Journal of
Mathematical
Inequalities
Volume 16, Number 3 (2022), 1005–1028 doi:10.7153/jmi-2022-16-68

DEFERRED STATISTICAL CONVERGENCE AND POWER SUMMABILITY METHOD FOR $\mathfrak{q}$–LAGUERRE POLYNOMIALS OPERATOR

P. N. AGRAWAL, BEHAR BAXHAKU* AND SOMPAL SINGH

(*Communicated by M. Mursaleen*)

Abstract. In the present article, we discuss the Korovkin type approximation thereoms and the rate of convergence with the aid of the modulus of continuity using deferred statistical convergence and the power series summability technique for an operator based on $\mathfrak{q}$-Laguerre polynomials introduced by Özarslan (Studia Sci. Math. Hungar., 44 (1), 65–80). We also define the $r-th$ order generalization of these operators by means of the Taylor polynomial to approximate functions in $\mathfrak{f}\in C^r[0,1]$ such that $\mathfrak{f}^{(r)}\in Lip_K\alpha$, $0<\alpha\leqslant 1$. Furthermore, we find an estimate of the rate of convergence of the $\mathfrak{q}$-Laguerre operator acting on $\mathfrak{f}$ at those points x where the one sided $\mathfrak{q}$-derivatives $\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}$ and $\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}$ exist.

1. Introduction

The generalized Laguerre polynomials or the associated Laguerre polynomials are the orthogonal polynomials that arise in the study of quantum mechanics and also in the treatment of quantum harmonic oscillator due to their relationship with the Hermite polynomials. They are the polynomial solutions of the second order ordinary linear differential equation

$$\varkappa u''+(\alpha+1-\varkappa)u'+mu=0,$$

where α is a real constant. In the special case $\alpha=0$, one obtains the simple Laguerre polynomials. The generating function for the generalized Laguerre polynomials $\mathscr{L}_k^{(\alpha)}(w)$ is given by

$$\sum_{k=0}^{\infty}\varkappa^k\mathscr{L}_k^{(\alpha)}(w)=\frac{1}{(1-\varkappa)^{\alpha+1}}\exp\left(\frac{-w\varkappa}{1-\varkappa}\right) \tag{1}$$

Taking $\alpha=m(\in\mathbb{N})$, Cheney and Sharma [1] introduced the following linear positive operators based on the generalized Laguerre polynomials

$$\mathfrak{G}_m(\mathfrak{f};\varkappa)=\sum_{k=0}^{\infty}\mathfrak{f}\left(\frac{k}{m+k}\right)\mathscr{L}_k^{(m)}(w)\varkappa^k(1-\varkappa)^{m+1}\exp\left(\frac{w\varkappa}{1-\varkappa}\right), \tag{2}$$

Mathematics subject classification (2020): 41A25, 41A36, 33C45, 26A15.

Keywords and phrases: Korovkin type theorem, deferred statistical convergence, modulus of continuity, rate of convergence, power series method.

* Corresponding author.

where $\varkappa \in [0,1)$ and $w \in (-\infty,0]$.

From (1), it is clear that if $w=0$ and $\alpha=m$ then $\mathscr{L}_k^{(m)}(0)=\binom{m+k}{k}$ and hence the operator given by

$$S_m(\mathfrak{f};\varkappa)=\sum_{k=0}^{\infty}\mathfrak{f}\left(\frac{k}{m+k}\right)\binom{m+k}{k}\varkappa^k(1-\varkappa)^{m+1},\ \ \varkappa\in[0,1)$$

is a particular case of the operator (2).

Now we present some basic definitions based on the $\mathfrak{q}$-calculus [12], which are used in this paper. Let $0<\mathfrak{q}<1$.

The q*-integer* $[c]_{\mathfrak{q}}$ $(c\in\mathbb{N})$ is defined as $[c]_{\mathfrak{q}}=\left\{\frac{1-\mathfrak{q}^c}{1-\mathfrak{q}}\right.$, or,

$$[c]_{\mathfrak{q}}=1+\mathfrak{q}+\ldots+\mathfrak{q}^{c-1}.$$

The q*-shifted factorial* $(1-\varkappa)_{\mathfrak{q}}^c$ $(c\in\mathbb{N}\cup\{0\})$ is defined as $(1-\varkappa)_{\mathfrak{q}}^c=\prod_{s=0}^{\infty}\frac{1-\varkappa\mathfrak{q}^s}{1-\varkappa\mathfrak{q}^{s+c}}$ and for $c=m\in\mathbb{N}$,

$$(1-\varkappa)_{\mathfrak{q}}^m=(1-\varkappa)(1-\mathfrak{q}\varkappa)\ldots(1-\mathfrak{q}^{m-1}\varkappa).$$

For the integers m,s such that $0\leqslant s\leqslant m$, the $\mathfrak{q}$*-binomial* is defined as

$$\binom{m}{s}_{\mathfrak{q}}=\frac{[m]_{\mathfrak{q}}!}{[s]_{\mathfrak{q}}![m-s]_{\mathfrak{q}}!}.$$

For any non-negative integer m, the $\mathfrak{q}$*-factorial* is defined as

$$[m]_{\mathfrak{q}}!=[m]_{\mathfrak{q}}[m-1]_{\mathfrak{q}}!\text{ if }m\geqslant 1$$

and $[m]_{\mathfrak{q}}!=1$ if $m=0$.

Let $\mathfrak{f}:\mathbb{R}\to\mathbb{R}$ be a function and let $\varkappa\in\mathbb{R}$. The $\mathfrak{q}$-derivative $\mathfrak{D}_{\mathfrak{q}}$ of a function $\mathfrak{f}$ at $\varkappa$ is given by

$$\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\varkappa)=\frac{\mathfrak{f}(\mathfrak{q}\varkappa)-\mathfrak{f}(\varkappa)}{\varkappa(\mathfrak{q}-1)},\ \ \varkappa\neq 0,\ \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(0)=\lim_{\varkappa\to 0}D_{\mathfrak{q}}\mathfrak{f}(\varkappa). \tag{3}$$

Notice that in the limit $\mathfrak{q}\to 1^-$ then $\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\varkappa)\to\mathfrak{f}'(\varkappa)$ provided $\mathfrak{f}'(\varkappa)$ exists.

The left and right $\mathfrak{q}$-derivatives of $\mathfrak{f}$ are given by

$$\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)=\frac{\mathfrak{f}(\mathfrak{q}\varkappa)-\mathfrak{f}(\varkappa)}{\varkappa(\mathfrak{q}-1)} \tag{4}$$

and

$$\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)=\frac{\mathfrak{f}(\varkappa)-\mathfrak{f}(\frac{\varkappa}{\mathfrak{q}})}{\varkappa(\mathfrak{q}-1)} \tag{5}$$

provided that $\varkappa\neq 0$.

The Jackson type $\mathfrak{q}$-integral introduced by Thomae [34] and Jackson [11] is defined as follows:

$$\int_0^c \mathfrak{f}(t)\mathrm{d}_{\mathfrak{q}}t = (1-\mathfrak{q})c\sum_{s=0}^{\infty}\mathfrak{f}(c\mathfrak{q}^s)\mathfrak{q}^s,$$

where $c \in [0,\infty)$. The $\mathfrak{q}$-analogues of the rules for differentiation of a product of functions and integration by parts are given by

$$\mathfrak{D}_{\mathfrak{q}}(\mathfrak{f}\mathfrak{g})(\varkappa) = \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\varkappa)\mathfrak{g}(\varkappa) + \mathfrak{f}(\mathfrak{q}\varkappa)\mathfrak{D}_{\mathfrak{q}}\mathfrak{g}(\varkappa) \tag{6}$$

and

$$\int_b^c \mathfrak{g}(\varkappa)\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(\varkappa)d_{\mathfrak{q}}\varkappa = \mathfrak{f}(c)\mathfrak{g}(c) - \mathfrak{f}(b)\mathfrak{g}(b) - \int_b^c \mathfrak{f}(\mathfrak{q}\varkappa)\mathfrak{D}_{\mathfrak{q}}\mathfrak{g}(\varkappa)d_{\mathfrak{q}}\varkappa \tag{7}$$

respectively.

