1}^{n} (X_{nk} - E(X_{nk}|\mathscr{F}_{n,k-1})) \right| > \varepsilon n/4 \right\} < \infty. \quad (2.8)$$

Note that $\{X_{nk} - E(X_{nk}|\mathscr{F}_{n,k-1}), 1 \leqslant k \leqslant n\}$ is a sequence of martingale differences for any $n \geqslant 1$. By the Markov inequality, Lemma 2.3, and the Jensen inequality, we have

that for any $s \geqslant 1$,

$$P\left\{\left|\sum_{k=1}^{n}(X_{nk} - E(X_{nk}|\mathscr{F}_{n,k-1}))\right| > \varepsilon n/4\right\}$$

$$\leqslant Cn^{-s}E\left|\sum_{k=1}^{n}(X_{nk} - E(X_{nk}|\mathscr{F}_{n,k-1}))\right|^{s}$$

$$\leqslant Cn^{-s}\left\{E\left(\sum_{k=1}^{n}E(X_{nk}^{2}|\mathscr{F}_{n,k-1})\right)^{s/2} + \sum_{k=1}^{n}EX_{nk}^{s}\right\}.$$

Noting that $X_{nk} = |a_{nk}X_{k}'|I(|a_{nk}X_{k}'| \leqslant n)$ and $X_{k}' = X_{k}I(|X_{k}| > M)$, we have by Lemma 2.1 that

$$E(X_{nk}^{2}|\mathscr{F}_{n,k-1}) \leqslant (1+\psi(1))E|a_{nk}X_{k}|^{2}I(|a_{nk}X_{k}| \leqslant n) \text{ a.s.},$$

and

$$EX_{nk}^{s} \leqslant E|a_{nk}X_{k}|^{s}I(|a_{nk}X_{k}| \leqslant n).$$

Therefore, for all $s \geqslant 1$,

$$P\left\{\left|\sum_{k=1}^{n}(X_{nk} - E(X_{nk}|\mathscr{F}_{n,k-1}))\right| > \varepsilon n/4\right\}$$

$$\leqslant Cn^{-s}\left\{\left(\sum_{k=1}^{n}E|a_{nk}X_{k}|^{2}I(|a_{nk}X_{k}| \leqslant n)\right)^{s/2} + \sum_{k=1}^{n}E|a_{nk}X_{k}|^{s}I(|a_{nk}X_{k}| \leqslant n)\right\}. \quad (2.9)$$

By the same argument as the proof of Theorem 3.1 in Chen and Sung (2018), we have that for some s (when $r = 1$, $s = 2$; when $r > 1$, $s > \max\{p, (r-1)q, 2(r-1)/(t-1)\}$, where $1 < t < \min\{2, p, r\}$),

$$\sum_{n=1}^{\infty}n^{r-2-s}\left(\sum_{k=1}^{n}E|a_{nk}X_{k}|^{2}I(|a_{nk}X_{k}| \leqslant n)\right)^{s/2} < \infty \quad (2.10)$$

and

$$\sum_{n=1}^{\infty}n^{r-2-s}\sum_{k=1}^{n}E|a_{nk}X_{k}|^{s}I(|a_{nk}X_{k}| \leqslant n) < \infty. \quad (2.11)$$

Hence (2.8) holds from (2.9)–(2.11).

Necessity. Set $a_{nk} = 0$ if $k = 1, \cdots, n-1$, and $a_{nn} = n^{1/p}$. Then (1.1) holds and we can rewrite (1.3) as

$$\sum_{n=1}^{\infty}n^{r-2}P\{n^{1/p}|X_{n}| > \varepsilon n\} < \infty, \quad \forall \varepsilon > 0,$$

which is equivalent to $E|X|^{(r-1)q} < \infty$.

It remains to show that $EX = 0$ and $E|X|^r < \infty$. To do these, set $a_{nk} = 1$ for all $1 \leqslant k \leqslant n$. Then (1.1) holds and we can rewrite (1.3) as

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1 \leqslant m \leqslant n} \left| \sum_{k=1}^{m} X_k \right| > \varepsilon n \right\} < \infty, \quad \forall \varepsilon > 0. \tag{2.12}$$

For the case $r = 1$, (2.12) is equivalent to $EX = 0$, which is due to Liu and Jin (2018).

