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ON MONOTONIC L_φ–SOLUTIONS FOR A CLASS OF QUADRATIC–URYSOHN INTEGRAL EQUATIONS

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Abstract. This article is devoted to study the existence of a.e. monotonic solutions of functional quadratic Urysohn integral equations in Orlicz spaces L_φ. Due to various continuity properties of the operators in Orlicz spaces, there are many different cases to discuss the considered problem. We focus on assumptions permitting us to consider strongly nonlinear operators and to combine the results of both standard and quadratic integral equations. We discuss the studied problem in three general and different cases when the function φ satisfies Δ', Δ_2, and Δ_3-conditions separately under a general set of assumptions.

1. Introduction

Quadratic functional integral equations have been applied in various branches such as, in neutron transport theory, radioactive transfer theory, the traffic theory, plasma physics, kinetic theory of gases, and in mathematical physics (cf. [2, 9, 10, 21]).

We dedicate to study the following quadratic-Urysohn integral equation

$$x(t) = g(t) + f\left(t, x(\eta(t)),\ \lambda \cdot G(x)(t) \cdot \int_a^b u(t,s,x(s))\, ds\right),\ \ t \in [a,b] \tag{1}$$

in Orlicz spaces L_φ, where G is a general operator acting on some Orlicz spaces.

We discuss equation (1) in three different cases when the function φ satisfies Δ', Δ_2, and Δ_3-conditions separately.

These are very large and interesting classes that permit us to get more general growth conditions (of exponential growth, for instance) basically more rapid than a polynomial growth on the considered operators (cf. [11, 20, 29]). Let me mention, the thermodynamical problem leads to the integral equation

$$x(t) + \int_I k(t,s) \cdot e^{x(s)}\, ds = 0,$$

which has exponential nonlinearities (cf. [32]). Also, the Chandrasekhar equation represents an important prototype of the quadratic integral equation which was discussed

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in the space of continuous solutions (cf. [9, 19]). This equation describes scattering through a homogeneous semi-infinite plane atmosphere. The discontinuous solutions are useful estimation of non-homogeneous atmosphere for such equation (cf. [8]), then we focus on solutions in Orlicz spaces (see also [14]).

Recall that, the discontinuous solutions of particular cases of equation (1) have been studied in Orlicz spaces with $G(x)=1$ and $f(t,x,y)=y$ (cf. [26, 27, 28, 30] for non-quadratic equations). Some additionally properties of solutions of such equations were also examined in Orlicz spaces like constant-sign solutions (see [1], for instance) and the results in generalized Orlicz spaces (Musielak-Orlicz spaces) can be found in [7, 28].

In [4], the existence of monotonic integrable solutions had checked, where $G(x)=1$, $f(t,x,y)=f(t,y)$, (see also [8, 17]) and in [25], this was done for equations with a perturbation term, where $f(t,x,y)=f_1(t,y)$, $G(x)=f_2(t,x)$, using the measure of noncompactness (see also [23]). The quadratic Hammerstein integral equations were discussed in [15] with $f(t,x,y)=y$ in Orlicz spaces L_φ when φ satisfies Δ' and Δ_3-conditions separately, and in [14] with $G(x)=x$.

In [13, 16] the authors studied the quadratic Hammerstein integral equation in Orlicz spaces with $G(x)=u(t,x)$, $f(t,x,y)=y$ and in [24] with linear perturbation of second kind while L_∞ is one of the intermediate spaces. The case when the function φ satisfies Δ_2-condition was checked in [12].

Another motivation of this work is to examine the monotonicity property of solutions which is studied in various previous articles (cf. [5, 6, 23], for instance).

This article extends the results presented in the previous studies by studying the solvability of functional quadratic-Urysohn integral equations in Orlicz spaces under a general set of assumptions with aid of Darbo fixed point theorem. We skip several restrictions like in [8, 14, 16, 24] by using the strategy discussed in [13] and assuming a triple of different Orlicz spaces (need not be Banach algebras). This allows us to unify the proof for quadratic and non-quadratic cases.

2. Notation and auxiliary facts

Let $\mathbb{R}$ be the field of real numbers and $I=[a,b]\subset\mathbb{R}$. Assume that M and N be complementary N-functions i.e. $N(x)=\sup_{y\geqslant 0}(xy-M(x))$, where $N:[0,+\infty)\to[0,+\infty)$ is continuous, convex and even with $\lim_{x\to 0}\frac{N(x)}{x}=0$, $\lim_{x\to\infty}\frac{N(x)}{x}=\infty$ and $N(x)>0$ if $x>0$ ($N(u)=0 \iff u=0$).

Denote by $L_M(I)$ the *Orlicz space* of all measurable functions $x:I\to\mathbb{R}$ for which

$$\|x\|_M=\inf_{\varepsilon>0}\left\{\int_I M\left(\frac{x(s)}{\varepsilon}\right)\,ds\leqslant 1\right\}.$$

Let $E_M(I)$ be the closure in $L_M(I)$ of the set of all bounded functions. Further, E_M-spaces be a class of functions from L_M having absolutely continuous norms.

Moreover, we have $E_M=L_M$ if M satisfies the Δ_2-condition, i.e.

$$(\Delta_2)\ \text{there exist } \omega,\ t_0\geqslant 0 \text{ such that } M(2t)\leqslant\omega M(t),\ t\geqslant t_0.$$

The N-function M is said to satisfy Δ'-condition if there exist $K,\ t_0 \geqslant 0$ such that for $t,s \geqslant t_0$, we have $M(ts) \leqslant KM(t)M(s)$.

Moreover, the N-function M is said to satisfy Δ_3-condition if there exist $K,\ t_0 \geqslant 0$ such that for $t \geqslant t_0$, we have $tM(t) \leqslant M(Kt)$.

DEFINITION 1. [20] Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions i.e. it is measurable in t for any $x \in \mathbb{R}$ and continuous in x for almost all $t \in I$. Then to every function $x(t)$ being measurable on I we may assign the function

$$F_f(x)(t) = f(t,x(t)),\ \ t \in I.$$

The operator F_f in such a way is called the superposition operator generated by the function f.

LEMMA 1. ([20, Theorem 17.5]) *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. Then*

$$M_2(f(s,x)) \leqslant a(s) + b \cdot M_1(x),$$

where $b \geqslant 0$ and $a \in L^1(I)$, if and only if the superposition operator F_f acts from $L_{M_1}(I)$ to $L_{M_2}(I)$.