In the last two decades, there has been a great deal of interest among researchers to define positive linear operators based on q-integers and investigate their various approximation properties. For some recent significant studies in this direction, we refer the reader to the papers ([21], [23], [24], [25], [29] and [35] etc.).

The $\mathfrak{q}$-Laguerre polynomials ([9], p. 29), ([10], p. 57) and ([18], p. 21) are given by

$$\mathscr{L}_n^{(a)}(\varkappa;\mathfrak{q}) = \frac{(\mathfrak{q}^{a+1};\mathfrak{q})_n}{(\mathfrak{q};\mathfrak{q})_n}\sum_{j=0}^{n}\frac{(\mathfrak{q}^{-n};\mathfrak{q})_j\,\mathfrak{q}^{(j)}}{(\mathfrak{q}^{a+1};\mathfrak{q})_j(\mathfrak{q};\mathfrak{q})_j}(1-\mathfrak{q})^j(\mathfrak{q}^{n+a+1}\varkappa)^j,$$

where the $\mathfrak{q}$-shifted factorial is defined as

$$(a;\mathfrak{q})_0 = 1;\;\; (a;\mathfrak{q})_n = \prod_{j=0}^{n-1}(1-a\mathfrak{q}^j),\; n\in\mathbb{N}$$

and $a \in \mathbb{C}$. Moak ([18], eq. (4.17), p. 29) found the generating function for the $\mathfrak{q}$-Laguerre polynomials as

$$\mathscr{G}_a(\varkappa,w) = \frac{(\varkappa\mathfrak{q}^{a+1};\mathfrak{q})_\infty}{(\varkappa;\mathfrak{q})_\infty}\sum_{m=0}^{\infty}\frac{\mathfrak{q}^{n^2+am}\{-(1-\mathfrak{q})\varkappa w\}^m}{(\mathfrak{q};\mathfrak{q})_m\,(\varkappa\mathfrak{q}^{a+1};\mathfrak{q})_m} = \sum_{j=0}^{\infty}\mathscr{L}_j^{(a)}(w;\mathfrak{q})\varkappa^j,\;(Re\;a>1) \tag{8}$$

where

$$(a;\mathfrak{q})_\infty = \prod_{k=0}^{\infty}(1-\mathfrak{q}^k a),\;|\mathfrak{q}|<1.$$

Evidently,

$$(a;\mathfrak{q})_m = \frac{(a;\mathfrak{q})_\infty}{(a\mathfrak{q}^m;\mathfrak{q})_\infty}.$$

In 2007, for $\mathfrak{f}\in C[0,1)$, Özarslan [27] considered the $\mathfrak{q}$-analogue of the Laguerre type operators as

$$(\mathfrak{G}_{m,\mathfrak{q}}\mathfrak{f}(u))(\varkappa;w) = \frac{1}{\mathscr{G}_m(\varkappa,w)}\sum_{k=0}^{\infty}\mathfrak{f}\left(\frac{[k]_{\mathfrak{q}}}{[m+k]_{\mathfrak{q}}}\right)\mathscr{L}_k^{(m)}(w;\mathfrak{q})\varkappa^k, \tag{9}$$

where $0 \leqslant \varkappa < 1$, $-\infty < w \leqslant 0$, $\mathfrak{q} \in (0,1)$ and $\mathscr{G}_m(\varkappa, w)$ is the generating function given by (8). In order to study the approximation of $\mathfrak{f} \in C[0,1]$, we modify the definition (9) as follows:

$$\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f}(u))(\varkappa, w) = \frac{1}{\mathscr{G}_m(\varkappa, w)} \sum_{k=0}^{\infty} \mathfrak{f}\left(\frac{[k]_\mathfrak{q}}{[m+k]_\mathfrak{q}}\right) \mathscr{L}_k^{(m)}(w;\mathfrak{q})\varkappa^k, \ \ 0 \leqslant \varkappa < 1 \quad (10)$$

and

$$\left(\mathfrak{G}_{m,\mathfrak{q}}\mathfrak{f}(u)\right)(1;w) = \mathfrak{f}(1), \ \varkappa = 1.$$

The purpose of the present article is to discuss the deferred weighted statistical convergence of the operators given by (10) and also investigate the convergence by the power summability method. We also consider the $r-th$ order generalization of these operators with the help of a Taylor polynomial. Lastly, we investigate the approximation of functions whose $\mathfrak{q}$-derivatives are of bounded variation on the interval $[0,1]$.

2. Preliminaries

We shall need the following basic result to establish the main results of the paper.

LEMMA 1. [27] *For the operators given by* (10)*, we have*

$$(i) \ \ \|(\mathfrak{G}_{m,\mathfrak{q}}u^2)(.;w) - (.)^2\| \leqslant \frac{2|w|}{[m]_\mathfrak{q}(1-\mathfrak{q}^{m+1})} + \frac{1}{[m]_\mathfrak{q}},$$

$$(ii) \ \ \|(\mathfrak{G}_{m,\mathfrak{q}}u)(.;w) - (.)\| \leqslant \frac{|w|}{[m]_\mathfrak{q}(1-\mathfrak{q}^{m+1})},$$

$$(iii) \ \ \|(\mathfrak{G}_{m,\mathfrak{q}}1)(.;w) - 1\| = 0.$$

where $\|.\|$ *denotes the sup-norm on* $[0,1]$.

In 1935, Zygmund [36] gave the concept of statistical convergence which was later formalized by Steinhaus [31] and Fast [7]. In the past two decades, the investigation of statistical convergence has become an active area of research. Many researchers have contributed in this direction (e.g. [3], [4], [8], [13], [19] and [22] etc.).

A real or complex valued sequence $\xi = \xi_i$ is called statistically convergent to l if for a given $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{m}\left|\left\{i : i \leqslant m, |\xi_i - l| \geqslant \varepsilon\right\}\right| = 0,$$

where the vertical bars represent the cardinality of the set. We denote this convergence by

$$st - \lim_{m\to\infty} \xi_m = l.$$

Let (c_m) and (d_m) be sequences of non negative integers such that $c_m < d_m$, $\forall\, m \in \mathbb{N}$ and

$$\lim_{m\to\infty} d_m = \infty.$$

Then we call a sequence $\xi = (\xi_i)$ to be deferred statistically convergent to l if for every $\varepsilon > 0$, we have

$$\lim_{m\to\infty} \frac{1}{(d_m - c_m)} \left| \left\{ i : c_m < i \leqslant d_m,\ |\xi_i - l| \geqslant \varepsilon \right\} \right| = 0. \tag{11}$$

It is denoted as

$$DS_{c,d} - \lim_{m\to\infty} \xi_m = l.$$

Let us note that if $d_m = m$ and $c_m = 0$, then the deferred statistical convergence reduces to the statistical convergence. However, if $d_m = \lambda_m$ and $c_m = 0$, where λ_m is a sequence of positive integers such that $\lambda_m < \lambda_{m+1}, \forall\, m \in \mathbb{N}$ then the definition (11) coincides with the λ-statistical convergence (see [20], [26]). Further, if $d_m = \eta_m$ and $c_m = \eta_{m-1}$ where $\eta_m \in \mathbb{N} \cup \{0\}$ and $\eta_m - \eta_{m-1} \to \infty$, as $m \to \infty$ then the definition (11) includes the lacunary statistical convergence.

