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ESTIMATES FOR THE NUMERICAL RADIUS OF $n \times n$ OPERATOR MATRICES

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Abstract. We present new upper bounds for the numerical radius of $n \times n$ operator matrices defined on a complex Hilbert space, i.e., operator matrices of the form $[T_{ij}]$, and illustrate with numerical examples that these bounds are better than the existing bounds.

1. Introduction

Let $\mathscr{H}_i$ and $\mathscr{H}_j$ be any two complex Hilbert spaces, and $B(\mathscr{H}_j, \mathscr{H}_i)$ denote the space of all bounded linear operators from the Hilbert space $\mathscr{H}_j$ to $\mathscr{H}_i$, if $\mathscr{H}_1 = \mathscr{H}_2 = \mathscr{H}$, then we write $\mathbb{B}(\mathscr{H}_1, \mathscr{H}_2) = \mathbb{B}(\mathscr{H})$. When $\dim \mathscr{H} = n$, $\mathbb{B}(\mathscr{H})$ means the full matrix algebra $M_n(\mathbb{C})$ of all $n \times n$ matrices in the complex field $\mathbb{C}$. For $T \in \mathbb{B}(\mathscr{H})$, the conjugate transpose of T is denoted by T^*, $T = Re(T) + iIm(T)$ is the Cartesian decomposition of T and the matrices $Re(T) = \frac{1}{2}(T + T^*)$ and $Im(T) = \frac{1}{2i}(T - T^*)$ are the real part and the imaginary part of T, respectively.

Let $T \in \mathbb{B}(\mathscr{H})$, $\omega(T)$ and $\|T\|$ be the numerical radius and operator norm of T respectively, defined as follows:

$$\omega(T) = \sup\{|\langle Tx, x\rangle| : x \in \mathscr{H}, \|x\| = 1\}$$

and

$$\|T\| = \sup\{|\langle Tx, y\rangle| : x \in \mathscr{H}, \|x\| = \|y\| = 1\}.$$

It is well known that $w(.)$ is a norm on $\mathbb{B}(\mathscr{H})$, and for $T \in \mathbb{B}(\mathscr{H})$, the following result

$$\frac{1}{2}\|T\| \leqslant \omega(T) \leqslant \|T\| \tag{1.1}$$

holds. These inequalities are sharp. The first inequality becomes an equality if $T^2 = 0$, i.e.,

$$\omega(T) = \frac{1}{2}\|T\|, \tag{1.2}$$

and if T is normal, the second inequality is an equality.

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Let $\sigma(T)$ denote the spectrum of T, and $r(T)$, the spectral radius of T, defined as

$$r(T) := \sup\{|\lambda| : \lambda \in \sigma(T)\}.$$

Then

$$r(T) \leqslant \omega(T) \leqslant \|T\| \tag{1.3}$$

holds, and equality holds when T is normal, i.e.,

$$r(T) = \omega(T) = \|T\|. \tag{1.4}$$

Over the years, many eminent mathematicians such as F. Kittaneh and S.S. Dragomir and others have obtained various refinements and generalization of (1.1), to cite a few of them are [1, 2, 3] and references therein. A more tempting question is to investigate the upper and lower bounds of the numerical radius for T, which is a 2×2 or $n\times n$ operator matrix, we refer the reader to [4, 5, 6] as a recent work treating such operator matrices.

Recently, Al-Dolat et al. [7] have obtained the following results for 2×2 operator matrices.

THEOREM 1.1. [7, Theorem 2.7] *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let $A \in \mathbb{B}(\mathscr{H}_1)$, $B \in \mathbb{B}(\mathscr{H}_2, \mathscr{H}_1)$, $C \in \mathbb{B}(\mathscr{H}_1, \mathscr{H}_2)$, and $D \in \mathbb{B}(\mathscr{H}_2)$. Then, for $t \in [0,1]$,*

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\|A\| + 2\omega(D) + \left\|t^2AA^* + BB^*\right\|^{\frac{1}{2}} + \left\|(1-t)^2AA^* + C^*C\right\|^{\frac{1}{2}}\right).$$

THEOREM 1.2. [7, Theorem 2.8] *Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be Hilbert spaces and let $A \in \mathbb{B}(\mathscr{H}_1)$, $B \in \mathbb{B}(\mathscr{H}_2, \mathscr{H}_1)$, $C \in \mathbb{B}(\mathscr{H}_1, \mathscr{H}_2)$, and $D \in \mathbb{B}(\mathscr{H}_2)$. Then for $t \in [0,1]$,*

$$\omega\left(\begin{bmatrix} A & B \\ C & D \end{bmatrix}\right) \leqslant \frac{1}{2}\left(\omega(A) + 2\omega(D) + \sqrt{t^2\omega^2(A) + \|B\|^2} + \sqrt{(1-t)^2\omega^2(A) + \|C\|^2}\right).$$

In this paper, following the idea of Al-Dolat et al. [9], we establish new upper bounds for the numerical radius of $n\times n$ operator matrices, i.e., operator matrices of the form $[T_{ij}]$, and we use new methods to obtain upper bounds of $n\times n$ operator matrices, which have not appeared in previous papers. Quite apart from that, specific examples are given to compare our results with existing results.

2. Main results

We begin this section with the following sequence of lemmas which will be used to reach our goal in this present article.

LEMMA 2.1. [8] *Let $T \in \mathbb{B}(\mathscr{H})$. Then*

$$\omega(T) = \sup_{\theta\in\mathbb{R}} \left\|Re\left(e^{i\theta}T\right)\right\|.$$

LEMMA 2.2. [9] *Let* $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ *be Hilbert spaces, and let* $T=[T_{ij}]$ *be an* $n\times n$ *operator matrix with* $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$*.Then*

$$r([T_{ij}])\leqslant r([\|T_{ij}\|]). \tag{2.1}$$

LEMMA 2.3. [10, Remark 2.8] *Let* $T\in\mathbb{B}(\mathscr{H})$*. Then*

$$\omega^2([T])\leqslant\|Re(T)\|^2+\|Im(T)\|^2.$$

LEMMA 2.4. $r(AB)=r(BA)$ *for every* $A,\ B\in\mathbb{B}(\mathscr{H})$*.*

Now, we are in a position to begin our main work. We give two novel estimates for the numerical radius of $n\times n$ operator matrices. Our findings depend mainly on results of Theorems 1.1 and 1.2.

