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MONOTONIC PROPERTIES FOR RATIO OF THE GENERALIZED (p,k)–POLYGAMMA FUNCTIONS

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Abstract. In this paper, we show monotonic properties for ratio of the generalized (p,k)-polygamma functions by using Mehrez-Sitnik method. The new inequalities extend the known result obtained by Feng Qi.

1. Introduction

The Euler gamma function is defined by

$$\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt, \quad x>0. \tag{1.1}$$

The logarithmic derivative of $\Gamma(x)$ is called the psi or digamma function. That is

$$\psi(x) = \frac{d}{dx}\ln\Gamma(x) = \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma - \frac{1}{x} + \sum_{n=1}^{\infty}\frac{x}{n(n+x)}, \tag{1.2}$$

where $\gamma = 0.5772\ldots$ is the Euler-Mascheroni constant, and $\psi^{(m)}(x)$ for $m\in\mathbb{N}$ are known as the polygamma functions. The gamma, digamma and polygamma functions play an important role in the theory of special functions, and have many applications in other many branches, such as statistics, fractional differential equations, mathematical physics and theory of infinite series. The reader may see references [5, 7, 9, 14]. some of the work about the complete monotonicity, convexity and concavity, and inequalities of these special functions may refer to [1, 2, 4, 8, 17, 18, 19, 20, 21, 22, 23, 24].

In [6], R. Diaz and E. Pariguan defined the k-analogue of the gamma function for $k>0$ and $x>0$ as

$$\Gamma_k(x) = \int_0^\infty t^{x-1}e^{-\frac{t^k}{k}}dt = \lim_{n\to\infty}\frac{n!k^n(nk)^{\frac{x}{k}-1}}{(x)_{n,k}} \tag{1.3}$$

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Paper JMI-16-62

where $\lim\limits_{k\to 1}\Gamma_k(x)=\Gamma(x)$. Similarly, the k-analogue of the digamma and polygamma functions are defined by $\psi_k(x)=\frac{d}{dx}\log\Gamma_k(x)=\frac{\Gamma_k'(x)}{\Gamma_k(x)}$ and $\psi_k^{(m)}(x)=\frac{d^m}{dx^m}\log\Gamma_k(x)$ for $x>0$, respectively.

Later, Nantomah, Prempeh and Twum [13] introduced a new $(p;k)$-analogue of gamma function with two parameters as follows: for $p,k>0$,

$$\Gamma_{p,k}(x)=\frac{(p+1)!k^{p+1}(pk)^{\frac{x}{k}-1}}{(x)_{p,k}},x>0 \tag{1.4}$$

where $(x)_{p,k}=x(x+k)(x+2k)\dots(x+pk)$ and $\lim\limits_{p\to\infty}\Gamma_{p,k}(x)=\Gamma_k(x)$. Furthermore, we naturally define the (p,k)- analogue of the digamma and polygamma functions as follows: $\psi_{p,k}(x)=\frac{\Gamma_{p,k}'(x)}{\Gamma_{p,k}(x)}$ and $\psi_{p,k}^{(m)}(x)=\frac{d^m}{dx^m}\psi_{p,k}(x)$. The functions $\psi_{p,k}(x)$ and $\psi_{p,k}^{(m)}(x)$ satisfy the following series and integral representations.

$$\begin{aligned}\psi_{p,k}(x)&=\frac{1}{k}\ln(pk)-\sum_{n=0}^{p}\frac{1}{nk+x}\\&=\frac{1}{k}\ln(pk)-\int_0^\infty\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}e^{-xt}dt\end{aligned} \tag{1.5}$$

and

$$\begin{aligned}\psi_{p,k}^{(m)}(x)&=(-1)^m m!\sum_{n=0}^{p}\frac{1}{(nk+x)^{m+1}}\\&=(-1)^{m+1}\int_0^\infty\frac{1-e^{-k(p+1)t}}{1-e^{-kt}}t^m e^{-xt}dt.\end{aligned} \tag{1.6}$$

For more details, the reader may see references [11, 12].