Let (b_m) be a positive non increasing sequence. Then the sequence (ξ_m) is said to converge deferred statistically to the number l with the rate $o(b_m)$ provided for every $\varepsilon > 0$,

$$\lim_{m\to\infty} \frac{1}{b_m} \left\{ \frac{1}{(d_m - c_m)} \left| \left\{ i : c_m < i \leqslant d_m,\ |\xi_i - l| \geqslant \varepsilon \right\} \right| \right\} = 0.$$

We denote it as $\xi_m - l = DS_{c,d} - o(b_m)$, as $m \to \infty$. The sequence (ξ_m) is called deferred -statistically bounded with the rate $O(b_m)$, if for every $\varepsilon > 0$,

$$\sup_m \frac{1}{b_m} \left\{ \frac{1}{(d_m - c_m)} \left| \left\{ i : c_m < i \leqslant d_m,\ |\xi_i| \geqslant \varepsilon \right\} \right| \right\} < \infty,$$

and it is denoted by $\xi_m = DS_{c,d} - O(b_m)$, as $m \to \infty$. For further details, please see ([5], [6]).

Next, we discuss the power series summability method, a member of the class of continuous summability methods, to study the convergence by q-Laguerre polynomials operator.

Let (ξ_i) be a sequence of non-negative real numbers such that $\xi_1 > 0$ and the radius of convergence r of the power series $\xi(z) = \sum\limits_{i=1}^{\infty} \xi_i z^{i-1}$ satisfy $0 < r \leqslant \infty$, then we say that the sequence $\eta = (\eta_i)$ is convergent to l in the sense of power series summability method if $\lim\limits_{z\to r-} \frac{1}{\xi(z)} \sum\limits_{i=1}^{\infty} \xi_i \eta_i z^{i-1} = l$.

The power series method is called regular if

$$\lim_{z\to r-} \frac{\xi_i z^{i-1}}{\xi(z)} = 0,\ \forall\, i \in \mathbb{N}.$$

In particular, if $\xi_i = 1$, $\forall\, i \in \mathbb{N}$, then $\xi(z) = \frac{1}{1-z}$ and $r = 1$, hence the power series method includes the Abel's method and if we choose $\xi_i = \frac{1}{(i-1)!}$, $\forall\, i \in \mathbb{N}$, then we obtain $\xi(z) = e^z$ and $r = \infty$ and so, the power series method turns into Borel method. Many researchers have contributed to the study of approximation by positive linear operators using the above methods. The interested reader may refer to ([30], [32] and [33] etc.).

3. Main results

Let us assume that $\langle q_m \rangle$ be a sequence in $(0,1)$ such that $q_m \to 0$, as $m \to \infty$ and $q_m^m \to a$, as $m \to \infty$, $0 \leqslant a < 1$.

3.1. Deferred statistical convergence

First, we show that the sequence of q-Laguerre operators converges deferred statistically to the function f, if f is continuous on $[0,1]$.

THEOREM 1. *For all* $\mathfrak{f} \in C[0,1]$*, we have*

$$DS_{c,d} - \lim_{m\to\infty} \|(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f})(.,w) - f\| = 0.$$

Proof. By the uniform continuity of $\mathfrak{f}$ on [0,1], for a given $\varepsilon > 0$ $\exists$ a $\delta > 0$ such that $|\mathfrak{f}(u) - \mathfrak{f}(\varkappa)| < \varepsilon$, whenever $|u - \varkappa| < \delta$. For $|u - \varkappa| \geqslant \delta$, we have $|\mathfrak{f}(u) - \mathfrak{f}(\varkappa)| \leqslant 2\|\mathfrak{f}\| \frac{(u-\varkappa)^2}{\delta^2}$.

Hence, for all $u, \varkappa \in [0,1]$, we can write

$$|\mathfrak{f}(u) - \mathfrak{f}(\varkappa)| < \varepsilon + \frac{2\|\mathfrak{f}\|}{\delta^2}(u-\varkappa)^2.$$

Thus, in view of $(\mathfrak{G}_{m,\mathfrak{q}_m}(1))(\varkappa,w) = 1$, we get

$$\begin{aligned}|(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f})(\varkappa;w) - \mathfrak{f}(\varkappa)| &\leqslant (\mathfrak{G}_{m,\mathfrak{q}_m}(|\mathfrak{f}(u) - \mathfrak{f}(\varkappa)|))(\varkappa;w)\\ &\leqslant \varepsilon + \frac{2\|\mathfrak{f}\|}{\delta^2}\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u-\varkappa)^2\right)(\varkappa;w).\end{aligned}$$

We may write

$$\begin{aligned}&\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u-\varkappa)^2\right)(\varkappa;w)\\ &= \left(\mathfrak{G}_{m,\mathfrak{q}_m}(u^2)\right)(\varkappa;w) - 2\varkappa\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u)\right)(\varkappa;w) + \varkappa^2\\ &= \left\{\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u^2)\right)(\varkappa;w) - \varkappa^2\right\} - 2\varkappa\left\{\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u)\right)(\varkappa;w) - \varkappa\right\},\end{aligned}$$

hence

$$\begin{aligned}\left\|\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u-(.))^2\right)(.;w)\right\| &\leqslant \left\|(\mathfrak{G}_{m,\mathfrak{q}_m}(u^2)(.;w) - (.)^2\right\|\\ &\quad + 2\left\|\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u)\right)(.;w) - (.)\right\|. \qquad (12)\end{aligned}$$

Consequently,

$$\begin{aligned}
&\left\|(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f})(;w)-\mathfrak{f}\right\| \\
&\leqslant \varepsilon+\frac{2\|\mathfrak{f}\|}{\delta^2}\left\{\left\|(\mathfrak{G}_{m,\mathfrak{q}_m}(u^2))(.;w)-(.)^2\right\|+2\left\|\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u)\right)(.;w)-(.)\right\|\right\} \\
&\leqslant \varepsilon+\frac{2\|\mathfrak{f}\|}{\delta^2}\left[\left\{\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right\}+\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\right] \\
&= \varepsilon+\frac{2\|\mathfrak{f}\|}{\delta^2}\left\{\frac{4|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right\}. \qquad (13)
\end{aligned}$$

Now, for any $\varepsilon'>0$, let us consider the sets

$$J_1=\frac{1}{d_m-c_m}\left\{k:c_m<k\leqslant d_m,\left\|(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f})(.;w)-\mathfrak{f}\right\|\geqslant\varepsilon'\right\};$$
$$J_2=\frac{1}{d_m-c_m}\left\{k:c_m<k\leqslant d_m,\frac{8\|\mathfrak{f}\||w|}{\delta^2[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\geqslant\frac{\varepsilon'-\varepsilon}{2}\right\}$$

and

$$J_3=\frac{1}{d_m-c_m}\left\{k:c_m<k\leqslant d_m,\frac{2\|\mathfrak{f}\|}{\delta^2[m]_{\mathfrak{q}_m}}\geqslant\frac{\varepsilon'-\varepsilon}{2}\right\}.$$

Then in view of (13), we obtain

$$J_1\subset J_2\cup J_3$$

which implies that

$$|J_1|\leqslant|J_2|+|J_3|.$$

Hence

$$\lim_{m\to\infty}|J_1|\leqslant\lim_{m\to\infty}|J_2|+\lim_{m\to\infty}|J_3|.$$

Since the sequences

$$\frac{8\|\mathfrak{f}\||w|}{\delta^2[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}$$

and

$$\frac{2\|\mathfrak{f}\|}{\delta^2[m]_{\mathfrak{q}_m}}$$

tend to zero, as $m\to\infty$, it follows that

$$DS_{c,d}-\lim_{m\to\infty}\frac{8\|\mathfrak{f}\||w|}{\delta^2[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}=0$$

and

$$DS_{c,d} - \lim_{m\to\infty} \frac{2\|\mathfrak{f}\|}{\delta^2[m]_{\mathfrak{q}_m}} = 0.$$

Hence, $\lim_{m\to\infty} |J_1| = 0$, whence the result. $\square$

In the following result, we determine the rate of deferred statistical convergence for the operators (10) by means of modulus of continuity.