THEOREM 2.5. *Let* $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ *be Hilbert spaces, and let* $T=[T_{ij}]$ *be an* $n\times n$ *operator matrix with* $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$*, where* $1\leqslant i,j\leqslant n$*. Then for* $t\in[0,1]$*, we have*

$$\omega(T)\leqslant\frac{1}{2}\left(D_1+D_2+\cdots+D_n\right),$$

where

$$\begin{aligned}
D_1&=\|T_{11}\|+\left\|t^2T_{11}T_{11}^*+T_{12}T_{12}^*+\cdots+T_{1n}T_{1n}^*\right\|^{\frac{1}{2}}\\
&\quad+\left\|(1-t)^2T_{11}T_{11}^*+T_{21}^*T_{21}+\cdots+T_{n1}^*T_{n1}\right\|^{\frac{1}{2}},\\
D_2&=\|T_{22}\|+\left\|t^2T_{22}T_{22}^*+T_{23}T_{23}^*+\cdots+T_{2n}T_{2n}^*\right\|^{\frac{1}{2}}\\
&\quad+\left\|(1-t)^2T_{22}T_{22}^*+T_{32}^*T_{32}+\cdots+T_{n2}^*T_{n2}\right\|^{\frac{1}{2}},\\
&\ \ \vdots\\
D_{n-1}&=\|T_{(n-1)(n-1)}\|+\left\|t^2T_{(n-1)(n-1)}T_{(n-1)(n-1)}^*+T_{(n-1)n}T_{(n-1)n}^*\right\|^{\frac{1}{2}}\\
&\quad+\left\|(1-t)^2T_{(n-1)(n-1)}T_{(n-1)(n-1)}^*+T_{n(n-1)}^*T_{n(n-1)}\right\|^{\frac{1}{2}},\\
D_n&=2\omega(T_{nn}).
\end{aligned}$$

Proof. By the triangle inequality of numerical radius, the first equality of (1.4) and Lemmas 2.2 and 2.4, we can get that

$$\begin{aligned}
&2\left\|Re\left(e^{i\theta}T\right)\right\|\\
&=2\omega\left(Re\left(e^{i\theta}T\right)\right)\\
&=\omega\left(\begin{bmatrix}
e^{i\theta}T_{11}+e^{-i\theta}T_{11}^* & e^{i\theta}T_{12}+e^{-i\theta}T_{21}^* & \cdots & e^{i\theta}T_{1n}+e^{-i\theta}T_{n1}^*\\
e^{i\theta}T_{21}+e^{-i\theta}T_{12}^* & e^{i\theta}T_{22}+e^{-i\theta}T_{22}^* & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^*\\
\vdots & \vdots & & \vdots\\
e^{i\theta}T_{n1}+e^{-i\theta}T_{1n}^* & e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^* & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^*
\end{bmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&\leqslant \omega\left(\begin{bmatrix} t\left(e^{i\theta}T_{11}+e^{-i\theta}T_{11}^*\right) & e^{i\theta}T_{12} & \cdots & e^{i\theta}T_{1n} \\ e^{-i\theta}T_{12}^* & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{-i\theta}T_{1n}^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad +\omega\left(\begin{bmatrix} (1-t)\left(e^{i\theta}T_{11}+e^{-i\theta}T_{11}^*\right) & e^{-i\theta}T_{21}^* & \cdots & e^{-i\theta}T_{n1}^* \\ e^{i\theta}T_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{i\theta}T_{n1} & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad +\omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{i\theta}T_{22}+e^{-i\theta}T_{22}^* & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^* \\ \vdots & \vdots & & \vdots \\ 0 & e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^* & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^* \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} tT_{11}^* & 0 & \cdots & e^{i\theta}I \\ T_{12}^* & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_{1n}^* & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ tT_{11} & T_{12} & \cdots & T_{1n} \end{bmatrix}\right) \\
&\quad +r\left(\begin{bmatrix} (1-t)T_{11}^* & 0 & \cdots & e^{i\theta}I \\ e^{2i\theta}T_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{2i\theta}T_{n1} & 0 & \cdots & 0 \end{bmatrix}\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (1-t)T_{11} & e^{-2i\theta}T_{21}^* & \ldots & e^{-2i\theta}T_{n1}^* \end{bmatrix}\right) \\
&\quad +\omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{i\theta}T_{22}+e^{-i\theta}T_{22}^* & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^* \\ \vdots & \vdots & & \vdots \\ 0 & e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^* & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^* \end{bmatrix}\right) \\
&= r\left(\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ tT_{11} & T_{12} & \cdots & T_{1n} \end{bmatrix}\begin{bmatrix} tT_{11}^* & 0 & \cdots & e^{i\theta}I \\ T_{12}^* & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_{1n}^* & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad +r\left(\begin{bmatrix} e^{-i\theta}I & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ (1-t)T_{11} & e^{-2i\theta}T_{21}^* & \ldots & e^{-2i\theta}T_{n1}^* \end{bmatrix}\begin{bmatrix} (1-t)T_{11}^* & 0 & \cdots & e^{i\theta}I \\ e^{2i\theta}T_{21} & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ e^{2i\theta}T_{n1} & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad +\omega\left(\begin{bmatrix} e^{i\theta}T_{22}+e^{-i\theta}T_{22}^* & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^* \\ \vdots & & \vdots \\ e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^* & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^* \end{bmatrix}\right)
\end{aligned}
$$