LEMMA 2. [24] *Assume that a function $f : I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions. The superposition operator F_f maps $E_\phi(I) \to E_\phi(I)$ is continuous and bounded if and only if*

$$|f(s,x)| \leqslant m(s) + b \cdot |x|,$$

where $b \geqslant 0$ and $m \in L_\phi(I)$ in which the N-function ϕ satisfies the Δ_2-condition.

For the product of operators in Orlicz spaces, we have the following lemma:

LEMMA 3. [22, Theorem 10.2] *Let φ_1, φ_2 and φ are arbitrary N-functions. The following conditions are equivalent:*

1. *For every functions $u \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}$, $u \cdot w \in L_\varphi(I)$.*
2. *There exists a constant $k > 0$ such that for all measurable u,w on I we have $\|uw\|_\varphi \leqslant k\|u\|_{\varphi_1}\|w\|_{\varphi_2}$.*
3. *There exists numbers $C > 0$, $u_0 \geqslant 0$ such that for all $s,t \geqslant u_0$, $\varphi\left(\frac{st}{C}\right) \leqslant \varphi_1(s) + \varphi_2(t)$.*
4. $\limsup_{t\to\infty} \frac{\varphi_1^{-1}(t)\varphi_2^{-1}(t)}{\varphi(t)} < \infty$.

Let $S = S(I)$ denotes the set of Lebesgue measurable functions on I and let "*meas*" refers to the Lebesgue measure in $\mathbb{R}$. The set S associated with the metric

$$d(x,y) = \inf_{a>0}[a + meas\{s : |x(s) - y(s)| \geqslant a\}]$$

be a complete space. The convergence in measure on I is equivalent to convergence with respect to d (cf. Proposition 2.14 in [31]). The compactness in such spaces is called a "compactness in measure".

LEMMA 4. [14] *Let X be a bounded subset of $L_M(I)$. Assume that, there is a family of subsets $(\Omega_c)_{0\leqslant c\leqslant b-a}$ of the interval I such that meas $\Omega_c = c$ for every $c\in[0,b-a]$, and for every $x\in X$,*

$$x(t_1)\geqslant x(t_2),\ \ (t_1\in\Omega_c,\ t_2\notin\Omega_c).$$

Then X is compact in measure in $L_M(I)$.

In what follows, assume that $(E,\|\cdot\|)$ be an arbitrary Banach space with zero element θ. Denote by B_r the closed ball centered at θ and with radius r and the symbol $B_r(E)$ is to point out the space. If $X\subset E$, then $\bar{X}$ and $convX$ indicate the closure and convex closure of X, respectively. By $\mathscr{M}_E$ denotes the family of all nonempty and bounded subsets of E and by $\mathscr{N}_E$ its subfamily consisting of all relatively compact subsets.

DEFINITION 2. [3] A mapping $\mu:\mathscr{M}_E\to[0,\infty)$ is said to be a measure of noncompactness in E if the following conditions hold:

(i) $\mu(X)=0 \Longleftrightarrow X\in\mathscr{N}_E$.
(ii) $X\subset Y \Longrightarrow \mu(X)\leqslant\mu(Y)$.
(iii) $\mu(\bar{X})=\mu(convX)=\mu(X)$.
(iv) $\mu(\lambda X)=|\lambda|\mu(X)$, for $\lambda\in\mathbb{R}$.
(v) $\mu(X+Y)\leqslant\mu(X)+\mu(Y)$.
(vi) $\mu(X\bigcup Y)=\max\{\mu(X),\mu(Y)\}$.
(vii) If X_n is a sequence of nonempty, bounded, closed subsets of E such that $X_{n+1}\subset X_n,\ n=1,2,3,\cdots$, and $\lim_{n\to\infty}\mu(X_n)=0$, then the set $X_\infty=\bigcap_{n=1}^{\infty}X_n$ is nonempty.

An example of such a mapping is the following:

DEFINITION 3. [3] Let X be a nonempty and bounded subset of E. The Hausdorff measure of noncompactness $\beta_H(X)$ is defined as

$$\beta_H(X)=inf\{r>0:\ \text{there exists a finite subset Y of E such that } x\subset Y+B_r\}.$$

For any $\varepsilon>0$, let c be a measure of equiintegrability of the set X in $L_M(I)$ (cf. Definition 3.9 in [31] or [18]):

$$c(X)=\lim_{\varepsilon\to 0}\ \sup_{mesD\leqslant\varepsilon}\ \sup_{x\in X}\|x\cdot\chi_D\|_{L_M(I)},$$

where χ_D denotes the characteristic function of a measurable subset $D\subset I$.

LEMMA 5. [18, 15] *Let X be a nonempty, bounded and compact in measure subset of $E_M(I)$. Then*

$$\beta_H(X)=c(X).$$

THEOREM 1. [3] *Let Q be a nonempty, bounded, closed and convex subset of E and let $V : Q \to Q$ be a continuous transformation which is a contraction with respect to the measure of noncompactness μ, i.e. there exists $k \in [0,1)$ such that*

$$\mu(V(X)) \leqslant k\mu(X),$$

for any nonempty subset X of E. Then V has at least one fixed point in the set Q.

3. Main results

Denote by B the operator associated with the right-hand side of equation (1) i.e.

$$x = B(x) = g + F_f\big(x(\eta), U(x)\big),$$

where

$$F_f(x(\eta), U(x)) = f\big(t, x(\eta), U(x)\big), \quad U(x) = \lambda G(x) \cdot A(x),$$

$$A(x)(t) = \int_a^b u(t,s,x(s))ds.$$

We will characterize three different cases, which permits us to get more general growth conditions on the considered functions. We shall stress on the assumptions of the considered functions to nominate the intermediate spaces, in which our results are in the target space $L_\varphi(I)$.

3.1. The case of Δ'-condition

Assume, that $\varphi, \varphi_1, \varphi_2$ are N-functions and that M and N are complementary N-functions, where φ satisfies Δ_2-condition. Moreover, put the following assumptions:

(G1) There exists a constant $k_1 > 0$ such that for every $v \in L_{\varphi_1}(I)$ and $w \in L_{\varphi_2}(I)$ we have $\|vw\|_\varphi \leqslant k_1 \|v\|_{\varphi_1} \|w\|_{\varphi_2}$,

(G2) $f(t,x,y) : I \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is measurable in t and continuous in x and y for almost all t. There exist constants $b_1, b_2 \geqslant 0$ and $a \in L_\varphi(I)$ such that

$$|f(t,x,y)| \leqslant a(t) + b_1\|x\|_\varphi + b_2\|y\|_\varphi.$$

Moreover, assume that $f(t,x,y)$ is nondecreasing with respect to each variable t, s and x separately,

(G3) $G : L_\varphi(I) \to L_{\varphi_1}(I)$, takes continuously $E_\varphi(I)$ into $E_{\varphi_1}(I)$ and there exists a constant $b_0 > 0$ such that $|G(x)| \leqslant b_0 \|x\|_\varphi$ and that G takes the set of all a.e. nondecreasing functions into itself. Moreover, assume that for any $x \in E_\varphi(I)$, we have $G(x) \in E_{\varphi_1}(I)$.