THEOREM 2. *Let* $\mathfrak{f} \in C[0,1]$ *and* $DS_{c,d} - \omega(\mathfrak{f};\sqrt{\nu_{m,2}}) = o(\zeta_m)$, *as* $m \to \infty$ *then*

$$DS_{c,d} - \|(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f})(.;w) - \mathfrak{f}\| = o(\zeta_m), \ \ as \ \ m \to \infty,$$

where $\nu_{m,2} = \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(u-(.))^2(.;w) \right\|.$

Proof. For any $\mathfrak{f} \in C[0,1]$, considering the elementary inequality

$$|\mathfrak{f}(u) - \mathfrak{f}(\varkappa)| \leqslant \left(1 + \frac{(u-\varkappa)^2}{\delta^2}\right)\omega(\mathfrak{f};\delta), \ \delta > 0$$

we have

$$|(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f})(\varkappa;w) - \mathfrak{f}(\varkappa)| \leqslant \left(1 + \frac{1}{\delta^2}\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u-\varkappa)^2\right)(\varkappa;w)\right)\omega(\mathfrak{f};\delta), \ \delta > 0$$

which implies that

$$\begin{aligned}\left\|\left(\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})\right)(.;w) - \mathfrak{f}\right\| &\leqslant \left(1 + \frac{1}{\delta^2}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(u-(.))^2(.;w)\right\|\right)\omega(\mathfrak{f};\delta) \\ &= 2\omega(\mathfrak{f};\sqrt{\nu_{m,2}}),\end{aligned}$$

where we choose $\delta = \sqrt{\nu_{m,2}}$. For any $\varepsilon > 0$, we consider

$$J_1 = \frac{1}{d_m - c_m}\left\{k : c_m < k \leqslant d_m, \left\|\left(\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})\right)(.;w) - \mathfrak{f}\right\| \geqslant \varepsilon\right\}$$

and

$$J_2 = \frac{1}{d_m - c_m}\left\{k : c_m < k \leqslant d_m, \ 2\omega(\mathfrak{f};\sqrt{\nu_{m,2}}) \geqslant \varepsilon\right\}.$$

It is clear that $J_1 \subset J_2$.

Consequently,

$$|J_1| \leqslant |J_2|$$

and hence

$$\lim_{m\to\infty} |J_1| \leqslant \lim_{m\to\infty} |J_2|.$$

Since

$$DS_{c,d}-\omega(\mathfrak{f};\sqrt{\nu_{m,2}})=o(\zeta_m),\quad \text{as}\quad m\to\infty,$$

it follows that

$$\frac{1}{\zeta_m}|J_2|\to 0,\quad \text{as}\quad m\to\infty$$

and hence

$$\lim_{m\to\infty}\frac{1}{\zeta_m}|J_1|=0,$$

which proves the theorem. □

In our further consideration, let us denote $\mu_{m,n}(\varkappa)=\Big(\mathfrak{G}_{m,\mathfrak{q}_m}(u-\varkappa)^n\Big)(\varkappa;w)$, $n\in\mathbb{N}\cup\{0\}$.

3.2. Power summability method

The following theorem is a Korovkin type approximation theorem for the operators (10) by using power summability method.

THEOREM 3. *For all* $\mathfrak{f}\in C[0,1]$, *the operators* (10) *verify*

$$\lim_{z\to R-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\Big\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\Big\|\xi_m z^{m-1}=0$$

Proof. Taking into account the inequality (13), we have

$$\begin{aligned}
&\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\Big\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\Big\|\xi_m z^{m-1}\\
&\leqslant \varepsilon+\frac{2\|\mathfrak{f}\|}{\delta^2}\Bigg\{\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\|(\mathfrak{G}_{m,\mathfrak{q}_m}(u^2))(.;w)-(.)^2\|\xi_m z^{m-1}\\
&\quad+\frac{2}{\xi(z)}\sum_{m=1}^{\infty}\|(\mathfrak{G}_{m,\mathfrak{q}_m}(u))(.;w)-(.)\|\xi_m z^{m-1}\Bigg\}\\
&\leqslant \varepsilon+\frac{2\|\mathfrak{f}\|}{\delta^2}\Bigg\{\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\Bigg\{\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\Bigg\}\xi_m z^{m-1}\\
&\quad+\frac{2}{\xi(z)}\sum_{m=1}^{\infty}\Bigg(\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\Bigg)\xi_m z^{m-1}\Bigg\}.
\end{aligned}\tag{14}$$

Now, let us show that

$$\lim_{z\to R-}\Bigg\{\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\Bigg\{\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\Bigg\}\xi_m z^{m-1}=0.\tag{15}$$

Since $\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\to 0$, as $m\to\infty$, for the given $\varepsilon>0$, $\exists\ m_0(\varepsilon)\in\mathbb{N}$ such that

$$\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}<\frac{\varepsilon}{2},\ \forall\, m>m_0(\varepsilon).$$

Therefore,

$$\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\left\{\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right\}\xi_m z^{m-1}\leqslant\frac{M}{\xi(z)}\sum_{m=1}^{m_0}\xi_m z^{m-1}+\frac{\varepsilon}{2},\tag{16}$$

where

$$M=\max_{1\leqslant m\leqslant m_0(\varepsilon)}\left\{\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right\}.$$

By the regularity condition $\exists\ \delta_m(\varepsilon)>0$ such that

$$\frac{\xi_m z^{m-1}}{\xi(z)}<\frac{\varepsilon}{2Mm_0(\varepsilon)},\ \forall\, r-\delta_m(\varepsilon)<z<r,$$

and $m=1,2,\ldots,m_0(\varepsilon)$. Let us choose

$$\delta(\varepsilon)=\min_{1\leqslant m\leqslant m_0(\varepsilon)}\delta_m(\varepsilon).$$

Then,

$$\frac{\xi_m z^{m-1}}{\xi(z)}<\frac{\varepsilon}{2Mm_0(\varepsilon)},\forall\, r-\delta(\varepsilon)<z<r \text{ and } m=1,2,\ldots,m_0(\varepsilon).$$

Hence from (16) it follows that

$$\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\left\{\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right\}\xi_m z^{m-1}<\varepsilon,\ \forall\, r-\delta(\varepsilon)<z<r.\tag{17}$$

Since $\varepsilon>0$, is arbitrary, the equation (15) is established.