$$
\begin{aligned}
&= r\left(\begin{bmatrix} te^{-i\theta}T_{11}^{*} & 0\cdots & I \\ 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots \\ t^2T_{11}T_{11}^{*}+T_{12}T_{12}^{*}+\cdots+T_{1n}T_{1n}^{*} & 0\cdots & te^{i\theta}T_{11}\end{bmatrix}\right) \\
&\quad +r\left(\begin{bmatrix} (1-t)e^{-i\theta}T_{11}^{*} & 0\cdots & I \\ 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots \\ (1-t)^2T_{11}T_{11}^{*}+T_{21}^{*}T_{21}+\cdots+T_{n1}^{*}T_{n1} & 0\cdots & (1-t)e^{i\theta}T_{11}\end{bmatrix}\right) \\
&\quad +\omega\left(\begin{bmatrix} e^{i\theta}T_{22}+e^{-i\theta}T_{22}^{*} & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^{*} \\ \vdots & & \vdots \\ e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^{*} & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^{*}\end{bmatrix}\right) \\
&\leqslant r\left(\begin{bmatrix} t\|T_{11}\| & 0\cdots & 1 \\ 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots \\ \|t^2T_{11}T_{11}^{*}+T_{12}T_{12}^{*}+\cdots+T_{1n}T_{1n}^{*}\| & 0\cdots & t\|T_{11}\|\end{bmatrix}\right) \\
&\quad +r\left(\begin{bmatrix} (1-t)\|T_{11}\| & 0\cdots & 1 \\ 0 & 0\cdots & 0 \\ \vdots & \vdots & \vdots \\ \|(1-t)^2T_{11}T_{11}^{*}+T_{21}^{*}T_{21}+\cdots+T_{n1}^{*}T_{n1}\| & 0\cdots & (1-t)\|T_{11}\|\end{bmatrix}\right) \\
&\quad +\omega\left(\begin{bmatrix} e^{i\theta}T_{22}+e^{-i\theta}T_{22}^{*} & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^{*} \\ \vdots & & \vdots \\ e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^{*} & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^{*}\end{bmatrix}\right) \\
&\leqslant \|T_{11}\|+\|t^2T_{11}T_{11}^{*}+T_{12}T_{12}^{*}+\cdots+T_{1n}T_{1n}^{*}\|^{\frac{1}{2}} \\
&\quad +\|(1-t)^2T_{11}T_{11}^{*}+T_{21}^{*}T_{21}+\cdots+T_{n1}^{*}T_{n1}\|^{\frac{1}{2}} \\
&\quad +\omega\left(\begin{bmatrix} e^{i\theta}T_{22}+e^{-i\theta}T_{22}^{*} & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^{*} \\ \vdots & & \vdots \\ e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^{*} & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^{*}\end{bmatrix}\right) \\
&\leqslant \cdots \\
&\leqslant D_1+D_2+\cdots+D_n,
\end{aligned}
$$

Thus, on account of Lemma 2.1, we have

$$
\omega(T)=\sup_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}T\right)\right\|\leqslant\frac{1}{2}\left(D_1+D_2+\cdots+D_n\right). \quad \square
$$

To make things a bit clearer, we list the next corollary which can be obtained immediately by setting $n=3$ in Theorem 2.5.

COROLLARY 2.6. *Let $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n$ be Hilbert spaces, and let*

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

be a 3×3 operator matrix with $T_{ij} \in \mathbb{B}(\mathscr{H}_j, \mathscr{H}_i)$, where $1 \leqslant i, j \leqslant 3$. Then, for $t \in [0,1]$, we have

$$\begin{aligned} \omega(T) \leqslant \frac{1}{2} \Big(& \|T_{11}\| + \left\|t^2 T_{11} T_{11}^* + T_{12} T_{12}^* + T_{13} T_{13}^*\right\|^{\frac{1}{2}} \\ & + \left\|(1-t)^2 T_{11} T_{11}^* + T_{21}^* T_{21} + T_{31}^* T_{31}\right\|^{\frac{1}{2}} + \|T_{22}\| + \|t^2 T_{22} T_{22}^* + T_{23} T_{23}^*\|^{\frac{1}{2}} \\ & + \|(1-t)^2 T_{22} T_{22}^* + T_{32}^* T_{32}\|^{\frac{1}{2}} + 2\omega(T_{33}) \Big), \end{aligned}$$

REMARK 2.7. Guelfen and Kittaneh [11, Corollary 2.8] have been obtained that if $T = [T_{ij}]$ is an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathscr{H}_j, \mathscr{H}_i)$, then

$$\omega(T) \leqslant \frac{1}{2} \sum_{i=1}^{n} \left(\|T_{ii}\| + \sqrt{\left\| T_{ii} T_{ii}^* + \sum_{j \neq i, j=1}^{n} T_{ij} T_{ij}^* \right\|} \right).$$

Considering the operator $T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$, where $T_{11} = T_{12} = T_{22} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $T_{13} = T_{21} = \begin{pmatrix} 0 & 2 \\ 2 & 0 \end{pmatrix}$, $T_{23} = T_{31} = \begin{pmatrix} 0 & 3 \\ 3 & 0 \end{pmatrix}$, $T_{32} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$, $T_{33} = \begin{pmatrix} 0 & 0 \\ 5 & 0 \end{pmatrix}$, it is easy to see that Theorem 2.5 which in the case $n = 3$, i.e., Corollary 2.6, gives

$$\omega(T) \leqslant \frac{1}{2}(10 + \sqrt{5} + \sqrt{14} + \sqrt{2}) \approx 8.6960$$

when $t = 0$, and

$$\omega(T) \leqslant \frac{1}{2}(8 + \sqrt{6} + \sqrt{13} + \sqrt{10}) \approx 8.6087$$

when $t = 1$ whereas the bound obtained by Guelfen and Kittaneh in [11, Corollary 2.8] gives

$$\omega(T) \leqslant \frac{1}{2}(7 + \sqrt{6} + \sqrt{14} + \sqrt{35}) \approx 9.5537$$

in the case $n = 3$. This indicates that for such operators the bound obtained by our result is better than that of Guelfen and Kittaneh.

A completely analogous argument of Theorem 2.5 leads to the following result.