(C1) $g \in E_\varphi(I)$ is nondecreasing a.e. on I, (C2) $u(t,s,x) : I \times I \times \mathbb{R} \to \mathbb{R}$ satisfies Carathéodory conditions (i.e. it is measurable in (t,s) for any $x \in \mathbb{R}$ and continuous in x for almost all $t,s \in I$. Further, $u(t,s,x)$ is assumed to be nondecreasing with respect to each variable t,s and x separately,

(C3) $|u(t,s,x)| \leqslant K(t,s)(b(s) + R(|x|)$ for $t,s \in I$ and $x \in \mathbb{R}$, where $b \in E_N(I)$ and R is nonnegative, nondecreasing, continuous function defined on $\mathbb{R}^+$, and $K(t,s) \geqslant 0$ for $t,s \in I$.

(C4) Let N satisfies the Δ'-condition and suppose that there exist ω, γ, $u_0 \geqslant 0$ for which

$$N(\omega(R(u))) \leqslant \gamma\varphi(u) \leqslant \gamma M(u) \;\text{ for }\; u \geqslant u_0,$$

(K1) $s \to K(t,s) \in L_M(I)$ for a.e. $t \in I$,

(K2) $K \in E_M(I^2)$ and $t \to K(t,s) \in E_{\varphi_2}(I)$ for a.e. $s \in I$,

(K3) $\eta : I \to I$ is an increasing absolutely continuous function and there is a positive constant Z such that $\eta' \geqslant Z$ a.e. on (a,b).

PROPOSITION 1.

(a) Assumption (K3) leads to $x(\eta(\cdot)) : E_\varphi(I) \to E_\varphi(I)$ *and gives the estimation*

$$\begin{aligned}\int_a^b \varphi\left(\frac{x(\eta(s))}{\varepsilon}\right) ds &\leqslant \int_a^b \varphi\left(\frac{x(\eta(s))}{\varepsilon}\right)\frac{\eta'(s)}{Z}\, ds \\ &= \frac{1}{Z}\int_{\eta(a)}^{\eta(b)} \varphi\left(\frac{x(v)}{\varepsilon}\right) dv \\ &\leqslant \frac{1}{Z}\int_a^b \varphi\left(\frac{x(v)}{\varepsilon}\right) dv,\end{aligned}$$

which yields that

$$\|x(\eta)\|_\varphi \leqslant \frac{1}{Z}\|x\|_\varphi. \tag{2}$$

(b) Let us recall, that $x \in E_\varphi(I)$ *iff for arbitrary* $\varepsilon > 0$ *there exists* $\delta > 0$ *such that* $\|x\chi_T\|_\varphi < \varepsilon$ *for every measurable subset* T *of* I *with the Lebesgue measure smaller that* δ *(i.e.* x *has absolutely continuous norm).*

THEOREM 2. *Let the assumptions (G1)–(G3), (C1)–(C4) and (K1)–(K3), be satisfied. If* $\left(\frac{b_1}{Z} + 2b_0k_1b_2 \cdot |\lambda| \cdot \|K\|_M \cdot (\|b\|_N + R(1))\right) < 1$*, then there exists a number* $\rho > 0$ *such that for all* $\lambda \in \mathbb{R}$ *with* $|\lambda| < \rho$ *there exists a solution* $x \in E_\varphi(I)$ *of (1) which is a.e. nondecreasing on* I.

Proof. The proof will be given in the next steps.

Step I. First of all observe that under assumption (G2) and Lemma 2 the operator F_f acts from $E_\varphi(I)$ into itself. Next, we need to prove, that the operator U maps the unit ball in $E_\varphi(I)$ into the space $E_\varphi(I)$ continuously. It is sufficient to examine that property for the operator A (see Lemma 3).

Since N is an N-function satisfying Δ'-condition and by (C3), we are able to use [20, Theorem 19.1]. From this there exists a constant C (not depending on the kernel) such that for any measurable $T \subset I$ and $x \in L_\varphi(I)$, $\|x\|_\varphi \leqslant 1$ we have

$$\|A(x)\chi_T\|_{\varphi_2} \leqslant C\|K\chi_{T\times I}\|_M. \tag{3}$$

Now, by the Hölder inequality and the assumption (C3), we get

$$|Ax(t)| \leqslant \|K(t,s)\| \cdot |(b(s)+R(|x(s)|))|$$

for $t,s \in I$. Put $k(t) = 2\|K(t,\cdot)\|_M$ for $t \in I$. As $K \in E_M(I^2)$ this function is integrable on I. By the assumptions (K1) and (K2) about the kernel K (cf. [30]) we obtain that

$$\|A(x)(t)\| \leqslant k(t)\cdot(\|b\|_N + \|R(|x(\cdot)|)\|_N) \ \text{ for a.e. } t \in I.$$

Whence for arbitrary measurable subset T of I and $x \in E_\varphi(I)$

$$\|A(x)\chi_T\|_{\varphi_2} \leqslant \|k\chi_T\|_{\varphi_2}\cdot(\|b\|_N + \|R(|x(\cdot)|\|_N).$$

Finally if t is such that $K(t,\cdot) \in E_M(I)$ and $x \in E_\varphi(I)$ we have

$$\int_T \|u(t,s,x(s))\|\, ds \leqslant 2\|K(t,\cdot)\chi_T\|_M\cdot(\|b\|_N + \|R(|x(\cdot)|)\|_N) \text{for a.e. } t \in I.$$

From this it follows that $A : B_1(E_\varphi(I)) \to E_{\varphi_2}(I)$.

Next, we will show that $A : B_1(E_\varphi(I)) \to E_{\varphi_2}(I)$ is continuous. Let $x_n, x_0 \in B_1(E_\varphi(I))$ be such that $\|x_n - x_0\|_\varphi \to 0$ as n tends to ∞. Suppose, contrary to our claim, that A is not continuous and the $\|A(x_n) - A(x_0)\|_{\varphi_2}$ does not converge to zero. Then there exists $\varepsilon > 0$ and a subsequence (x_{n_j}) such that

$$\|A(x_{n_j}) - A(x_0)\|_{\varphi_2} > \varepsilon \ \text{ for } j = 1,2,\ldots \tag{4}$$

and the subsequence is a.e. convergent to x_0. Since (x_n) is a subset of the ball the sequence $(\int_a^b \varphi(|x_n(t)|)dt)$ is bounded. As the space $E_\varphi(I)$ is regular the balls are norm-closed in $L_1(I)$ so the sequence $(\int_a^b |x_n(t)|dt)$ is also bounded.