By a similar reasoning, we can show that

$$\lim_{z\to R-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\xi_m z^{m-1}=0.$$

Hence, from (14), we have

$$\lim_{z\to R-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\|(\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f}))(.;w)-\mathfrak{f}\|\xi_m z^{m-1}=0.\quad\square$$

The following theorem yields the rate of convergence of $(\mathfrak{G}_{m,\mathfrak{q}_m}\mathfrak{f}(u))(\varkappa;w)$ to $\mathfrak{f}$ by the power series method.

THEOREM 4. *For* $\mathfrak{f}\in C[0,1]$, *there holds*

$$\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\|\xi_m z^{m-1}\leqslant 2\omega(\mathfrak{f};\delta),$$

where $\delta=\left\{\frac{1}{\xi(z)}\sum\limits_{m=1}^{\infty}\xi_m z^{m-1}\nu_{m,2}\right\}^{\frac{1}{2}}$ *and* $\nu_{m,2}$ *is defined as in Theorem 2.*

Proof. Since $\mathfrak{f}\in C[0,1]$, for any $z\in(0,R)$ and $\delta>0$,

$$\begin{aligned}
&\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\|\xi_m z^{m-1}\\
&\leqslant\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\left\{1+\frac{1}{\delta^2}\left\|\left(\mathfrak{G}_{m,\mathfrak{q}_m}(u-(.))^2\right)(.;w)\right\|\right\}\omega(\mathfrak{f};\delta)\xi_m z^{m-1}\\
&=\omega(\mathfrak{f};\delta)\left\{1+\frac{1}{\delta^2}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\nu_{m,2}\right\}\\
&=2\omega(\mathfrak{f};\delta),
\end{aligned}$$

where $\delta=\left\{\frac{1}{\xi(z)}\sum\limits_{m=1}^{\infty}\xi_m z^{m-1}\nu_{m,2}\right\}^{\frac{1}{2}}$. $\square$

The following example shows that our Theorem 3, is a non-trivial generalization of the classical Korovkin result given in [17].

EXAMPLE 1. Let us consider the $\mathfrak{q}$-Laguerre operators $\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)$ defined by (10). Using these operators, for $\mathfrak{f}\in C[0,1]$, we define the following positive linear operators on $C[0,1]$:

$$\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(\mathfrak{f})(.;w)=(1+\eta_m)\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w), \tag{18}$$

where $\langle\eta_m\rangle=\begin{cases}1, \text{ if } m=p^3,\ p\in\mathbb{N}\,,\\ 0, \quad\ \text{otherwise}\end{cases}$. It is clear that the sequence (η_m) diverges in the classical sense. From the definition (18), we have

$$\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(h_j)(.;w)=(1+\eta_m)\mathfrak{G}_{m,\mathfrak{q}_m}(h_j)(.;w)\ \text{ where }\ h_j=u^j,\ j=0,1,2.$$

Now if we take $\xi_m=1$, for all $m\in\mathbb{N}$, then we obtain $\xi(z)=\sum_{m=1}^{\infty}\xi_m z^{m-1}=\frac{1}{1-z}$, $|z|<1$ which implies that $R=1$. Further, we obtain

$$\lim_{z\to 1-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\eta_m=\lim_{z\to 1-}\frac{(1-z)}{z}\sum_{p=1}^{\infty}z^{p^3}. \tag{19}$$

From Exercise 35 on page 54 of [G. Pólya and G. Szegö, Problems and theorems in analysis I: Series. Integral Calculus. Theory of Functions, Springer-Verlag, 1972], we have

$$\lim_{z\to 1^-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\eta_m=\lim_{z\to 1-}\frac{(1-z)}{z}\sum_{p=1}^{\infty}z^{p^3}=0. \tag{20}$$

Using the definition (18) of auxiliary operators and (19), we conclude that

$$\lim_{z\to 1^-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}||\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}(\mathrm{h}_0)}(.;w)-h_0||=0.$$

Moreover, from Lemma 1 we have

$$||\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(h_1)(.;w)-h_1||\leqslant\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\left(1+\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\right)\eta_m.$$

Since $\left(\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\right)\to 0,$ as $m\to\infty,$ $\exists\, M_1>0$ such that $\left(1+\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}\right)<M_1, \forall\ m\in\mathbb{N}.$ Therefore, in view of the equation (19) we get

$$\lim_{z\to 1^-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}||\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(h_1)(.;w)-h_1||=0.$$

Finally, again using Lemma 1 and the definition of the operators $\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}$ given by (18) we have

$$||\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(h_2)(.;w)-h_2||\leqslant\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}$$
$$+\eta_m\left(\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}+1\right). \tag{21}$$

Since for $m\to\infty,$ $\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}$ converges to 0, it is also convergent to 0 in the sense of power series method. On the other hand, since $\left(\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}+1\right)\leqslant(M_2+1)$, for some $M_2>0,$ in view of (20) it follows that

$$\lim_{z\to 1^-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\eta_m\left(\frac{2|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}+1\right)$$
$$\leqslant\lim_{z\to 1^-}\frac{(M_2+1)}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\eta_m=0.$$

Hence from (21), we obtain

$$\lim_{z\to 1^-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}||\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(h_2)(.;w)-h_2||=0.$$

So, we can say that our operator defined in relation (18) satisfies all the conditions of Theorem 3 and therefore we have

$$\lim_{z\to 1^-}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}||\overline{\mathfrak{G}_{\mathrm{m},\mathfrak{q}_\mathrm{m}}}(\mathfrak{f})(.;w)-\mathfrak{f}||=0.$$

Thus the auxiliary operator $\overline{\mathfrak{G}_{m,\mathfrak{q}_m}}(\mathfrak{f})$ converges to $\mathfrak{f}$ in power summability method but, since (η_m) is not convergent to 0, as $m \to \infty$, we can say that the classical Korovkin theorem does not hold true for these operators.

The next theorem yields the rate of convergence by the operators (10) with the aid of power summability method for continuously differentiable functions on $[0,1]$.

THEOREM 5. *For any $\mathfrak{f}' \in C[0,1]$ and for all z sufficiently close to R from left side, the following inequalities are satisfied:*

$$\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\| \leqslant \|\mathfrak{f}'\|\lambda_{w,1}(z)+2\lambda_{w,2}(z)\omega(\mathfrak{f}';\lambda_{w,2}(z))$$

where

$$\lambda_{w,1}(z)=\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(u)(.;w)-\varkappa\right\|$$

and

$$\lambda_{w,2}(z)=\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\sqrt{\left\|\mathfrak{G}_{m,\mathfrak{q}_m}((u-(.))^2)(.;w)\right\|}.$$

Proof. Following Shisha and Mond Theorem [28] and applying Cauchy-Schwarz inequality we see that, for any $\delta > 0$

$$\begin{aligned}
&\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-f\right\| \\
&\leqslant \|\mathfrak{f}\|\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(1)(.;w)-1\right\| \\
&\quad+\|\mathfrak{f}'\|\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(u)(.;w)-(.)\right\| \\
&\quad+\omega(\mathfrak{f}';\delta)\left(\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\sqrt{\left\|\mathfrak{G}_{m,\mathfrak{q}_m}((u-(.))^2)(.;w)\right\|}\right) \\
&\quad\times\left[\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\sqrt{\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(1)(.;w)\right\|}\right. \\
&\quad\left.+\frac{1}{\delta}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\sqrt{\left\|\mathfrak{G}_{m,\mathfrak{q}_m}((u-(.))^2)(.;w)\right\|}\right].
\end{aligned} \tag{22}$$

Now, we use Lemma 1 in (22) and choose

$$\delta=\lambda_{w,2}(z)=\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\sqrt{\left\|\mathfrak{G}_{m,\mathfrak{q}_m}((u-(.))^2)(.;w)\right\|}. \qquad \square$$

Now, we determine the rate of convergence of the sequence $\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(.;w)$ to $\mathfrak{f}$ with the help of power summability method for the elements of the Lipschitz class $Lip_K\alpha$, for $0<\alpha\leqslant 1$.