THEOREM 2.8. *Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be Hilbert spaces, and let $T=[T_{ij}]$ be an $n\times n$ operator matrix with $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, where $1\leqslant i,j\leqslant n$. Then, for $t\in[0,1]$, we have*

$$\omega(T)\leqslant\frac{1}{2}\left(E_1+E_2+\cdots+E_n\right),$$

where

$$\begin{aligned}
E_1&=\omega(T_{11})+\sqrt{t^2\omega^2(T_{11})+\|T_{12}\|^2+\cdots+\|T_{1n}\|^2}\\
&\quad+\sqrt{(1-t)^2\omega^2(T_{11})+\|T_{21}\|^2+\cdots+\|T_{n1}\|^2},\\
E_2&=\omega(T_{22})+\sqrt{t^2\omega^2(T_{22})+\|T_{23}\|^2+\cdots+\|T_{2n}\|^2}\\
&\quad+\sqrt{(1-t)^2\omega^2(T_{22})+\|T_{32}\|^2+\cdots+\|T_{n2}\|^2},\\
&\vdots\\
E_{n-1}&=\omega(T_{(n-1)(n-1)})+\sqrt{t^2\omega^2(T_{(n-1)(n-1)})+\|T_{(n-1)n}\|^2}\\
&\quad+\sqrt{(1-t)^2\omega^2(T_{(n-1)(n-1)})+\|T_{n(n-1)}\|^2},\\
E_n&=2\omega(T_{nn}).
\end{aligned}$$

Proof. By the triangle inequality of numerical radius, the first equality of (1.4) and Lemma 2.2, we can get that

$$\begin{aligned}
&2\left\|Re\left(e^{i\theta}T\right)\right\|\\
&=2\omega\left(Re\left(e^{i\theta}T\right)\right)\\
&=\omega\left(\begin{bmatrix}
e^{i\theta}T_{11}+e^{-i\theta}T_{11}^* & e^{i\theta}T_{12}+e^{-i\theta}T_{21}^* & \cdots & e^{i\theta}T_{1n}+e^{-i\theta}T_{n1}^*\\
e^{i\theta}T_{21}+e^{-i\theta}T_{12}^* & e^{i\theta}T_{22}+e^{-i\theta}T_{22}^* & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^*\\
\vdots & \vdots & & \vdots\\
e^{i\theta}T_{n1}+e^{-i\theta}T_{1n}^* & e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^* & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^*
\end{bmatrix}\right)\\
&\leqslant\omega\left(\begin{bmatrix}
t\left(e^{i\theta}T_{11}+e^{-i\theta}T_{11}^*\right) & e^{i\theta}T_{12} & \cdots & e^{i\theta}T_{1n}\\
e^{-i\theta}T_{12}^* & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots\\
e^{-i\theta}T_{1n}^* & 0 & \cdots & 0
\end{bmatrix}\right)\\
&\quad+\omega\left(\begin{bmatrix}
(1-t)\left(e^{i\theta}T_{11}+e^{-i\theta}T_{11}^*\right) & e^{-i\theta}T_{21}^* & \cdots & e^{-i\theta}T_{n1}^*\\
e^{i\theta}T_{21} & 0 & \cdots & 0\\
\vdots & \vdots & & \vdots\\
e^{i\theta}T_{n1} & 0 & \cdots & 0
\end{bmatrix}\right)
\end{aligned}$$

$$
\begin{aligned}
&+\omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 \\ 0 & e^{i\theta}T_{22}+e^{-i\theta}T_{22}^{*} & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^{*} \\ \vdots & \vdots & & \vdots \\ 0 & e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^{*} & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^{*} \end{bmatrix}\right) \\
\leqslant\, & r\left(\begin{bmatrix} t\|e^{i\theta}T_{11}+e^{-i\theta}T_{11}^{*}\| & \|T_{12}\| & \cdots & \|T_{1n}\| \\ \|T_{12}\| & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \|T_{1n}\| & 0 & \cdots & 0 \end{bmatrix}\right) \\
&+r\left(\begin{bmatrix} (1-t)\|e^{i\theta}T_{11}+e^{-i\theta}T_{11}^{*}\| & \|T_{21}\| & \cdots & \|T_{n1}\| \\ \|T_{21}\| & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ \|T_{n1}\| & 0 & \cdots & 0 \end{bmatrix}\right) \\
&+\omega\left(\begin{bmatrix} e^{i\theta}T_{22}+e^{-i\theta}T_{22}^{*} & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^{*} \\ \vdots & & \vdots \\ e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^{*} & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^{*} \end{bmatrix}\right) \\
\leqslant\, & \omega(T_{11})+\sqrt{t^{2}\omega^{2}(T_{11})+\|T_{12}\|^{2}+\cdots+\|T_{1n}\|^{2}} \\
&+\sqrt{(1-t)^{2}\omega^{2}(T_{11})+\|T_{21}\|^{2}+\cdots+\|T_{n1}\|^{2}} \\
&+\omega\left(\begin{bmatrix} e^{i\theta}T_{22}+e^{-i\theta}T_{22}^{*} & \cdots & e^{i\theta}T_{2n}+e^{-i\theta}T_{n2}^{*} \\ \vdots & & \vdots \\ e^{i\theta}T_{n2}+e^{-i\theta}T_{2n}^{*} & \cdots & e^{i\theta}T_{nn}+e^{-i\theta}T_{nn}^{*} \end{bmatrix}\right), \\
\leqslant\, & \cdots \\
\leqslant\, & E_1+E_2+\cdots+E_n.
\end{aligned}
$$

Thus, on account of Lemma 2.1, we have

$$\omega(T)=\sup_{\theta\in\mathbb{R}}\left\|Re\left(e^{i\theta}T\right)\right\|\leqslant\frac{1}{2}\left(E_1+E_2+\cdots+E_n\right). \quad \square$$

To make things a bit clearer, we list the next corollary which can be obtained immediately by setting $n=3$ in Theorem 2.8.