Moreover, by (C3) and (C4) there exist r, ω, γ, $u_0 > 0$, s.t. (cf. [20, p. 196])

$$\begin{aligned}
\|R(|x(\cdot)|)\|_N &= \frac{1}{\omega}\|\omega R(|x(\cdot)|)\|_N \\
&\leqslant \frac{1}{\omega}\inf_{r>0}\left\{\int N(\omega R(|x(t)|)/r)dt \leqslant 1\right\} \\
&\leqslant \frac{1}{\omega}\left(1 + \int_a^b N(\omega R(|x(t)|))dt\right) \\
&\leqslant \frac{1}{\omega}\left(1 + N(\omega R(u_0))\cdot(b-a) + \gamma\int_a^b \varphi(|x(t)|)dt\right),
\end{aligned}$$

whenever $x \in L_\varphi(I)$ with $\|x\|_\varphi \leqslant 1$.

Thus

$$\begin{aligned}\int_T \|u(t,s,x_n(s))\|\,ds &\leqslant 2\|K(t,\cdot)\chi_T\|_M \cdot (\|b\|_N + \|R(|x_n(\cdot)|)\|_N)\\ &\leqslant 2\|K(t,\cdot)\chi_T\|_M \cdot \bigg(\|b\|_N + \frac{1}{\omega}\bigg(1+N(\omega R(u_0))\cdot(b-a)\\ &\qquad + \gamma\int_a^b \varphi(|x_n(t)|)dt\bigg)\bigg)\end{aligned}$$

and then the sequence $(\|u(t,s,x_n(s))\|)$ is equiintegrable on I for a.e. $t\in I$. By the continuity of $u(t,s,\cdot)$ we get $\lim_{j\to\infty} u(t,s,x_{n_j}(s)) = u(t,s,x_0(s))$ for a.e. $s\in I$. Now, applying the Vitali convergence theorem we obtain that

$$\lim_{j\to\infty} A(x_{n_j})(t) = A(x_0)(t) \quad \text{for a.e. } t\in I.$$

But the equation (3) implies that $A(x_{n_j})$ is a subset of $E_{\varphi_2}(I)$ and then $\lim_{j\to\infty} A(x_{n_j})(t) = A(x_0)(t)$ which contradicts the inequality (4). Since A is continuous between indicated spaces. By assumption (G3) the operator G is continuous from $B_1(E_\varphi(I))$ into $E_{\varphi_1}(I)$ and then by (G1) the operator U has the same property and then $U : B_1(E_\varphi(I)) \to E_\varphi(I)$ is continuous. Finally, by the assumption (C1) the operator $B : B_1(E_\varphi(I)) \to E_\varphi(I)$ is continuous.

Step II. We will construct the invariant ball for our operator i.e. $B_1(E_\varphi(I))$.

Let x be an arbitrary element from $B_1(E_\varphi(I))$. By using our assumptions and recalling the estimation (2) and the formula (3), then for sufficiently small λ (i.e. $|\lambda| < \rho$), where

$$\rho = \frac{1-\|g\|_\varphi - \|a\|_\varphi - \frac{b_1}{Z}}{2b_0k_1b_2\cdot C\cdot\|K\|_M},$$

we have

$$\begin{aligned}\|B(x)\|_\varphi &\leqslant \|g\|_\varphi + \|f(t,x(\eta),U(x))\|_\varphi\\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + b_1\|x(\eta)\|_\varphi + b_2\|Ux\|_\varphi\\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + b_2\|\lambda G(x)\cdot A(x)\|_\varphi\\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + b_2k_1|\lambda|\|G(x)\|_{\varphi_1}\cdot\|A(x)\|_{\varphi_2}\\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + b_2k_1\cdot b_0\cdot\|x\|_\varphi|\lambda|\cdot\left\|\int_a^b u(t,s,x(s))\,ds\right\|_{\varphi_2}\\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + 2k_1b_2b_0\|x\|_\varphi\cdot|\lambda|\cdot C\cdot\|K\|_M\\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z} + 2b_0k_1b_2\cdot\rho\cdot C\cdot\|K\|_M \leqslant 1,\end{aligned}$$

whenever $\|x\|_\varphi \leqslant 1$. Then we have $B : B_1(E_\varphi(I)) \to E_\varphi(I)$ is continuous.

Step III. Let $Q_1 \subset B_1(E_\varphi(I))$ consisting of all functions that are a.e. nondecreasing on I. This set is nonempty, bounded, convex and closed set in $L_\varphi(I)$ see [15]. Moreover, the set Q_1 is compact in measure due to Lemma 4.

Step IV. Now, we will show, that B preserves the monotonicity of functions. Take $x \in Q_1$, then x and $x(\eta)$ is a.e. nondecreasing on I and consequently $A(x)$ is a.e. nondecreasing on I thanks for the assumption (C2). Since the pointwise product of a.e. monotone functions are still of the same type and by (G3), the operator U is a.e. nondecreasing on I. Further, $F_f(x(\eta),U(x))$ is also of the same type in virtue of the assumption (G2). Moreover, the assumption (C1) permits us to deduce that the operator B is also a.e. nondecreasing on I. This gives us that $B: Q_1 \to Q_1$ is continuous.

Step V. We will prove that B is a contraction concerning the measure of noncompactness μ. Assume that $\phi \neq X \subset Q_1$ and let $\varepsilon > 0$ be fixed arbitrary constant. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\operatorname{meas} D \leqslant \varepsilon$, we have

$$
\begin{aligned}
&\|B(x)\cdot\chi_D\|_\varphi \leqslant \|g\cdot\chi_D\|_\varphi + \left\| F_f\Big(x(\eta),Ux\Big)\cdot\chi_D \right\|_\varphi \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + b_1\|x(\eta)\cdot\chi_D\|_\varphi + b_2\|U(x)\cdot\chi_D\|_\varphi \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi + b_2\|\lambda\cdot G(x)\cdot A(x)\cdot\chi_D\|_\varphi \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi + b_2k_1\cdot|\lambda|\cdot\|G(x)\cdot\chi_D\|_{\varphi_1}\cdot\|A(x)\cdot\chi_D\|_{\varphi_2} \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi + b_2k_1\cdot|\lambda|\cdot b_0\cdot\|x\chi_D\|_\varphi\cdot\left\|\int_D u(t,s,x(s))\,ds\right\|_{\varphi_2} \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi + b_2k_1\cdot|\lambda|\cdot b_0\cdot\|x\chi_D\|_\varphi\cdot 2\|K\|_M\|b+R(r)\|_N \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi + 2b_0k_1b_2\cdot|\lambda|\cdot\|x\chi_D\|_\varphi\cdot\|K\|_M(\|b\|_N+R(1)).
\end{aligned}
$$