We recall that a function $\mathfrak{f}\in Lip_K\alpha$ if the inequality

$$|\mathfrak{f}(u)-\mathfrak{f}(\varkappa)|\leqslant K|u-\varkappa|^{\alpha} \tag{23}$$

holds for all $u,\varkappa\in[0,1]$.

THEOREM 6. *Let $\mathfrak{f}\in Lip_K\alpha$ then for all $z\in(0,R)$, the operators (10) verify the following inequality:*

$$\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\|\leqslant K\delta_{\alpha}(z)$$

where $\delta_{\alpha}(z)=\frac{1}{\xi(z)}\sum\limits_{m=1}^{\infty}\xi_m z^{m-1}\left[\frac{4|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right]^{\frac{\alpha}{2}}$.

Proof. Let $\mathfrak{f}\in Lip_K(\alpha)$, $\alpha\in(0,1]$. By linearity and monotonicity of $\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)$ and (23), we write

$$|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(u;w)-\mathfrak{f}(\varkappa)|\leqslant K(\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(|u-\varkappa|^{\alpha};w)).$$

By the Hölder inequality for $r=\frac{2}{\alpha}$ and $s=\frac{2}{2-\alpha}$, and if we take maximum on $[0,1]$ and use (12) and Lemma 1, then we have

$$\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\|\leqslant K\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left[\frac{4|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})}+\frac{1}{[m]_{\mathfrak{q}_m}}\right]^{\frac{\alpha}{2}}$$

whence the result. □

In the following theorem, our focus is to determine the degree of approximation for the operators (10) by using power summability method in terms of second order modulus of continuity via Peetre's K-functional.

Let $C^2[0,1]:=\{\mathfrak{f}\in C[0,1]:\mathfrak{f}',\mathfrak{f}''\in C[0,1]\}$ with the norm defined by

$$\|\mathfrak{f}\|_{C^2[0,1]}=\|\mathfrak{f}\|+\|\mathfrak{f}'\|+\|\mathfrak{f}''\|.$$

For any $\delta>0$ and $\mathfrak{f}\in C[0,1]$, the Peetre's $\mathscr{K}$-functional is given by

$$\mathscr{K}(\mathfrak{f};\delta)=\inf_{\mathfrak{g}\in C^2[0,1]}\left\{\|\mathfrak{f}-\mathfrak{g}\|+\delta\|\mathfrak{g}\|_{C^2[0,1]}\right\}.$$

From Devore and Lorentz [2], there holds

$$\mathscr{K}(\mathfrak{f};\delta)\leqslant C\omega_2(\mathfrak{f};\sqrt{\delta}), \tag{24}$$

where $C>0$ is a constant.

THEOREM 7. *For any* $\mathfrak{f} \in C[0,1]$ *and for all* $z \in (0,R)$, *there holds the inequality*

$$\frac{1}{\xi(z)} \sum_{m=1}^{\infty} \xi_m z^{m-1} \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w) - \mathfrak{f} \right\| \leqslant C\omega_2((f);\sqrt{\delta(z)}$$

where $\delta(z) = \frac{1}{\xi(z)} \sum_{m=1}^{\infty} \left(\frac{5|w|}{2[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})} + \frac{1}{2[m]_{\mathfrak{q}_m}} \right) \xi_m z^{m-1}$.

Proof. Let $\mathfrak{g} \in C^2[0,1]$ be arbitrary. By the Taylor's expansion, we have

$$\begin{aligned} |\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{g})(\varkappa;w) - \mathfrak{g}(\varkappa)| \leqslant\ & |\mathfrak{G}_{m,\mathfrak{q}_m}(u-\varkappa)(\varkappa;w)||\mathfrak{g}'(\varkappa)| \\ & + \frac{1}{2}\mathfrak{G}_{m,\mathfrak{q}_m}((u-\varkappa)^2|\mathfrak{g}''(\xi)|)(\varkappa;w), \end{aligned} \tag{25}$$

where ξ lies between u and $\varkappa$. Then using Lemma 1 in (25), we get

$$\begin{aligned} &\frac{1}{\xi(z)} \sum_{m=1}^{\infty} \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{g})(.;w) - \mathfrak{g} \right\| \xi_m z^{m-1} \\ &\leqslant \|\mathfrak{g}'\| \frac{1}{\xi(z)} \sum_{m=1}^{\infty} \left(\frac{|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})} \right) \xi_m\, z^{m-1} \\ &\quad + \frac{\|\mathfrak{g}''\|}{2} \frac{1}{\xi(z)} \sum_{m=1}^{\infty} \left(\left\| (\mathfrak{G}_{m,\mathfrak{q}_m}(u^2))(.;w) - (.)^2 \right\| + 2\left\| \left(\mathfrak{G}_{m,\mathfrak{q}_m}(u)\right)(.;w) - (.) \right\| \right) \xi_m z^{m-1} \\ &\leqslant \frac{\|\mathfrak{g}\|_{C^2[0,1]}}{\xi(z)} \sum_{m=1}^{\infty} \left(\frac{3|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})} + \frac{1}{2[m]_{\mathfrak{q}_m}} \right) \xi_m z^{m-1}. \end{aligned} \tag{26}$$

On the other hand

$$\left\| \mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w) - \mathfrak{f} \right\| \leqslant 2\|\mathfrak{f} - \mathfrak{g}\| + \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{g})(.;w) - \mathfrak{g} \right\|. \tag{27}$$

From (26) and (27), we have

$$\begin{aligned} &\frac{1}{\xi(z)} \sum_{m=1}^{\infty} \xi_m z^{m-1} \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w) - \mathfrak{f} \right\| \\ &\leqslant \left(2\|\mathfrak{f} - \mathfrak{g}\| + \|\mathfrak{g}\|_{C^2[0,1]} \frac{1}{\xi(z)} \sum_{m=1}^{\infty} \left(\frac{3|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})} + \frac{1}{2[m]_{\mathfrak{q}_m}} \right) \xi_m z^{m-1} \right). \end{aligned}$$

Taking infimum over all $\mathfrak{g} \in C^2[0,1]$ on the right side of the last inequality, and choosing $\delta = \delta(z) = \frac{1}{\xi(z)} \sum_{m=1}^{\infty} \left(\frac{3|w|}{[m]_{\mathfrak{q}_m}(1-\mathfrak{q}_m^{m+1})} + \frac{1}{2[m]_{\mathfrak{q}_m}} \right) \xi_m z^{m-1}$, we obtain

$$\frac{1}{\xi(z)} \sum_{m=1}^{\infty} \xi_m z^{m-1} \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w) - \mathfrak{f} \right\| \leqslant 2\mathscr{K}(\mathfrak{f};\delta(z)).$$