COROLLARY 2.9. *Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be Hilbert spaces, and let*

$$T=\begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

be a 3×3 operator matrix with $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, where $1\leqslant i,j\leqslant 3$. Then, for $t\in$

$[0,1]$, *we have*

$$\omega(T)\leqslant\frac{1}{2}\Bigg(\omega(T_{11})+\sqrt{t^2\omega^2(T_{11})+\|T_{12}\|^2+\|T_{13}\|^2}+\sqrt{(1-t)^2\omega^2(T_{11})+\|T_{21}\|^2+\|T_{31}\|^2}+\omega(T_{22})+\sqrt{t^2\omega^2(T_{22})+\|T_{23}\|^2}+\sqrt{(1-t)^2\omega^2(T_{22})+\|T_{32}\|^2}+2\omega(T_{33})\Bigg).$$

REMARK 2.10. Guelfen and Kittaneh [11, Corollary 2.6] has obtained that if $T=[T_{ij}]$ is an $n\times n$ operator matrix with $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, then

$$\omega(T)\leqslant\frac{1}{2}\sum_{i=1}^{n}\left(\omega(T_{ii})+\sqrt{\omega^2(T_{ii})+\sum_{j\neq i,j=1}^{n}\left\|T_{ij}\right\|^2}\right).$$

Considering the operator $T=\begin{bmatrix}T_{11}&T_{12}&T_{13}\\T_{21}&T_{22}&T_{23}\\T_{31}&T_{32}&T_{33}\end{bmatrix}$, where $T_{11}=T_{22}=\begin{pmatrix}0&6\\6&0\end{pmatrix}$ and $T_{12}=T_{13}=T_{21}=T_{23}=T_{31}=T_{32}=T_{33}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$, it is easy to see that Theorem 2.8 which in the case $n=3$, i.e., Corollary 2.9, gives $\omega(T)\leqslant 7+\sqrt{10}+\sqrt{11}\approx 13.4790$ when $t=\frac{1}{2}$ whereas the bound obtained by Guelfen and Kittaneh in [11, Corollary 2.6] gives $\omega(T)\leqslant 6.5+\sqrt{38}+\frac{1}{2}\sqrt{3}\approx 13.5305$ in the case $n=3$. This indicates that for such operators the bound obtained by our results is better than that of Guelfen and Kittaneh.

LEMMA 2.11. *Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be Hilbert spaces, $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, where $1\leqslant i,j\leqslant n$, and let* $T=\begin{bmatrix}0&\cdots&0&T_{1n}\\0&\cdots&T_{2(n-1)}&0\\\vdots&&\vdots&\vdots\\T_{n1}&\cdots&0&0\end{bmatrix}$ *be an $n\times n$ operator matrix. Then, for $t\in[0,1]$, we have*

$$\omega(T)\leqslant\frac{1}{2}\sqrt{\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}+T^*_{(n+1-i)i}\right\|^2\right\}+\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}-T^*_{(n+1-i)i}\right\|^2\right\}}.$$

Proof. First, we calculate that

$$\begin{aligned}\|Re(T)\|&=\frac{1}{2}\|T+T^*\|\\&=\frac{1}{2}\left\|\begin{bmatrix}0&\cdots&0&T_{1n}+T^*_{n1}\\0&\cdots&T_{2(n-1)}+T^*_{(n-1)2}&0\\\vdots&&\vdots&\vdots\\T_{n1}+T^*_{1n}&\cdots&0&0\end{bmatrix}\right\|\\&=\frac{1}{2}\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}+T^*_{(n+1-i)i}\right\|\right\}.\end{aligned}$$

For the same reason, we have

$$\|Im(T)\| = \frac{1}{2} \max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} - T^*_{(n+1-i)i} \right\| \right\}.$$

Thus, utilizing Lemma 2.3, we can get that

$$\begin{aligned}
\omega(T) &\leqslant \sqrt{\frac{1}{4} \max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} + T^*_{(n+1-i)i} \right\| \right\}^2 + \frac{1}{4} \max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} - T^*_{(n+1-i)i} \right\| \right\}^2} \\
&= \frac{1}{2} \sqrt{\max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} + T^*_{(n+1-i)i} \right\|^2 \right\} + \max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} - T^*_{(n+1-i)i} \right\|^2 \right\}}.
\end{aligned}$$

□

The next result is a direct application of Lemma 2.11, which gives a new method to deal with $n \times n$ operator matrices.

THEOREM 2.12. *Let $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n$ be Hilbert spaces, and let $T = [T_{ij}]$ be an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathscr{H}_j, \mathscr{H}_i)$, where $1 \leqslant i, j \leqslant n$. Then, for $t \in [0,1]$, we have*

$$\omega(T) \leqslant F_1 + F_2 + \cdots + F_n,$$

where

$$\begin{aligned}
F_1 &= \frac{1}{2} \sqrt{\max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} + T^*_{(n+1-i)i} \right\|^2 \right\} + \max_{i=1,2,\cdots,n} \left\{ \left\| T_{i(n+1-i)} - T^*_{(n+1-i)i} \right\|^2 \right\}}, \\
F_2 &= \frac{1}{2} \sqrt{\max_{i=1,2,\cdots,n-1} \left\{ \left\| T_{i(n-i)} + T^*_{(n-i)i} \right\|^2 \right\} + \max_{i=1,2,\cdots,n-1} \left\{ \left\| T_{i(n-i)} - T^*_{(n-i)i} \right\|^2 \right\}} \\
&+ \frac{1}{2} \sqrt{\max_{i=1,\cdots,n-1} \left\{ \left\| T_{(i+1)(n+1-i)} + T^*_{(n+1-i)(i+1)} \right\|^2 \right\} + \max_{i=1,\cdots,n-1} \left\{ \left\| T_{(i+1)(n+1-i)} - T^*_{(n+1-i)(i+1)} \right\|^2 \right\}}, \\
&\vdots \\
F_{n-1} &= \frac{1}{2} \sqrt{\max\{\|T_{12} + T^*_{21}\|^2, \|T_{21} + T^*_{12}\|^2\} + \max\{\|T_{12} - T^*_{21}\|^2, \|T_{21} - T^*_{12}\|^2\}} \\
&+ \frac{1}{2} [\max\{\|T_{(n-1)n} + T^*_{n(n-1)}\|^2, \|T_{n(n-1)} + T^*_{(n-1)n}\|^2\} \\
&+ \max\{\|T_{(n-1)n} - T^*_{n(n-1)}\|^2, \|T_{n(n-1)} - T^*_{(n-1)n}\|^2\}]^{\frac{1}{2}} \\
&= \frac{1}{2} \sqrt{\left\| T_{12} + T^*_{21} \right\|^2 + \left\| T_{12} - T^*_{21} \right\|^2} + \frac{1}{2} \sqrt{\left\| T_{(n-1)n} + T^*_{n(n-1)} \right\|^2 + \left\| T_{(n-1)n} - T^*_{n(n-1)} \right\|^2}, \\
F_n &= \omega(T_{11}) + \omega(T_{nn}).
\end{aligned}$$