Hence, taking into account that $g, a \in E_\varphi$, we have

$$
\lim_{\varepsilon\to 0}\{\sup_{mes\ D\leqslant\varepsilon}\ [\sup_{x\in X}\{\|g\chi_D\|_\varphi + \|a\chi_D\|_\varphi = 0\}]\}.
$$

Thus by definition of $c(x)$ and by taking the supremum over all $x \in X$ and all measurable subsets D with $measD \leqslant \varepsilon$ we get

$$
c(B(X)) \leqslant \left(\frac{b_1}{Z} + 2b_0k_1b_2\cdot|\lambda|\cdot\|K\|_M\cdot(\|b\|_N + R(1))\right)c(X).
$$

Since $X \subset Q_r$ is a nonempty, bounded and compact in measure subset of E_φ, we can use Lemma 5 and get

$$
\beta_H(B(X)) \leqslant \left(\frac{b_1}{Z} + 2b_0k_1b_2\cdot|\lambda|\cdot\|K\|_M\cdot(\|b\|_N + R(1))\right)\beta_H(X).
$$

Since $\left(\frac{b_1}{Z} + 2b_0k_1b_2\cdot|\lambda|\cdot\|K\|_M\cdot(\|b\|_N+R(1))\right) < 1$, we can apply Theorem 1, which accomplishes the proof. □

3.2. The case of Δ_3-condition

Let us consider the case of N-functions satisfying Δ_3-condition with the growth essentially more rapid than a polynomial. Note, that an N-function M determines the properties of the Orlicz spaces $L_M(I)$, and then the less restrictive rate of the growth of this function implies the "worser" properties of the space. By ϑ we will denote the norm of the identity operator from $L_\varphi(I)$ into $L^1(I)$ i.e. $\sup\{\|x\|_1 : x \in B_1(L_\varphi(I))\}$.

THEOREM 3. *Assume, that $\varphi, \varphi_1, \varphi_2$ are N-functions and that M and N are complementary N-functions, where φ satisfies Δ_2-condition and that (G1)–(G3), (C1)–(C3), (K1), and (K3) hold true. Moreover, put the following assumptions:*

(C5) 1. *N satisfies the Δ_3-condition,*
2. *$K \in E_M(I^2)$ and $t \to K(t,s) \in E_{\varphi_2}(I)$ for a.e. $s \in I$,*
3. *There exist β, $u_0 > 0$ such that*

$$R(u) \leqslant \beta \frac{M(u)}{u}, \ \ \textit{for} \ \ u \geqslant u_0,$$

(K4) *φ_2 is an N-function satisfying*

$$\iint\limits_{I^2} \varphi_2(M(|K(t,s)|))\, dtds < \infty$$

and

$$\left(\frac{b_1}{Z} + 2b_0k_1b_2|\lambda| \cdot (2+(b-a)(1+\varphi_2(1))) \cdot \|K\|_{\varphi_2 \circ M} \cdot \big(\|b\|_N + R(r_0)\big)\right) < 1,$$

where

$$r_0 = \frac{1}{\vartheta}\left(\frac{\omega}{\frac{b_1}{Z} + 2b_0k_1b_2|\lambda| \cdot (2+(b-a)(1+\varphi_2(1))) \cdot \|K\|_{\varphi_2 \circ M}} - \|b\|_N\right).$$

Then there exist a number $\rho > 0$ and a number $\varpi > 0$ such that for all $\lambda \in \mathbb{R}$ with $|\lambda| < \rho$ and for all $g, a \in E_\varphi(I)$ with $(\|g\|_\varphi + \|a\|_\varphi) < \varpi$, there exists a solution $x \in E_\varphi(I)$ of (1) which is a.e. nondecreasing on I.

Proof. We will introduce only the steps of the proof when they are unlike in Theorem 2.

Step I'. In this case, we will study the operator B on the whole $E_\varphi(I)$.

By [20, Lemma 15.1 and Theorem 19.2] and the assumption (K4):

$$\|A(x)\chi_T\|_{\varphi_2} \leqslant 2 \cdot (2+(b-a)(1+\varphi_2(1))) \cdot \|K \cdot \chi_{T\times I}\|_{\varphi_2 \circ M} (\|b\|_N + \|R(|x(\cdot)|)\|_N) \quad (5)$$

for arbitrary $x \in L_\varphi(I)$ and arbitrary measurable subset T of I.

Let us note, that the assumption (C5) 3. implies that there exist constants $\omega, u_0 > 0$ and $\eta_0 > 1$ such that $N(\omega R(u)) \leqslant \eta_0 u$ for $u \geqslant u_0$.

Thus for $x \in L_\varphi(I)$

$$\begin{aligned} \|R(|x(\cdot)|)\|_N &\leqslant \frac{1}{\omega}\left(1+\int_I N(\omega R(|x(s)|)\,ds\right) \\ &\leqslant \frac{1}{\omega}\left(1+\eta_0 u_0(b-a)+\eta_0\int_I |x(s)|\,ds\right). \end{aligned}$$

The remaining estimations can be derived as in Theorem 2 and then we obtain, that $A: E_\varphi(I) \to E_{\varphi_2}(I)$, so by the properties of G and F_f, we get $B: E_\varphi(I) \to E_\varphi(I)$.