Now, we apply the relation (24) to obtain the required result. $\square$

3.3. q-Laguerre-Taylor operators

It is well known that the linear positive operators have a weak convergence and do not respond to the smoothness of the function. In order to overcome this problem, Kirov and Popova [16] considered the generalization of the operator by means of the Taylor polynomial. Following their idea, for $r \in \mathbb{N}_0$, $\mathfrak{f} \in C^r[0,1]$ and $m \in \mathbb{N}$, we define the $\mathfrak{q}$-Laguerre-Taylor operators as

$$\mathfrak{G}_{m,r,\mathfrak{q}}(\mathfrak{f}(u))(\varkappa,w) = \sum_{k=0}^{\infty} \frac{\mathscr{L}_k^{(m)}(w;\mathfrak{q})}{\mathscr{G}_m(\varkappa,w)} \varkappa^k \sum_{j=0}^{r} \frac{\mathfrak{f}^{(r)}\left(\frac{[k]_\mathfrak{q}}{[m+k]_\mathfrak{q}}\right)}{j!} \left(\varkappa - \frac{[k]_\mathfrak{q}}{[m+k]_\mathfrak{q}}\right)^j, \; 0 \leqslant \varkappa < 1 \tag{28}$$

and

$$\left(\mathfrak{G}_{m,r,\mathfrak{q}}\mathfrak{f}(u)\right)(1;w) = \mathfrak{f}(1), \; \varkappa = 1.$$

Clearly $\mathfrak{G}_{m,0,\mathfrak{q}}(\mathfrak{f}(u))(\varkappa,w) = \mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f}(u))(\varkappa,w)$ for every $\mathfrak{f} \in C[0,1]$ $\varkappa \in [0,1]$, $m \in \mathbb{N}$.

THEOREM 8. *For any $r \in \mathbb{N}$, $\mathfrak{f} \in C^r[0,1]$ such that $\mathfrak{f}^{(r)} \in Lip_K(\alpha)$, $0 < \alpha \leqslant 1$ and all $z \in (0,R)$, we have*

$$\begin{aligned} &\frac{1}{\xi(z)} \sum_{m=1}^{\infty} \xi_m z^{m-1} \left\| \mathfrak{G}_{m,r,\mathfrak{q}_m}(\mathfrak{f})(.;w) - \mathfrak{f} \right\| \\ &\leqslant K \frac{\beta(\alpha+1,r)}{(r-1)!} \frac{1}{\xi(z)} \sum_{m=1}^{\infty} \xi_m z^{m-1} \left\| \mathfrak{G}_{m,\mathfrak{q}_m}(|u-(.)|^{\alpha+r})(.;w) \right\|. \end{aligned}$$

Proof. For $\mathfrak{f} \in C^r[0,1]$, we apply the following Taylor formula [16] about the point $\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}} \in [0,1]$:

$$\begin{aligned} \mathfrak{f}(\varkappa) = &\sum_{j=0}^{r} \frac{\mathfrak{f}^{(j)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)}{j!} \left(\varkappa - \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)^j + \frac{\left(\varkappa - \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)^r}{(r-1)!} \\ &\times \int_0^1 (1-u)^{r-1} \left[\mathfrak{f}^{(r)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}} + u\left(\varkappa - \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)\right) - \mathfrak{f}^{(r)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)\right] du. \end{aligned}$$

From the Taylor formula and the definition of the operators given by (28), we write the difference between $\mathfrak{G}_{m,r,\mathfrak{q}_m}(\mathfrak{f})(.;w)$ and $\mathfrak{f}(\varkappa)$ as follow:

$$\begin{aligned} &\mathfrak{f}(\varkappa) - \mathfrak{G}_{m,r,\mathfrak{q}_m}(\mathfrak{f})(.;w) \\ &= \frac{1}{\mathscr{G}_m(\varkappa,w)} \sum_{k=0}^{\infty} \mathscr{L}_k^{(m)}(w;\mathfrak{q}_m) \varkappa^k \frac{\left(\varkappa - \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)^r}{(r-1)!} \mathfrak{Q}_m\left(\varkappa, \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right), \end{aligned} \tag{29}$$

where

$$\begin{aligned} &\mathfrak{Q}_m\left(\varkappa, \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right) \\ &= \int_0^1 (1-u)^{r-1} \left[\mathfrak{f}^{(r)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}} + u\left(\varkappa - \frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)\right) - \mathfrak{f}^{(r)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)\right] du. \end{aligned}$$

Since $\mathfrak{f}^{(r)} \in Lip_K(\alpha)$, we get

$$|\mathfrak{f}^{(r)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}+u\left(\varkappa-\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)\right)-\mathfrak{f}^{(r)}\left(\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right)| \leqslant Ku^{\alpha}\left|\varkappa-\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right|^{\alpha},$$

thus from (29), we have

$$|\mathfrak{G}_{m,r,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}(\varkappa)| \leqslant K\frac{1}{\mathscr{G}_m(\varkappa,w)}\sum_{k=0}^{\infty}\mathscr{L}_k^{(m)}(w;\mathfrak{q}_m)\varkappa^k\frac{\left|\varkappa-\frac{[k]_{\mathfrak{q}_m}}{[m+k]_{\mathfrak{q}_m}}\right|^{\alpha+r}}{(r-1)!}\beta(\alpha+1,r). \tag{30}$$

Hence,

$$\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\| \leqslant K\frac{\beta(\alpha+1,r)}{(r-1)!}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(|u-(.)|^{\alpha+r})\right\|.$$

Thus,

$$\begin{aligned}&\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f})(.;w)-\mathfrak{f}\right\|\\ &\leqslant K\frac{\beta(\alpha+1,r)}{(r-1)!}\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{G}_{m,\mathfrak{q}_m}(|u-(.)|^{\alpha+r})\right\|.\end{aligned}$$

This completes the proof. $\square$

REMARK 1. Consider the function $\mathfrak{f}^*$ defined as

$$\mathfrak{f}^*(u)=|u-\varkappa|^{\alpha+r}.$$

Using the well known inequality

$$|a^{\tau}-b^{\tau}| \leqslant \tau|a-b|,\ for\ 0 \leqslant a,b \leqslant 1\ and\ \tau \geqslant 1,$$

it follows that $\mathfrak{f}^* \in Lip_{\alpha+r}1$. Since $\mathfrak{f}^*(\varkappa)=0$, we have

$$\|\mathfrak{G}_{m,\mathfrak{q}_m}(\mathfrak{f}^*)\| \leqslant \left(1+\frac{\mu_{m,2}}{\delta^2}\right)\omega(\mathfrak{f}^*;\delta)$$

where $v_{m,2}=\|\mathfrak{G}_{m,\mathfrak{q}_m}(u-(.))^2(.;w)\|$. Choosing $\delta=\sqrt{v_{m,2}}$, from Theorem 8, we obtain

$$\begin{aligned}&\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\left\|\mathfrak{f}-\mathfrak{G}_{m,r,\mathfrak{q}_m}(\mathfrak{f})\right\|\\ &\leqslant \frac{2K}{(r-1)!}\alpha\beta(\alpha,r)\frac{1}{\xi(z)}\sum_{m=1}^{\infty}\xi_m z^{m-1}\sqrt{v_{m,2}}.\end{aligned}$$

3.4. Functions with derivatives of bounded variation

In this section, we shall estimate the degree of approximation of functions whose $\mathfrak{q}$-derivatives are of bounded variation. First we shall need the following result:

LEMMA 2. *For m sufficiently large and* $\varkappa \in (0,1)$, *the following inequality holds:*
(i) $\rho_m^{\mathfrak{q}}(\varkappa,w,s) = \int_0^s (\mathfrak{D}_{\mathfrak{q},y}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,y))d_{\mathfrak{q}}y \leqslant \frac{\mu_{m,2}(\varkappa)}{(\varkappa-s)^2}$, $0<s<\varkappa$ *and*
(ii) $1-\rho_m^{\mathfrak{q}}(\varkappa,w,z) = \int_z^1 (\mathfrak{D}_{\mathfrak{q},y}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,y))d_{\mathfrak{q}}y \leqslant \frac{\mu_{m,2}(\varkappa)}{(\varkappa-s)^2}$, $\varkappa<z<1$, *where*

$$\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,y) = \begin{cases} \sum\limits_{k\leqslant \frac{ym}{1-y}} \frac{\mathscr{L}_k^{(m)}(w;\mathfrak{q})\varkappa^k}{\mathscr{G}_m(\varkappa,w)}, & 0<y<1 \\ 0\ , & y=0. \\ 1 & y=1. \end{cases}$$