Let $\alpha=(\alpha_1,\alpha_2,\cdots,\alpha_n)$ and $\beta=(\beta_1,\beta_2,\cdots,\beta_n)\in\mathbb{R}^n$. A n-tuple α is said to strictly majorize β (in symbols $\alpha\succ\beta$) if $(\alpha_{[1]},\alpha_{[2]},\cdots,\alpha_{[n]})\neq(\beta_{[1]},\beta_{[2]},\cdots,\beta_{[n]})$, $\sum_{i=1}^k\alpha_{[i]}\geqslant\sum_{i=1}^k\beta_{[i]}$ for $1\leqslant k\leqslant n-1$, and $\sum_{i=1}^n\alpha_i=\sum_{i=1}^n\beta_i$ where $\alpha_{[1]}\geqslant\alpha_{[2]}\geqslant\cdots\geqslant\alpha_{[n]},\beta_{[1]}\geqslant\beta_{[2]}\geqslant\cdots\geqslant\beta_{[n]}$ are rearrangements of α and β in a descending order.

In [3], it was discovered that if $m\geqslant 2$, then

$$\frac{m-1}{m}<\frac{\left[\psi^{(m)}(x)\right]^2}{\psi^{(m-1)}(x)\psi^{(m+1)}(x)}<\frac{m}{m+1} \tag{1.7}$$

holds true for $x>0$.

In [[15], Remark 6.2], F. Qi guessed that,

(1) for $m,n\in\mathbb{N}$, the function

$$Q_{m,n}(x)=\frac{\psi^{(m+n)}(x)}{\psi^{(n)}(x)\psi^{(m)}(x)} \tag{1.8}$$

should be decreasing on $(0,\infty)$;

(2) for $m,n,p,q\in\mathbb{N}$ such that $(p,q)\succ(m,n)$, the function

$$Q_{m,n;p,q}(x)=\frac{\psi^{(m)}(x)\psi^{(n)}(x)}{\psi^{(p)}(x)\psi^{(q)}(x)} \tag{1.9}$$

should be decreasing on $(0,\infty)$;

Later, F. Qi [16] proved the above decreasing properties by virtue of convolution theorem for the Laplace transforms, with the aid of three monotonicity rules for the ratios of two functions, of two definite integrals, and of two Laplace transforms, in terms of the majorization, and in the light of other analytic techniques.

Motivated by the work of F. Qi, we generalized the inequality (1.9) to a decreasing monotonic property for ratio of the generalized (p,k)-polygamma functions. It is worth noting that our proof is different, and very simple.

THEOREM 1.1. *For $p,k>0$ and every $s,t,\lambda,\mu\in\mathbb{N}$ such that $(\lambda,\mu)\succ(s,t)$, the function*

$$Q_{s,t,\lambda,\mu,p,k}(x)=\frac{\psi_{p,k}^{(s)}(x)\psi_{p,k}^{(t)}(x)}{\psi_{p,k}^{(\lambda)}(x)\psi_{p,k}^{(\mu)}(x)}$$

is decreasing on $(0,\infty)$ onto the interval $\left(\frac{(s-1)!(t-1)!}{(\lambda-1)!(\mu-1)!},\frac{s!t!}{\lambda!\mu!}\right)$. As a result, for $p,k>0$ and every $s,t,\lambda,\mu\in\mathbb{N}$ such that $(\lambda,\mu)\succ(s,t)$, we have

$$\frac{(s-1)!(t-1)!}{(\lambda-1)!(\mu-1)!}<Q_{s,t,\lambda,\mu,p,k}(x)<\frac{s!t!}{\lambda!\mu!} \tag{1.10}$$

is valid on $(0,\infty)$.

REMARK 1.1. By taking $p\to\infty$ and $k=1$ in Theorem 1.1, we obtain the Theorem 4.1 in [16].