We now consider the case $r > 1$. Clearly (2.12) implies that

$$\sum_{n=1}^{\infty} n^{r-2} P\left\{ \max_{1 \leqslant k \leqslant n} |X_k| > \varepsilon n \right\} < \infty, \quad \forall \varepsilon > 0, \tag{2.13}$$

which also implies that

$$P\left\{ \max_{1 \leqslant k \leqslant n} |X_k| > n \right\} \to 0 \tag{2.14}$$

as $n \to \infty$. We can assume that $0 \leqslant \psi(1) < \infty$ by the subsequence method, since $\psi(n) \to 0$ as $n \to \infty$. Note that if (2.12) holds for $r > 1$, then it also holds for $r = 1$ and hence $EX = 0$ from the above case $r = 1$. By the definition of ψ-mixing, and the Markov inequality, we obtain that for any $i \neq j$,

$$P\{|X_i| > n, |X_j| > n\} \leqslant (1 + \psi(1)) P\{|X_i| > n\} P\{X_j| > n\}$$
$$= (1 + \psi(1)) P^2\{|X| > n\}$$
$$\leqslant \frac{(1 + \psi(1)) E|X|}{n} P\{|X| > n\}.$$

It follows that

$$Var\left(\sum_{k=1}^{n} I(|X_k| > n) \right) \leqslant E\left(\sum_{k=1}^{n} I(|X_k| > n) \right)^2$$

$$= \sum_{k=1}^{n} P\{|X_k| > n\} + \sum_{1 \leqslant i \neq j \leqslant n} P\{|X_i| > n, |X_j| > n\}$$

$$\leqslant \sum_{k=1}^{n} P\{|X_k| > n\} + \frac{(1 + \psi(1)) E|X|}{n} \sum_{1 \leqslant i \neq j \leqslant n} P\{|X| > n\}$$

$$= \sum_{k=1}^{n} P\{|X_k| > n\} + \frac{(1 + \psi(1)) E|X|}{n} \cdot n(n-1) P\{|X| > n\}$$

$$\leqslant \sum_{k=1}^{n} P\{|X_k| > n\} + (1 + \psi(1)) E|X| \cdot n P\{|X| > n\}$$

$$= (1 + (1 + \psi(1)) E|X|) \sum_{k=1}^{n} P\{|X_k| > n\}.$$

By Lemma A.6 in Zhang and Wen (2001),

$$\left(1 - P\left\{ \max_{1 \leqslant k \leqslant n} |X_k| > n \right\} \right)^2 \sum_{k=1}^{n} P\{|X_k| > n\} \leqslant (1 + (1 + \psi(1)) E|X|) P\left\{ \max_{1 \leqslant k \leqslant n} |X_k| > n \right\},$$

which, together with (2.14), implies that for all large n

$$nP\{|X| > n\} = \sum_{k=1}^{n} P\{|X_k| > n\} \leqslant 2\left(1 + (1 + \psi(1))E|X|\right)P\left\{\max_{1 \leqslant k \leqslant n} |X_k| > n\right\}.$$

Thus, by (2.13),

$$\sum_{k=1}^{\infty} n^{r-1}P\{|X| > n\} < \infty,$$

which is equivalent to $E|X|^r < \infty$. The condition $EX = 0$ has already been obtained above. So we complete the proof. $\square$

3. Applications

From the complete convergence, we can define the complete consistency. If a sequence $\{\hat{\theta}_n, n \geqslant 1\}$ of statistical estimators converges completely to a parameter θ, then one says that this sequence is completely consistent, and write $\hat{\theta}_n \to \theta$ completely.