Proof. It is well know that if A in $\mathbb{B}(\mathscr{H})$, $\omega\left(\begin{bmatrix} A & 0 \\ 0 & 0 \end{bmatrix}\right) = \omega(A)$, and by the triangle inequality of numerical radius, we can get that

$$\begin{aligned}
\omega(T) &\leqslant \omega\left(\begin{bmatrix} 0 & \cdots & 0 & T_{1n} \\ 0 & \cdots & T_{2(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ T_{n1} & \cdots & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1(n-1)} & 0 \\ T_{21} & T_{22} & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ T_{(n-1)1} & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & T_{2n} \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & T_{(n-1)(n-1)} & T_{(n-1)n} \\ 0 & T_{n2} & \cdots & T_{n(n-1)} & T_{nn} \end{bmatrix}\right) \\
&= \omega\left(\begin{bmatrix} 0 & \cdots & 0 & T_{1n} \\ 0 & \cdots & T_{2(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ T_{n1} & \cdots & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} T_{11} & T_{12} & \cdots & T_{1(n-1)} \\ T_{21} & T_{22} & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ T_{(n-1)1} & 0 & \cdots & 0 \end{bmatrix}\right) \\
&\quad + \omega\left(\begin{bmatrix} 0 & \cdots & 0 & T_{2n} \\ \vdots & & \vdots & \vdots \\ 0 & \cdots & T_{(n-1)(n-1)} & T_{(n-1)n} \\ T_{n2} & \cdots & T_{n(n-1)} & T_{nn} \end{bmatrix}\right) \\
&\leqslant \cdots \\
&\leqslant E_1 + E_2 + \cdots + E_n,
\end{aligned}$$

where

$$\begin{aligned}
E_1 &= \omega\left(\begin{bmatrix} 0 & \cdots & 0 & T_{1n} \\ 0 & \cdots & T_{2(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ T_{n1} & \cdots & 0 & 0 \end{bmatrix}\right), \\
E_2 &= \omega\left(\begin{bmatrix} 0 & \cdots & 0 & T_{1(n-1)} \\ 0 & \cdots & T_{2(n-2)} & 0 \\ \vdots & & \vdots & \vdots \\ T_{(n-1)1} & \cdots & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & \cdots & 0 & T_{2n} \\ 0 & \cdots & T_{3(n-1)} & 0 \\ \vdots & & \vdots & \vdots \\ T_{n2} & \cdots & 0 & 0 \end{bmatrix}\right), \\
&\ \ \vdots \\
E_{n-1} &= \omega\left(\begin{bmatrix} 0 & T_{12} \\ T_{21} & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & T_{(n-1)n} \\ T_{n(n-1)} & 0 \end{bmatrix}\right), \\
E_n &= \omega(T_{11}) + \omega(T_{nn}).
\end{aligned}$$

The expected result can be obtained by using lemma 2.11. □

In the forthcoming, we give the following result, which is a special case of Theorem 2.12, when $n=3$.

COROLLARY 2.13. *Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be Hilbert spaces, and let*

$$T=\begin{bmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}$$

be a 3×3 operator matrix with $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, where $1\leqslant i,j\leqslant 3$. Then

$$\begin{aligned}\omega(T)\leqslant{}&\max\left\{\|T_{13}+T_{31}^*\|^2,\|T_{22}+T_{22}^*\|^2\right\}+\max\left\{\|T_{13}-T_{31}^*\|^2,\|T_{22}-T_{22}^*\|^2\right\}^{\frac{1}{2}}\\&+\frac{1}{2}\left(\sqrt{\left\|T_{23}+T_{32}^*\right\|^2+\left\|T_{23}-T_{32}^*\right\|^2}+\sqrt{\left\|T_{12}+T_{21}^*\right\|^2+\left\|T_{12}-T_{21}^*\right\|^2}\right)\\&+\omega(T_{11})+\omega(T_{33}).\end{aligned}$$

REMARK 2.14. Sahoo et al. [12, Theorem 2.9] have been obtained that if $T=[T_{ij}]$ is an $n\times n$ operator matrix with $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, then

$$\omega(T)\leqslant\frac{1}{2}\sum_{i=1}^{n}\omega(T_{ii})+\frac{n}{4}+\frac{1}{4}\sum_{i,j=1}^{n}\left\|T_{ij}\right\|^2.$$

Considering the operator $T=\begin{bmatrix} T_{11} & T_{12} & T_{13}\\ T_{21} & T_{22} & T_{23}\\ T_{31} & T_{32} & T_{33}\end{bmatrix}$, where $T_{11}=T_{12}=T_{22}=\begin{pmatrix}0&1\\1&0\end{pmatrix}$ and $T_{13}=\begin{pmatrix}0&2\\2&0\end{pmatrix}$, $T_{23}=\begin{pmatrix}0&3\\3&0\end{pmatrix}$, $T_{33}=\begin{pmatrix}2&1\\1&0\end{pmatrix}$, $T_{21}=T_{31}=T_{32}=\begin{pmatrix}0&0\\0&0\end{pmatrix}$, it is easy to see that Theorem 2.12 which in the case $n=3$, i.e., Corollary 2.13, gives $\omega(T)\leqslant 4\sqrt{2}+2\approx 7.6569$ whereas the bound obtained by Sahoo et al. in [12, Theorem 2.9] gives $\omega(T)\leqslant 7+\sqrt{2}\approx 8.4143$ in the case $n=3$. This indicates that for such operators the bound obtained by our results is better than that of Sahoo et al.