Step II'. We will study the operator B on the ball $B_r(E_\varphi(I))$, where r is a positive number satisfying

$$\begin{aligned} &\|g\|_\varphi + \|a\|_\varphi \\ &+\left(\frac{b_1}{Z}+2b_0k_1b_2|\lambda|C\|K\|_{\varphi_2\circ M}\right)\cdot r\left(\|b\|_N+\frac{1}{\omega}\left(1+\eta_0u_0(b-a)+\eta_0\vartheta r\right)\right) \leqslant r, \end{aligned} \tag{6}$$

where $C = (2+(b-a)(1+\varphi(1)))$. The above inequality has two positive solutions $r_1 < r_2$ for sufficiently small $\lambda < \rho$, (see [15]), where

$$\rho = \frac{1}{\left(\frac{b_1}{Z}+2b_0k_1b_2\cdot C\cdot\|K\|_{\varphi_2\circ M}\right)\left(\|b\|_N+\frac{1}{\omega}(1+\eta_0u_0(b-a))\right)}.$$

The following assumption about the discriminant implies the existence of solution of (6).

$$\begin{aligned} \frac{4(\|g\|_\varphi+\|a\|_\varphi)\eta_0\vartheta}{\omega} &< \left(\|b\|_N+\frac{1}{\omega}(1+\eta_0u_0(b-a)) - \frac{1}{\frac{b_1}{Z}+2b_0k_1b_2|\lambda|C\|K\|_{\varphi_2\circ M}}\right)^2 \\ &\quad\times\left(\frac{b_1}{Z}+2b_0k_1b_2|\lambda|C\|K\|_{\varphi_2\circ M}\right) \\ i.e. \quad \varpi &= \left(\|b\|_N+\frac{1}{\omega}(1+\eta_0u_0(b-a)) - \frac{1}{\frac{b_1}{Z}+2b_0k_1b_2|\lambda|C\|K\|_{\varphi_2\circ M}}\right)^2 \\ &\quad\times\frac{\frac{b_1}{Z}+2b_0k_1b_2|\lambda|C\|K\|_{\varphi_2\circ M}}{4\eta_0\vartheta}. \end{aligned}$$

For $x \in B_r(E_\varphi(I))$, we have the following estimation:

$$\begin{aligned} \|B(x)\|_\varphi &\leqslant \|g\|_\varphi + \|f(t,x(\eta),Ux)\|_\varphi \\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + b_1\|x(\eta)\|_\varphi + b_2\|Ux\|_\varphi \\ &\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + b_2k_1|\lambda|\|G(x)\|_{\varphi_1}\cdot\|A(x)\|_{\varphi_2} \end{aligned}$$

$$
\begin{aligned}
&= \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + b_2k_1|\lambda|\|G(x)\|_{\varphi_1} \cdot \left\| \int_a^b u(t,s,x(s))\,ds \right\|_{\varphi_2} \\
&\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + 2b_2k_1 \cdot C \cdot b_0 \cdot |\lambda| \cdot \|x\|_\varphi \|K\|_{\varphi_2\circ M}\Bigg(\|b\|_N \\
&\quad + \frac{1}{\omega}\left(1 + N(\omega R(u_0))\cdot(b-a) + \eta_0 \int_I |x(s)|\,ds\right)\Bigg) \\
&\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + 2b_0k_1b_2|\lambda| \cdot C \cdot \|x\|_\varphi \|K\|_{\varphi_2\circ M}\Bigg(\|b\|_N \\
&\quad + \frac{1}{\omega}\left(1 + N(\omega R(u_0))\cdot(b-a) + \eta_0\|x\|_1\right)\Bigg) \\
&\leqslant \|g\|_\varphi + \|a\|_\varphi + \frac{b_1}{Z}\|x\|_\varphi + 2b_0k_1b_2|\lambda| \cdot C \cdot \|x\|_\varphi \|K\|_{\varphi_2\circ M}\Bigg(\|b\|_N \\
&\quad + \frac{1}{\omega}\left(1 + N(\omega R(u_0))\cdot(b-a) + \eta_0\vartheta\|x\|_\varphi\right)\Bigg) \\
&\leqslant \|g\|_\varphi + \|a\|_\varphi + \left(\frac{b_1}{Z} + 2b_0k_1b_2|\lambda| \cdot C \cdot \|K\|_{\varphi_2\circ M}\right) \cdot r \\
&\quad \times \left(\|b\|_N + \frac{1}{\omega}\left(1 + \eta_0 u_0(b-a) + \eta_0\vartheta r\right)\right) \leqslant r.
\end{aligned}
$$

Then $B : B_r(E_\varphi(I)) \to B_r(E_\varphi(I))$ is continuous.

Step III' and *Step IV'* of our proof are similar to those from Theorem 2 for a subset $Q_r \subset B_r(E_\varphi(I))$.

Step V'. We will prove that B is a contraction concerning a measure of noncompactness. Assume that $\phi \neq X \subset Q_r$ and let $\varepsilon > 0$ be fixed arbitrary constant. Then for an arbitrary $x \in X$ and for a set $D \subset I$, $\operatorname{meas} D \leqslant \varepsilon$, we obtain

$$
\begin{aligned}
\|B(x)\cdot\chi_D\|_\varphi &\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\| \\
&\quad + b_2k_1 \cdot |\lambda| \cdot b_0 \cdot \|x\cdot\chi_D\|_\varphi \cdot \left\| \int_D u(t,s,x(s))\,ds \right\|_{\varphi_2} \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\| \\
&\quad + b_2k_1 \cdot |\lambda| \cdot b_0 \cdot \|x\cdot\chi_D\|_\varphi \cdot \left\| \int_D |K(\cdot,s)|(b(s) + R(|x(s)|))\,ds \right\|_{\varphi_2} \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\| \\
&\quad + b_2k_1 \cdot |\lambda| \cdot b_0 \cdot \|x\cdot\chi_D\|_\varphi \cdot \Bigg(\left\| \int_D |K(\cdot,s)|b(s)\,ds \right\|_{\varphi_2} \\
&\quad + \left\| \int_D |K(\cdot,s)|R(|x(s)|))\,ds \right\|_{\varphi_2} \Bigg)
\end{aligned}
$$

$$\begin{aligned}
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\| \\
&\quad +2b_2\cdot C\cdot k_1\cdot b_0\cdot|\lambda|\cdot\|x\cdot\chi_D\|_\varphi\cdot\vartheta\cdot\|K\|_{\varphi_2\circ M}\|b\|_N \\
&\quad +2b_2\cdot C\cdot k_1\cdot b_0\cdot|\lambda|\cdot\|x\cdot\chi_D\|_\varphi\cdot\left\|\int_D |K(\cdot,s)|R(|x(s)|)\,ds\right\|_{\varphi_2} \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\| \\
&\quad +2b_0k_1b_2|\lambda|\cdot C\cdot\|x\cdot\chi_D\|_\varphi\cdot\|K\|_{\varphi_2\circ M}\Big(\|b\|_N + R(r)\Big) \\
&\leqslant \|g\cdot\chi_D\|_\varphi + \|a\cdot\chi_D\|_\varphi + \frac{b_1}{Z}\|x\cdot\chi_D\| \\
&\quad +2b_0k_1b_2|\lambda|\cdot C\cdot\|x\cdot\chi_D\|_\varphi\cdot\|K\|_{\varphi_2\circ M}\Big(\|b\|_N + R(r_0)\Big),
\end{aligned}$$

where

$$r_0 = \frac{1}{\vartheta}\left(\frac{\omega}{\frac{b_1}{Z} + 2b_0k_1b_2|\lambda|\cdot(2+(b-a)(1+\varphi_2))\cdot\|K\|_{\varphi_2\circ M}} - \|b\|_N\right).$$

Let us note, that r_0 is an upper bound for solutions of (6).