In the section, we will consider the complete consistency of the least-squares (LS) estimators in the EV regression model with ψ-mixing errors. Recall that the simple linear EV regression model is

$$\eta_k = \theta + \beta x_k + \varepsilon_k, \quad \xi_k = x_k + \delta_k, \quad 1 \leqslant k \leqslant n, \tag{3.1}$$

where $\theta, \beta, x_1, \cdots, x_n$ are unknown parameters or constants, the errors $(\varepsilon_k, \delta_k), 1 \leqslant k \leqslant n$, are random vectors and $\xi_k, \eta_k, 1 \leqslant k \leqslant n$, are observable random variables. From (3.1), we have

$$\eta_k = \theta + \beta \xi_k + (\varepsilon_k - \beta \delta_k), \quad 1 \leqslant k \leqslant n.$$

As a usual regression model of η_k on ξ_k with the errors $\varepsilon_k - \beta \delta_k$, we can get LS estimators of β and θ as

$$\hat{\beta}_n = \frac{\sum_{k=1}^{n}(\xi_k - \bar{\xi}_n)(\eta_k - \bar{\eta}_n)}{\sum_{k=1}^{n}(\xi_k - \bar{\xi}_n)^2}, \quad \hat{\theta}_n = \bar{\eta}_n - \hat{\beta}_n \bar{\xi}_n,$$

where $\bar{\xi}_n = n^{-1}\sum_{k=1}^{n} \xi_k$ and $\bar{\eta}_n = n^{-1}\sum_{k=1}^{n} \eta_k$. The notations $\bar{\delta}_n$ and $\bar{x}_n$ are defined in the same way. Based on these notations, we have

$$\hat{\beta}_n - \beta = \frac{\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^{n}(x_k - \bar{x}_n)(\varepsilon_k - \beta \delta_k) - \beta \sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2}{\sum_{k=1}^{n}(\xi_k - \bar{\xi}_n)^2} \tag{3.2}$$

and

$$\hat{\theta}_n - \theta = \bar{x}_n(\beta - \hat{\beta}_n) + (\beta - \hat{\beta}_n)\bar{\delta}_n + \bar{\varepsilon}_n - \beta \bar{\delta}_n. \tag{3.3}$$

The model (3.1) is called the EV model or measurement error model which was poposed by Deaton (1985) to correct the effects of the sampling errors and is more practical than the ordinary regression model. Fuller (1987) summarized many early works

for the EV models. The last two decades, the studies for the EV model have attracted much attention due to its simple form and wide applicability. For the EV model with i.i.d. errors $\{(\varepsilon_n, \delta_n), n \geqslant 1\}$, Liu and Chen (2005) obtained the weak and strong consistency of the LS estimators. They proved that a necessary and sufficient condition for $\hat{\beta}_n$ being weakly and strongly consistent is $s_n/n \to \infty$, where $s_n = \sum_{k=1}^n (x_k - \bar{x}_n)^2$. They also proved that a necessary and sufficient condition for $\hat{\theta}_n$ being weakly consistent is $n\bar{x}_n/s_n^* \to 0$, where $s_n^* = \max\{n, s_n\}$. Miao et al. (2011) obtained convergence rates of $\hat{\beta}_n \to \beta$ a.s. and $\hat{\theta}_n \to \theta$ a.s. under some conditions. They proved that if $s_n/n \to \infty$, then $\sqrt{s_n} n^{-1/2}(\hat{\beta}_n - \beta) \to 0$ a.s., which improves the result of Liu and Chen (2005). For the EV model with dependent errors, the strong consistency of the LS estimators was studied by Wang et al. (2015) and Hu et al. (2017).

As applications of Theorem 1.1, we can obtain the complete consistency of the LS estimators for the unknown parameters, which complement the results of Liu and Chen (2005). The first one gives the complete consistency for the unknown parameter β.

THEOREM 3.1. *Under the model* (3.1), *assume that* $\{(\varepsilon, \delta), (\varepsilon_n, \delta_n), n \geqslant 1\}$ *is a sequence of identically distributed ψ-mixing random vectors with $E\varepsilon = E\delta = 0$, $E\varepsilon^4 < \infty$ and $E\delta^4 < \infty$. Further, assume that $E(\varepsilon\delta) - \beta E\delta^2 \neq 0$. Then*

$$\hat{\beta}_n \to \beta \ completely \ if \ and \ only \ if \ s_n/n \to \infty,$$

where $s_n = \sum_{k=1}^n (x_k - \bar{x}_n)^2$.