The next result is another application of Lemma 2.11.

THEOREM 2.15. *Let $\mathscr{H}_1,\mathscr{H}_2,\cdots,\mathscr{H}_n$ be Hilbert spaces, and let $T=[T_{ij}]$ be an $n\times n$ operator matrix with $T_{ij}\in\mathbb{B}(\mathscr{H}_j,\mathscr{H}_i)$, where $1\leqslant i,j\leqslant n$. Then for $t\in[0,1]$, we have* *(i) when n is even,*

$$\omega(T)\leqslant \max\left(\omega(T_{11}),\omega(T_{22}),\cdots,\omega(T_{nn})\right)$$
$$+\frac{1}{2}\sqrt{\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}+T^*_{(n+1-i)i}\right\|^2\right\}+\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}-T^*_{(n+1-i)i}\right\|^2\right\}}$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{\substack{j=1\\ j\neq i,j\neq n+1-i}}^{n}T_{ij}T^*_{ij}\right\|}.$$

(ii) when n is odd,

$$\omega(T)\leqslant \max\left(\omega(T_{11}),\omega(T_{22}),\cdots,t\omega\left(T_{(\frac{n+1}{2})(\frac{n+1}{2})}\right),\cdots,\omega(T_{nn})\right)+\frac{1}{2}\sqrt{\alpha+\beta}$$
$$+\frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{\substack{j=1\\ j\neq i,j\neq n+1-i}}^{n}T_{ij}T^*_{ij}\right\|},$$

where

$$\alpha=\max\left\{\max_{\substack{i=1,2,\cdots,n\\ i\neq\frac{n+1}{2}}}\left\|T_{i(n+1-i)}+T^*_{(n+1-i)i}\right\|^2,(1-t)^2\left\|T_{(\frac{n+1}{2})(\frac{n+1}{2})}+T^*_{(\frac{n+1}{2})(\frac{n+1}{2})}\right\|^2\right\},$$
$$\beta=\max\left\{\max_{\substack{i=1,2,\cdots,n\\ i\neq\frac{n+1}{2}}}\left\|T_{i(n+1-i)}-T^*_{(n+1-i)i}\right\|^2,(1-t)^2\left\|T_{(\frac{n+1}{2})(\frac{n+1}{2})}-T^*_{(\frac{n+1}{2})(\frac{n+1}{2})}\right\|^2\right\}.$$

Proof. First observe that

$$\begin{bmatrix}0 & T_{12} & \cdots & T_{1(n-1)} & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}^2=\begin{bmatrix}0 & 0 & 0 & \cdots & 0 & 0 & 0\\ T_{21} & 0 & T_{23} & \cdots & T_{2(n-2)} & 0 & T_{2n}\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots\\ 0 & 0 & 0 & \cdots & 0 & 0 & 0\end{bmatrix}^2$$
$$=\cdots=\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\\ 0 & T_{n2} & \cdots & T_{n(n-1)} & 0\end{bmatrix}^2$$
$$=\begin{bmatrix}0 & 0 & \cdots & 0 & 0\\ 0 & 0 & \cdots & 0 & 0\\ \vdots & \vdots & & \vdots & \vdots\\ 0 & 0 & \cdots & 0 & 0\end{bmatrix}.$$

Let

$$T_3 = \begin{bmatrix} 0 & T_{12} & T_{13} & \cdots & T_{1(n-2)} & T_{1(n-1)} & 0 \\ T_{21} & 0 & T_{23} & \cdots & T_{2(n-2)} & 0 & T_{2n} \\ T_{31} & T_{32} & 0 & \cdots & 0 & T_{3(n-1)} & T_{3n} \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & T_{n2} & T_{n3} & \cdots & T_{n(n-2)} & T_{n(n-1)} & 0 \end{bmatrix},$$

by using $\|A\|^2 = \|AA^*\|$ for any operator A and the equality (1.2), we have

$$\begin{aligned}
\omega(T_3) &\leqslant \omega\left(\begin{bmatrix} 0 & T_{12} & \cdots & T_{1(n-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right) + \omega\left(\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ T_{21} & 0 & T_{23} & \cdots & T_{2(n-2)} & 0 & T_{2n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}\right) \\
&\quad + \cdots + \omega\left(\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & T_{n2} & \cdots & T_{n(n-1)} & 0 \end{bmatrix}\right) \\
&= \frac{1}{2}\left\|\begin{bmatrix} 0 & T_{12} & \cdots & T_{1(n-1)} & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix}\right\| + \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ T_{21} & 0 & T_{23} & \cdots & T_{2(n-2)} & 0 & T_{2n} \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 0 & 0 \end{bmatrix}\right\| \\
&\quad + \cdots + \frac{1}{2}\left\|\begin{bmatrix} 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ 0 & T_{n2} & \cdots & T_{n(n-1)} & 0 \end{bmatrix}\right\| \\
&= \frac{1}{2}\sqrt{\left\|T_{12}T_{12}^* + \cdots + T_{1(n-1)}T_{1(n-1)}^*\right\|} \\
&\quad + \frac{1}{2}\sqrt{\left\|T_{21}T_{21}^* + T_{23}T_{23}^* + \cdots + T_{2(n-2)}T_{2(n-2)}^* + T_{2n}T_{2n}^*\right\|} \\
&\quad + \cdots \\
&\quad + \frac{1}{2}\sqrt{\left\|T_{n2}T_{n2}^* + \cdots + T_{n(n-1)}T_{n(n-1)}^*\right\|} \\
&= \frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{\substack{j=1 \\ j\neq i, j\neq n+1-i}}^{n} T_{ij}T_{ij}^*\right\|}.
\end{aligned}$$