Similarly, as in the previous theorem, we get

$$\beta_H(B(X)) \leqslant \left(\frac{b_1}{Z} + 2b_0k_1b_2|\lambda|\cdot C\cdot\|K\|_{\varphi_2\circ M}\cdot\big(\|b\|_N + R(r_0)\big)\right)\beta_H(X).$$

Since $\left(\frac{b_1}{Z} + 2b_0k_1b_2|\lambda|\cdot C\cdot\|K\|_{\varphi_2\circ M}\cdot\big(\|b\|_N + R(r_0)\big)\right) < 1$, we can apply the Theorem 1, which accomplishes the proof. □

3.3. The case of Δ_2-condition

In this section, we will discuss the case when N-function satisfies Δ_2-condition.

THEOREM 4. *Assume, that $\varphi, \varphi_1, \varphi_2$ are N-functions and that M and N are complementary N-functions and that (G1)–(G3), (C1)–(C3), and (K3) hold true. Moreover, put the following assumptions:*

(C6) Assume the N-functions φ and N satisfy the Δ_2-condition.

1. *There exist $\gamma \geqslant 0$ such that*

$$R(u) \leqslant \gamma N^{-1}(\varphi(u)) \quad \textit{for } u \geqslant 0.$$

2. *$s \to K(t,s) \in L_M(I)$ for a.e. $t \in I$ and $p(t) = \|K(t,\cdot)\|_M \in E_{\varphi_2}(I)$.*

Assume that for some $q>0$, *there exists* $r^*>0$ *on the interval* I *such that*

$$\int_I \varphi\left(|g(t)|+|a(t)|+\frac{b_1}{Z}\cdot r^*+b_0k_1b_2\cdot q\cdot r^*\cdot|p(t)|\left(\|b\|_N+\gamma\cdot r^*\right)\right)dt\leqslant r^*.$$

If $\left(\frac{b_1}{Z}+b_0k_1b_2\cdot|\lambda|\cdot\|p\|_{\varphi_2}\cdot\left(\|b\|_N+\gamma\cdot r^*\right)\right)<1$, *then there exists a number* $\rho>0$ *such that for all* $\lambda\in\mathbb{R}$ *with* $|\lambda|<\rho$, *there exists a solution* $x\in E_\varphi(I)$ *of* (1) *which is a.e. nondecreasing on* I.

Proof. Step I". First of all observe that by the assumptions (C2) and (C3), (cf. [20, Lemma 16.3 and Theorem 16.3] (with $M_1=N, M_2=\varphi_2$ and $N_1=M$) implies that the operator A is continuous mappings from the unit ball $B_1(E_\varphi(I))$ into $E_{\varphi_2}(I)$. By our assumption (G3) the operator G is continuous from $B_1(E_\varphi(I))$ into $E_{\varphi_1}(I)$ and then by (G1) the operator U is a continuous mapping from $B_1(E_\varphi(I))$ into the space $E_\varphi(I)$. Finally, by assumptions (C1), (G2), and (K3) we can deduce that the operator $B:B_1(E_\varphi(I))\to E_\varphi(I)$ is continuous.

Step II". We will construct an invariant set $V\subset B_1(E_\varphi(I))$ for the operator B is bounded in $L_\varphi(I)$.

Fix $\lambda\in\mathbb{R}$ with $\lambda<\rho$ and let $\rho=\sup Q$, where Q is the set of all positive numbers q for which there exists $r^*>0$ such that

$$\int_I \varphi\left(|g(t)|+|a(t)|+\frac{b_1}{Z}\cdot r^*+b_0k_1b_2\cdot q\cdot r^*\cdot|p(t)|\left(\|b\|_N+\gamma\cdot r^*\right)\right)dt\leqslant r^*.$$

Let V denote the closure of the set $\{x\in E_\varphi(I): \int_a^b\varphi(|x(s)|)\,ds\leqslant r^*-1\}$. Clearly V is not a ball in $E_\varphi(I)$, but $V\subset B_{r^*}(E_\varphi(I))$ (cf. [20, p. 222]). Notice that $\overline{V}$ is a bounded closed and convex subset of $E_\varphi(I)$.

Take an arbitrary $x\in V$. By using ([20, Theorem 10.5 with $k=1$]), we obtain that for any $t\in I$

$$\|R(|x|)\|_N\leqslant\gamma\left\|N^{-1}\left(\varphi\left(|x|\right)\right)\right\|_N\leqslant\gamma+\gamma\int_a^b\varphi\left(|x(s)|\right)\,ds \tag{7}$$

and then by the Hölder inequality and our assumptions we get

$$|A(x)(t)|\leqslant|p(t)|\left(\|b\|_N+\|R(|x|)\|_N\right).$$

Thus for any measurable subset T of I. For arbitrary $x\in V$ and $t\in I$, we have

$$\begin{aligned}|B(x)(t)|&\leqslant|g(t)|+|a(t)|+b_1\|x(\eta)\|_\varphi+b_2k_1|\lambda|\cdot|G(x)|\cdot|A(x)(t)|\\&\leqslant|g(t)|+|a(t)|+\frac{b_1}{Z}\|x\|_\varphi+b_2k_1|\lambda|b_0\|x\|_\varphi\cdot|p(t)|\left(\|b\|_N+\|R(|x|)\|_N\right)\end{aligned}$$

$$\leqslant |g(t)|+|a(t)|+\frac{b_1}{Z}\left(1+\int_I \varphi(|x(t)|)\,dt\right)$$
$$+b_2k_1|\lambda|b_0\left(1+\int_a^b \varphi(|x(t)|)\,dt\right)\cdot|p(t)|\left(\|b\|_N+\gamma+\gamma\int_a^b \varphi(|x(s)|)\,ds\right)$$
$$\leqslant |g(t)|+|a(t)|+\frac{b_1}{Z}\cdot r^*+b_0k_1b_2|\lambda|\cdot r^*\cdot|p(t)|\Big(\|b\|_N+\gamma+\gamma(r^*-1)\Big).$$

Therefore,

$$\int_I \varphi(B(x)(t))\,dt \leqslant \int_I \varphi\Big(|g(t)|+|a(t)|$$
$$+\frac{b_1}{Z}\cdot r^*+b_0k_1b_2|\lambda|\cdot r^*\cdot|p(t)|\Big(\|b\|_N+\gamma\cdot r^*\Big)\Big)dt.$$

By the definition of r^* we get $\int_I \varphi(B(x)(t))\,dt \leqslant r^*$ and then $B(V)\subset V$. Consequently $B(\overline{V})\subset\overline{B(V)}\subset\overline{V}=V$. Then $B: V\to V$ is continuous on $V\subset B_{r^*}(E_\varphi(I))$.