Proof. Sufficiency. Assume that $s_n/n \to \infty$. Noting that

$$s_n^{-1} \sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2 = 1 + 2s_n^{-1} \sum_{k=1}^n (x_k - \bar{x}_n)\delta_k + s_n^{-1} \sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2,$$

we have from (3.2) that for any $\Delta > 0$,

$$
\begin{aligned}
&P\{|\hat{\beta}_n - \beta| > \Delta\} \\
&= P\left\{|\hat{\beta}_n - \beta| > \Delta, \left|s_n^{-1}\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2 - 1\right| \leqslant 1/2\right\} \\
&\quad + P\left\{|\hat{\beta}_n - \beta| > \Delta, \left|s_n^{-1}\sum_{k=1}^n (\xi_k - \bar{\xi}_n)^2 - 1\right| > 1/2\right\} \\
&\leqslant P\left\{s_n^{-1}\left|\sum_{k=1}^n (\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^n (x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) - \beta\sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right| > \Delta/2\right\} \\
&\quad + P\left\{s_n^{-1}\left|2\sum_{k=1}^n (x_k - \bar{x}_n)\delta_k + \sum_{k=1}^n (\delta_k - \bar{\delta}_n)^2\right| > 1/2\right\}.
\end{aligned}
$$

Therefore, to prove $\hat{\beta}_n \to \beta$ completely, it suffices to prove that

$$\sum_{n=1}^{\infty} P\left\{ s_n^{-1} \left| \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k \right| > \Delta \right\} < \infty, \quad \forall \Delta > 0, \tag{3.4}$$

$$\sum_{n=1}^{\infty} P\left\{ s_n^{-1} \left| \sum_{k=1}^{n} (x_k - \bar{x}_n)\varepsilon_k \right| > \Delta \right\} < \infty, \quad \forall \Delta > 0, \tag{3.5}$$

$$\sum_{n=1}^{\infty} P\left\{ s_n^{-1} \left| \sum_{k=1}^{n} (x_k - \bar{x}_n)\delta_k \right| > \Delta \right\} < \infty, \quad \forall \Delta > 0, \tag{3.6}$$

$$\sum_{n=1}^{\infty} P\left\{ s_n^{-1} \left| \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)^2 \right| > \Delta \right\} < \infty, \quad \forall \Delta > 0. \tag{3.7}$$

Since $s_n/n \to \infty$, we have that for all large n,

$$P\left\{ s_n^{-1} \left| \sum_{k=1}^{n} (\delta_k - \bar{\delta}_n)\varepsilon_k \right| > \Delta \right\}$$

$$\leqslant P\left\{ (n/s_n) \cdot n^{-1} \left| \sum_{k=1}^{n} \delta_k \varepsilon_k \right| > \Delta/2 \right\} + P\left\{ (n/s_n) \cdot |\bar{\varepsilon}_n \bar{\delta}_n| > \Delta/2 \right\}$$

$$\leqslant P\left\{ n^{-1} \left| \sum_{k=1}^{n} (\delta_k \varepsilon_k - E\delta_k \varepsilon_k) \right| > \Delta/4 \right\} + P\left\{ |\bar{\varepsilon}_n \bar{\delta}_n| > \Delta/2 \right\}.$$

Noting that

$$E|\varepsilon\delta|^2 \leqslant (E\varepsilon^4)^{1/2}(E\delta^4)^{1/2} < \infty,$$

we have by Corollary 1.1 that

$$\sum_{n=1}^{\infty} P\left\{ n^{-1} \left| \sum_{k=1}^{n} (\delta_k \varepsilon_k - E\delta_k \varepsilon_k) \right| > \Delta/4 \right\} < \infty.$$

By Corollary 1.1 again, we also have

$$\sum_{n=1}^{\infty} P\left\{ |\bar{\varepsilon}_n \bar{\delta}_n| > \Delta/2 \right\} \leqslant \sum_{n=1}^{\infty} P\left\{ |\bar{\varepsilon}_n| > \sqrt{\Delta/2} \right\} + \sum_{n=1}^{\infty} P\left\{ |\bar{\delta}_n| > \sqrt{\Delta/2} \right\} < \infty.$$

Thus (3.4) holds.