2. Lemmas

LEMMA 2.1. ([10, 26]) *Let $\{a_n\}$ and $\{b_n\},(n=0,1,2,\cdots)$ be real numbers such that $b_n>0$ and $\{\frac{a_n}{b_n}\}_{n\geqslant 0}$ is increasing(decreasing), then $\{\frac{a_0+a_1+\cdots+a_n}{b_0+b_1+\cdots+b_n}\}$ is increasing(decreasing).*

LEMMA 2.2. ([25]) *For $p,k,x>0$ and every $m\geqslant 1$, the following limit identity holds true:*

$$\lim_{x\to 0^+}x^{m+1}\psi_{p,k}^{(m)}(x)=\frac{(-1)^m(m-1)!}{k}.$$

3. Proof of Theorem 1.1

Without losing the generality, we suppose $\lambda \leqslant \mu$. By using (1.6) and direct computation, we have

$$\begin{aligned}\frac{\psi_{p,k}^{(s)}(x)\,\psi_{p,k}^{(t)}(x)}{\psi_{p,k}^{(\lambda)}(x)\,\psi_{p,k}^{(\mu)}(x)} &= \frac{s!t!\sum_{n=0}^{p}\frac{1}{(nk+x)^{s+1}}\cdot\sum_{n=0}^{p}\frac{1}{(nk+x)^{t+1}}}{\lambda!\mu!\sum_{n=0}^{p}\frac{1}{(nk+x)^{\lambda+1}}\cdot\sum_{n=0}^{p}\frac{1}{(nk+x)^{\mu+1}}}\\ &= \frac{s!t!}{\lambda!\mu!}\frac{\sum_{n=0}^{p^2}\sum_{i=0}^{n}\frac{1}{(ik+x)^{s+1}[(n-i)k+x]^{t+1}}}{\sum_{n=0}^{p^2}\sum_{i=0}^{n}\frac{1}{(ik+x)^{\lambda+1}[(n-i)k+x]^{\mu+1}}}\\ &= \frac{s!t!}{\lambda!\mu!}\frac{\sum_{n=0}^{p^2}A_n(x)}{\sum_{n=0}^{p^2}B_n(x)}\end{aligned}$$

where

$$A_n(x)=\sum_{i=0}^{n}\frac{1}{(ik+x)^{s+1}\,[(n-i)\,k+x]^{t+1}}$$

and

$$B_n(x)=\sum_{i=0}^{n}\frac{1}{(ik+x)^{\lambda+1}\,[(n-i)\,k+x]^{\mu+1}}.$$

Let us define sequences $\{\alpha_i\}_{i\geqslant 0}, \{\beta_i\}_{i\geqslant 0}$ and $\{\omega_i\}_{i\geqslant 0}$ by

$$\alpha_i=\frac{1}{(ik+x)^{s+1}\,[(n-i)\,k+x]^{t+1}}$$

$$\beta_i=\frac{1}{(ik+x)^{\lambda+1}\,[(n-i)\,k+x]^{\mu+1}}$$

and

$$\omega_i=\frac{\alpha_i}{\beta_i}=\frac{[(n-i)k+x]^{\mu-t}}{(ik+x)^{s-\lambda}}.$$

It follows that

$$\frac{\omega_{i+1}}{\omega_i}=\frac{[(n-i-1)\,k+x]^{\mu-t}\,(ik+x)^{s-\lambda}}{[(i+1)\,k+x]^{s-\lambda}\,[(n-i)\,k+x]^{\mu-t}}$$

Noting the fact $\lambda+\mu=s+t$ and $\mu\geqslant t$, we easily obtain $\frac{\omega_{i+1}}{\omega_i}\leqslant 1$ is equivalent to

$$[(n-i-1)\,k+x]^{\mu-t}\,(ik+x)^{\mu-t}\leqslant[(i+1)\,k+x]^{\mu-t}\,[(n-i)\,k+x]^{\mu-t}.$$

If $\mu \neq t$, this is equivalent to

$$[(n-i-1)k+x](ik+x) < [(i+1)k+x][(n-i)k+x]$$
$$\Leftrightarrow -nk^2 - 2kx < 0.$$