Step III" and *Step IV"* of our proof are similar to those from Theorem 2 for a subset $Q_{r^*}\subset B_{r^*}(E_\varphi(I))$.

Step V". Assume that $X\subset Q_{r^*}$ is a nonempty and let $\varepsilon>0$ be arbitrary fixed constant. Then for an arbitrary $x\in X$ and for a set $D\subset I$, meas $D\leqslant\varepsilon$, we obtain

$$\|B(x)\cdot\chi_D\|_\varphi \leqslant \|g\cdot\chi_D\|_\varphi+\|a\cdot\chi_D\|_\varphi+\frac{b_1}{Z}\|x\cdot\chi_D\|$$
$$+b_0k_1b_2\cdot|\lambda|\cdot\|x\cdot\chi_D\|_\varphi\cdot\left\|\int_D |K(\cdot,s)|(b(s)+R(|x(s)|))\,ds\right\|_{\varphi_2}$$
$$\leqslant \|g\cdot\chi_D\|_\varphi+\|a\cdot\chi_D\|_\varphi+\frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi$$
$$+b_0k_1b_2\cdot|\lambda|\cdot\|x\cdot\chi_D\|_\varphi\cdot\|p\|_{\varphi_2}\left(\|b\|_N+\gamma+\gamma\int_a^b \varphi(|x(s)|)\,ds\right)$$
$$\leqslant \|g\cdot\chi_D\|_\varphi+\|a\cdot\chi_D\|_\varphi+\frac{b_1}{Z}\|x\cdot\chi_D\|_\varphi$$
$$+b_0k_1b_2\cdot|\lambda|\cdot\|x\cdot\chi_D\|_\varphi\cdot\|p\|_{\varphi_2}\Big(\|b\|_N+\gamma r^*\Big).$$

Similarly as done in Theorem 2, we have

$$\beta_H(B(X))\leqslant\left(\frac{b_1}{Z}+b_0k_1b_2\cdot|\lambda|\cdot\|p\|_{\varphi_2}\cdot\left(\|b\|_N+\gamma\cdot r^*\right)\right)\beta_H(X).$$

Since $\left(\frac{b_1}{Z}+b_0k_1b_2\cdot|\lambda|\cdot\|p\|_{\varphi_2}\cdot\left(\|b\|_N+\gamma\cdot r^*\right)\right)<1$, we can apply Theorem 1, which accomplishes the proof. □

4. Particular cases and examples

Let us present some particular cases and examples of equation (1) that illustrate the applicability of our results.

Assume that, N_1, N_2 the complementary functions for M_1, M_2, respectively. Let $M_1(u) = \exp|u| - |u| - 1$, $N_1(u) = (1+|u|)\cdot\ln(1+|u|) - |u|$ and $M_2(u) = \frac{u^2}{2} = N_2(u)$, where M_1 satisfies the Δ_3-condition and N_1 satisfies the Δ'-condition. If we define an N-function either as $\Psi(u) = M_2[N_1(u)]$ or $\Psi(u) = N_1[M_2(u)]$, then by choosing arbitrary kernel K from the space $L_\Psi(I)$ we are able to apply [20, Theorem 15.4]. Thus $(Hx)(t) = \int_a^b K(t,s)x(s)\,ds : L_{M_1}(I) \to L_{M_2}(I)$ is continuous and it is useful in applying our results.

Note that, a full discussion about the continuity and acting conditions for the operator $G(x) = l(t)\cdot x(t)$, $l \in L_\varphi$, between different Orlicz spaces are presented in [20, Theorem 18.2] (cf. our assumption (G)).

EXAMPLE 1. Let $f(t,x,y) = f(t,y)$, $G(x) = 1$ in equation (1), we have the functional integral equation

$$x(t) = g(t) + f\left(t, \int_0^1 K(t,s)x(\eta(s))\,ds\right), \quad t \in [0,1],$$

where the existence of monotonic integrable solutions of this equation discussed in [4] see also [17].

EXAMPLE 2. The classical-Urysohn integral equations have been studied in Orlicz spaces in [26, 27, 28] with $f(t,x,y) = y$, $G(x) = 1$ i.e.

$$x(t) = g(t) + \int_I u(t,s,x(s))\,ds, \quad t \in I.$$

The case of classical Hammerstein integral equations were also discussed in Orlicz spaces in [20, 30].

EXAMPLE 3. The existence of L_1-solution of functional-quadratic integral equation with a perturbation term can be found in [25], where $f(t,x,y) = f_1(t,y)$, $G(x) = f_2(t,x)$ which takes the form

$$x(t) = g(t,x(\eta_3(t))) + f_1\left(t, f_2(t,x(\eta_2(t)))\cdot\int_0^t u(t,s,x(\eta_1(s)))\,ds\right), \quad t \in \mathbb{R}^+.$$

The author used the measure of noncompactness to obtain the results see also [23].

EXAMPLE 4. The authors in [12, 15, 16] discussed the quadratic-Hammerstein integral equation in Orlicz spaces under various set of assumptions with $f(t,x,y) = y$

$$x(t) = g(t) + G(x)(t)\int_a^b K(t,s)f_1(s,x(s))\,ds, \quad t \in I.$$

This has been done for equations with linear perturbation of the second kind in Orlicz spaces (cf. [24] see also [14]).

EXAMPLE 5. Let $G(x)(t) = l(t)\cdot x(t)$, then we have the quadratic integral equations

$$x(t) = g(t) + f\left(t, x(t), l(t)\cdot x(t)\cdot \int_0^1 u(t,s,x(s))\, ds\right), \ \ t\in[a,b],$$

which represent a particular case of equation (1) with a suitable form of the functions g and f.

EXAMPLE 6. In case of $g(t) = 1,\ f(t,x,y) = y$ and $G(x) = \lambda\cdot x$ in equation (1), we have a general form of Chandrasekhar equation studied in [2, 10, 19]

$$x(t) = 1 + \lambda\, x(t)\int_0^1 \frac{t}{t+s} e^{-s}(b(s) + \log(1+|x(s)|^\alpha))\, ds,$$

where $R(x) = \log(1+|x(s)|^\alpha)$.

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