To prove (3.5), set $a_{nk} = n(x_k - \bar{x}_n)/s_n$ for $n \geqslant 1$ and $1 \leqslant k \leqslant n$. Then

$$\sup_{n \geqslant 1} n^{-1} \sum_{k=1}^{n} |a_{nk}|^2 = \sup_{n \geqslant 1} n/s_n < \infty.$$

By Theorem 1.1 with $r = p = q = 2$,

$$\sum_{n=1}^{\infty} P\left\{ s_n^{-1} \left| \sum_{k=1}^{n} (x_k - \bar{x}_n)\varepsilon_k \right| > \Delta \right\} = \sum_{n=1}^{\infty} P\left\{ \left| \sum_{k=1}^{n} a_{nk}\varepsilon_k \right| > \Delta n \right\} < \infty,$$

which implies (3.5). Similarly, (3.6) holds.

Since $s_n/n \to \infty$, we have that for all large n,

$$P\left\{s_n^{-1}\left|\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2\right| > \Delta\right\} = P\left\{s_n^{-1}\left|\sum_{k=1}^{n}\delta_k^2 - n\bar{\delta}_n^2\right| > \Delta\right\}$$

$$= P\left\{s_n^{-1}\left|\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) + nE\delta^2 - n\bar{\delta}_n^2\right| > \Delta\right\}$$

$$\leqslant P\left\{s_n^{-1}\left|\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2)\right| > \Delta/2\right\} + P\left\{s_n^{-1}n\left|E\delta^2 - \bar{\delta}_n^2\right| > \Delta/2\right\}$$

$$\leqslant P\left\{n^{-1}\left|\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2)\right| > \Delta/2\right\} + P\left\{\bar{\delta}_n^2 > \Delta/4\right\}.$$

By Corollary 1.1,

$$\sum_{n=1}^{\infty}P\left\{n^{-1}\left|\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2)\right| > \Delta/2\right\} < \infty$$

and $\sum_{n=1}^{\infty}P\left\{\bar{\delta}_n^2 > \Delta/4\right\} < \infty$, and hence (3.7) holds.

Necessity. By the Borel-Cantelli lemma, $\hat{\beta}_n \to \beta$ completely implies

$$\hat{\beta}_n \to \beta \text{ a.s.}$$

Thus $s_n/n \to \infty$ by Hu et al (2017). The proof is completed. $\quad\square$

The following theorem provides the complete consistency for the unknown parameter θ.

THEOREM 3.2. *Under the assumptions of Theorem 3.1, further assume that* $\sup_{n \geqslant 1} n\bar{x}_n^2/s_n^* < \infty$ *and* $E(\varepsilon\delta) - \beta E\delta^2 \neq 0$, *where* $s_n^* = \max\{n, s_n\}$. *Then*

$$\hat{\theta}_n \to \theta \text{ completely if and only if } n\bar{x}_n/s_n^* \to 0.$$

Proof. Sufficiency. Assume that $n\bar{x}_n/s_n^* \to 0$. Note that

$$\frac{1}{s_n^*}\sum_{k=1}^{n}(\xi_k - \bar{\xi}_n)^2 = \frac{s_n + nE\delta^2}{s_n^*} + \frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}$$

and $(s_n + nE\delta^2)/s_n^* \geqslant \min\{1, E\delta^2\}$ from the definition of s_n^*. If

$$\left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| \leqslant \Delta',$$

then $\sum_{k=1}^{n}(\xi_k - \bar{\xi}_n)^2/s_n^* \geqslant \Delta'$, where $\Delta' = \min\{1, E\delta^2\}/2$. Therefore by (3.2), for any $\Delta > 0$

$$
P\{|\bar{x}_n(\beta - \hat{\beta}_n)| > \Delta\}
$$

$$
\leqslant P\left\{|\bar{x}_n(\beta - \hat{\beta}_n)| > \Delta, \left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| \leqslant \Delta'\right\}
$$

$$
+ P\left\{|\bar{x}_n(\beta - \hat{\beta}_n)| > \Delta, \left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| > \Delta'\right\}
$$