If $\mu = t$, we easily obtain $\lambda = s$, and $\omega_{i+1} = \omega_i$. So, we conclude that the sequence $\{\omega_i\}_{i\geqslant 0}$ is decreasing and consequently the sequence $\left\{\frac{A_n(x)}{B_n(x)}\right\}_{n\geqslant 0}$ is also decreasing by Lemma 2.1. Hence using Lemma 2.1 again, we prove that the function $Q_{s,t;\lambda,\mu,p,k}(x)$ is decreasing on $(0,\infty)$. From the identity

$$\psi_{p,k}^{(m)}(x+k) = (-1)^m \frac{m!}{x^{m+1}} - (-1)^m \frac{m!}{(x+pk+k)^{m+1}} + \psi_{p,k}^{(m)}(x),$$

and Lemma 2.2, we easily obtain

$$\lim_{x\to 0^+} Q_{s,t;\lambda,\mu,p,k}(x) = \frac{s!t!}{\lambda!\mu!},$$

and

$$\lim_{x\to\infty} Q_{s,t;\lambda,\mu,p,k}(x) = \frac{(s-1)!(t-1)!}{(\lambda-1)!(\mu-1)!}.$$

This completes the proof.

4. A remark

In [27], Li Yin et. al. proved the following result: For $p,k>0$ and every positive integer $m \geqslant 4$, the function

$$\phi_{m,p,k}(x) = \frac{\left[\psi_{p,k}^{(m)}(x)\right]^4}{\psi_{p,k}^{(m-3)}(x)\psi_{p,k}^{(m-1)}(x)\psi_{p,k}^{(m+1)}(x)\psi_{p,k}^{(m+3)}(x)}$$

is strictly decreasing on $(0,\infty)$ with

$$\lim_{x\to\infty} \phi_{m,p,k}(x) = \frac{(m-3)(m-2)(m-1)^2}{m^2(m+1)(m+2)} \tag{4.1}$$

and

$$\lim_{x\to 0} \phi_{m,p,k}(x) = \frac{(m-2)(m-1)m^2}{(m+1)^2(m+2)(m+3)}. \tag{4.2}$$

As a result, for $p,k,x>0$ and every positive integer $m \geqslant 4$, we have

$$\begin{aligned}\frac{(m-3)(m-2)(m-1)^2}{m^2(m+1)(m+2)} &< \frac{\left[\psi_{p,k}^{(m)}(x)\right]^4}{\psi_{p,k}^{(m-3)}(x)\psi_{p,k}^{(m-1)}(x)\psi_{p,k}^{(m+1)}(x)\psi_{p,k}^{(m+3)}(x)} \\ &< \frac{(m-2)(m-1)m^2}{(m+1)^2(m+2)(m+3)}.\end{aligned}$$

Naturally, we can conjecture Theorem 1 to the more general case.

CONJECTURE 4.1. *For $p,k>0$ and every $\alpha_1,\alpha_2,\cdots\alpha_n,\beta_1,\beta_2,\cdots\beta_n\in\mathbb{N}$ such that $(\alpha_1,\alpha_2,\cdots,\alpha_n)\prec(\beta_1,\beta_2,\cdots,\beta_n)$, the function*

$$Q_{\alpha_1,\alpha_2,\cdots,\alpha_n;\beta_1,\beta_2,\cdots,\beta_n,p,k}(x)=\frac{\psi_{p,k}^{(\alpha_1)}(x)\cdots\psi_{p,k}^{(\alpha_n)}(x)}{\psi_{p,k}^{(\beta_1)}(x)\cdots\psi_{p,k}^{(\beta_n)}(x)}$$

is decreasing on $(0,\infty)$ onto the interval $\left(\frac{(\alpha_1-1)!\cdots(\alpha_n-1)!}{(\beta_1-1)!\cdots(\beta_n-1)!},\frac{(\alpha_1)!\cdots(\alpha_n)!}{(\beta_1)!\cdots(\beta_n)!}\right)$.

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