(i) When n is even, for the sake of simplicity, we let

$$T_1=\begin{bmatrix} T_{11} & 0 & \cdots & 0\\ 0 & T_{22} & \cdots & 0\\ \vdots & \vdots & & \vdots\\ 0 & 0 & \cdots & T_{nn}\end{bmatrix},\quad T_2=\begin{bmatrix} 0 & \cdots & 0 & T_{1n}\\ 0 & \cdots & T_{2(n-1)} & 0\\ \vdots & & \vdots & \vdots\\ T_{n1} & \cdots & 0 & 0\end{bmatrix},$$

thus, by the triangle inequality of numerical radius and Lemma 2.11, we can get that

$$\begin{aligned}\omega(T)&\leqslant \omega(T_1)+\omega(T_2)+\omega(T_3)\\ &\leqslant \max\left(\omega(T_{11}),\omega(T_{22}),\cdots,\omega(T_{nn})\right)\\ &\quad+\frac{1}{2}\sqrt{\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}+T^*_{(n+1-i)i}\right\|^2\right\}+\max_{i=1,2,\cdots,n}\left\{\left\|T_{i(n+1-i)}-T^*_{(n+1-i)i}\right\|^2\right\}}\\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{\substack{j=1\\ j\neq i,j\neq n+1-i}}^{n}T_{ij}T^*_{ij}\right\|}.\end{aligned}$$

(ii) When n is odd, for the sake of simplicity, we let

$$T_1'=\begin{bmatrix} T_{11} & & & & \\ & \ddots & & & \\ & & tT_{(\frac{n+1}{2})(\frac{n+1}{2})} & & \\ & & & \ddots & \\ & & & & T_{nn}\end{bmatrix},$$

$$T_2'=\begin{bmatrix} & & & & T_{1n}\\ & & & \iddots & \\ & & (1-t)T_{(\frac{n+1}{2})(\frac{n+1}{2})} & & \\ & \iddots & & & \\ T_{n1} & & & & \end{bmatrix},$$

thus, by the triangle inequality of numerical radius and Lemma 2.11, we can get that

$$\begin{aligned}\omega(T)&\leqslant \omega\left(T_1'\right)+\omega\left(T_2'\right)+\omega(T_3)\\ &\leqslant \max\left(\omega(T_{11}),\omega(T_{22}),\cdots,t\omega\left(T_{(\frac{n+1}{2})(\frac{n+1}{2})}\right),\cdots,\omega(T_{nn})\right)+\frac{1}{2}\sqrt{\alpha+\beta}\\ &\quad+\frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{\substack{j=1\\ j\neq i,j\neq n+1-i}}^{n}T_{ij}T^*_{ij}\right\|}.\qquad\square\end{aligned}$$

The following corollary can be obtained immediately by setting $n=3$ in Theorem 2.15.

COROLLARY 2.16. *Let $\mathscr{H}_1, \mathscr{H}_2, \cdots, \mathscr{H}_n$ be Hilbert spaces, and let*

$$T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$$

be a 3×3 operator matrix with $T_{ij} \in \mathbb{B}(\mathscr{H}_j, \mathscr{H}_i)$, where $1 \leqslant i, j \leqslant 3$. Then, for $t \in [0,1]$, we have

$$\begin{aligned} \omega(T) \leqslant \max\left(\omega(T_{11}), t\omega(T_{22}), \omega(T_{33})\right) & \\ + \frac{1}{2}\Big(\max\left\{\|T_{13}+T_{31}^*\|^2, (1-t)^2\|T_{22}+T_{22}^*\|^2\right\} & \\ + \max\left\{\|T_{13}-T_{31}^*\|^2, (1-t)^2\|T_{22}-T_{22}^*\|^2\right\}\Big)^{\frac{1}{2}} & \\ + \frac{1}{2}\left(\|T_{12}\| + \|T_{21}T_{21}^* + T_{23}T_{23}^*\|^{\frac{1}{2}} + \|T_{32}\|\right). & \end{aligned}$$

REMARK 2.17. Guelfen and Kittaneh [11, Theorem 2.9] has obtained that if $T = [T_{ij}]$ is an $n \times n$ operator matrix with $T_{ij} \in \mathbb{B}(\mathscr{H}_j, \mathscr{H}_i)$, then

$$\omega(T) \leqslant \max\left(\omega(T_{11}), \omega(T_{22}), \cdots, \omega(T_{nn})\right) + \frac{1}{2}\sum_{i=1}^{n}\sqrt{\left\|\sum_{j \neq i, j=1}^{n} T_{ij}T_{ij}^*\right\|}.$$

Considering the operator $T = \begin{bmatrix} T_{11} & T_{12} & T_{13} \\ T_{21} & T_{22} & T_{23} \\ T_{31} & T_{32} & T_{33} \end{bmatrix}$, where $T_{11} = T_{33} = \begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}$ and $T_{12} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $T_{13} = T_{31} = \begin{pmatrix} 3 & 0 \\ 0 & 3 \end{pmatrix}$, $T_{21} = T_{23} = T_{32} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$, $T_{22} = \begin{pmatrix} 0 & 4 \\ 4 & 0 \end{pmatrix}$, it is easy to see that Theorem 2.15 which in the case $n = 3$, i.e., Corollary 2.16, gives $\omega(T) \leqslant 6 + \frac{1}{2}\sqrt{2} \approx 6.7072$ when $t = \frac{1}{2}$ whereas the bound obtained by Guelfen and Kittaneh in [11, Theorem 2.9] gives $\omega(T) \leqslant 4 + \sqrt{10} + \frac{1}{2}\sqrt{2} \approx 7.8694$ in the case $n = 3$. This indicates that for such operators the bound obtained by our results is better than that of Guelfen and Kittaneh.

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