$$
\leqslant P\left\{\left|\frac{\bar{x}_n}{s_n^*}\left(\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^{n}(x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) - \beta\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2\right)\right| > \Delta'\Delta\right\}
$$

$$
+ P\left\{\left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| > \Delta'\right\},
$$

and similarly,

$$
P\left\{|(\beta - \hat{\beta}_n)\bar{\delta}_n| > \Delta\right\}
$$

$$
\leqslant P\left\{\left|\frac{\bar{\delta}_n}{s_n^*}\left(\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^{n}(x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) - \beta\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2\right)\right| > \Delta'\Delta\right\}
$$

$$
+ P\left\{\left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| > \Delta'\right\}.
$$

By the same argument as the proof of Theorem 3.1, we get that for all $\Delta > 0$,

$$
\sum_{n=1}^{\infty}P\left\{\left|\frac{\bar{x}_n}{s_n^*}\left(\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)\varepsilon_k + \sum_{k=1}^{n}(x_k - \bar{x}_n)(\varepsilon_k - \beta\delta_k) - \beta\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2\right)\right| > \Delta\right\} < \infty
$$

and

$$
\sum_{n=1}^{\infty}P\left\{\left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| > \Delta\right\} < \infty,
$$

which imply that $\sum_{n=1}^{\infty}P\{|\bar{x}_n(\beta - \hat{\beta}_n)| > \Delta\} < \infty, \forall\Delta > 0$.

Noting that

$$
\frac{\bar{\delta}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)\varepsilon_k = \frac{n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k\varepsilon_k - E\delta_k\varepsilon_k)\cdot\bar{\delta}_n - \frac{n}{s_n^*}\cdot\bar{\delta}_n^2\bar{\varepsilon}_n + \frac{nE\delta\varepsilon}{s_n^*}\cdot\bar{\delta}_n,
$$

$$
\frac{\bar{\delta}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2 = \frac{n}{s_n^*}\cdot\frac{1}{n}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2)\cdot\bar{\delta}_n - \frac{n}{s_n^*}\cdot\bar{\delta}_n^3 + \frac{nE\delta^2}{s_n^*}\cdot\bar{\delta}_n,
$$

we can easily prove that

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{\bar{\delta}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)\varepsilon_k\right| > \Delta\right\} < \infty, \ \forall \Delta > 0,$$

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{\bar{\delta}_n}{s_n^*}\sum_{k=1}^{n}(\delta_k - \bar{\delta}_n)^2\right| > \Delta\right\} < \infty, \ \forall \Delta > 0.$$

We can also easily prove that

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{\bar{\delta}_n}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x_n})(\varepsilon_k - \beta\delta_k)\right| > \Delta\right\} < \infty, \ \forall \Delta > 0,$$

$$\sum_{n=1}^{\infty} P\left\{\left|\frac{2}{s_n^*}\sum_{k=1}^{n}(x_k - \bar{x}_n)\delta_k + \frac{1}{s_n^*}\sum_{k=1}^{n}(\delta_k^2 - E\delta_k^2) - \frac{n\bar{\delta}_n^2}{s_n^*}\right| > \Delta\right\} < \infty, \ \forall \Delta > 0.$$

Hence, $\sum_{n=1}^{\infty} P\{|\bar{\delta}_n(\beta - \hat{\beta}_n)| > \Delta\} < \infty, \ \forall \Delta > 0$.

Obviously, we have by Corollary 1.1 that

$$\sum_{n=1}^{\infty} P\{|\bar{\varepsilon}_n - \beta\bar{\delta}_n| > \Delta\} < \infty, \ \forall \Delta > 0.$$

Therefore, by (3.3),

$$\sum_{n=1}^{\infty} P\{|\hat{\theta}_n - \theta| > \Delta\} < \infty, \ \forall \Delta > 0.$$

Necessity. It is obvious that $\hat{\theta}_n \to \theta$ completely implies

$$\hat{\theta}_n \to \theta \ \text{a.s.}$$

by the Borel-Cantelli lemma. Then $n\bar{x}_n/s_n^* \to 0$ by Hu et al. (2017). Hence the proof is completed. □

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