Proof. (i) For $0<s<\varkappa$, we obtain

$$\begin{aligned} \rho_m^{\mathfrak{q}}(\varkappa,w,s) &= \int_0^s \mathfrak{D}_{\mathfrak{q},y}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,y)d_q y \\ &\leqslant \int_0^s \mathfrak{D}_{\mathfrak{q},y}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,y)\left(\frac{\varkappa-y}{\varkappa-s}\right)^2 d_{\mathfrak{q}}y \\ &\leqslant \frac{1}{(\varkappa-s)^2}\int_0^s \mathfrak{D}_{\mathfrak{q},y}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,y)\,(\varkappa-y)^2 d_{\mathfrak{q}}y \\ &\leqslant \frac{\nu_{m,2}}{(\varkappa-s)^2}. \end{aligned}$$

Similarly, we can prove the second assertion (ii). □

The rate of convergence for functions of BV and those having derivatives of BV for $\mathfrak{q}$-operators was not resolved until 2019. Karsli [14, 15] proved this and studied the rate of convergence for BV functions for $\mathfrak{q}$-Bernstein-Schurer operators and $\mathfrak{q}$-Bernstein Durrmeyer operators.

Now, we shall obtain the rate of convergence of $\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(.;w)$ for functions having $\mathfrak{q}$-derivatives of bounded variation on $[0,1]$. We show that the points $\varkappa$ where $\mathfrak{D}_q^+\mathfrak{f}(\varkappa)$ and $\mathfrak{D}_{\mathfrak{q}}^-\mathfrak{f}(\varkappa)$ exist, the operators $\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(\varkappa;w)$ converge to the function $\mathfrak{f}(\varkappa)$ in the space of $\mathfrak{D}_{\mathfrak{q}}BV$. Let $\mathfrak{D}_{\mathfrak{q}}BV[0,1]$ be the space of all $\mathfrak{q}$-differentiable functions on $[0,1]$ which have a $\mathfrak{q}$-derivative of bounded variation on $[0,1]$. Such a function $f \in \mathfrak{D}_{\mathfrak{q}}BV[0,1]$ can be represented as

$$\mathfrak{f}(\varkappa) = \int_0^{\varkappa} \xi(s)d_{\mathfrak{q}}s + C,\ 0 \leqslant \varkappa \leqslant 1$$

where, $\xi \in BV[0,1]$ and C is a constant. It is clear that

$$\mathfrak{D}_{\mathfrak{q}}BV[0,1] = \{\mathfrak{f} : \mathfrak{D}_{\mathfrak{q}}\mathfrak{f} = \xi \in BV[0,1]\}.$$

THEOREM 9. *Let $\mathfrak{f} \in \mathfrak{D}_{\mathfrak{q}}BV[0,1]$ and let the one sides limits $\mathfrak{q}$-derivatives $\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}$ and $\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}$ exist at a fixed point $\varkappa \in (0,1)$. Then for m sufficiently large and $\varkappa \in (0,1)$, we have the following estimate:*

$$\begin{aligned}
&|\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(\varkappa;w) - \mathfrak{f}(\varkappa)| \\
\leqslant\ & \frac{\mu_{m,2}(\varkappa)}{\varkappa} \sum_{k=1}^{[\sqrt{[m]_{\mathfrak{q}}}]} \left(\bigvee_{\varkappa-\frac{\varkappa}{k}}^{\varkappa} \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa} \right) + \left| \frac{1}{2}\left(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa) - \mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa) \right| \sqrt{\mu_{m,2}(\varkappa)} \\
&+ \frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}} \left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa} \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa} \right) + \bigvee_{\varkappa}^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}} (\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}) \frac{1-\varkappa}{\sqrt{[m]_{\mathfrak{q}}}} \\
&+ \frac{\mu_{m,2}(\varkappa)}{1-\varkappa} \sum_{k=1}^{\sqrt{[m]_{\mathfrak{q}}}} \bigvee_{\varkappa}^{\varkappa+\frac{1-\varkappa}{k}} (\mathfrak{D}_{\mathfrak{q}}f_{\varkappa}) + \left| \frac{1}{2}\left(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa) + D_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa) \right| |\mu_{m,1}(\varkappa)| \qquad (31)
\end{aligned}$$

where

$$\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s) = \begin{cases} \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(s) - \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}^{+}(\varkappa), & \varkappa < s \leqslant 1 \\ 0\ , & s = \varkappa \\ \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(s) - \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}^{-}(\varkappa), & 0 \leqslant s < \varkappa. \end{cases},$$

$\bigvee_{c}^{d}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa})$ is the total variation of $\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}$ on $[c,d]$.

Proof. By the hypothesis, we may write the $\mathfrak{q}$-derivative $\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}$ in the form

$$\begin{aligned}
\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(s) =\ & \frac{1}{2}\left(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa) + \mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa) \right) + \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s) + \frac{1}{2}\left(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa) - \mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa) \right) sgn(s-\varkappa) \\
&+ \delta_{\varkappa}(s)\left(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(s) - \frac{1}{2}\left(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa) + \mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa) \right) \right), \qquad (32)
\end{aligned}$$

where

$$\delta_{\varkappa}(s) = \begin{cases} 1\ , & s = \varkappa \\ 0\ , & s \neq x. \end{cases}$$

Applying Lemma 2, we have

$$\begin{aligned}
&\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(\varkappa;w) - \mathfrak{f}(\varkappa) \\
&= \int_0^1 (\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))\mathfrak{f}(s)d_{\mathfrak{q}}s - \mathfrak{f}(\varkappa) = \int_0^1 (\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))(\mathfrak{f}(s) - \mathfrak{f}(\varkappa))d_{\mathfrak{q}}s \\
&= \int_0^{\varkappa} (\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))(\mathfrak{f}(s) - \mathfrak{f}(\varkappa))d_{\mathfrak{q}}s + \int_{\varkappa}^1 (\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_n^{\mathfrak{q}}(\varkappa,w,s))(\mathfrak{f}(s) - \mathfrak{f}(\varkappa))d_{\mathfrak{q}}s \\
&= -\int_0^{\varkappa} \left(\int_s^{\varkappa} \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(y)d_{\mathfrak{q}}y \right) (\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s \\
&\quad + \int_{\varkappa}^1 \left(\int_{\varkappa}^s \mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(y)d_{\mathfrak{q}}y \right) (\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s \\
&= \mathfrak{L}_1(\varkappa) + \mathfrak{L}_2(\varkappa),\ \textit{say}.
\end{aligned}$$

By using equation (32), we get

$$\begin{aligned}
\mathfrak{L}_1(\varkappa) = &-\int_0^{\varkappa}\int_s^{\varkappa}\Bigg(\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)+\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)+\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)\\
&+\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)-\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)sgn(y-\varkappa)\\
&+\delta_{\varkappa}(y)\Big(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}(y)-\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)+\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\Big)\Bigg)d_{\mathfrak{q}}y(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s.
\end{aligned}$$

Since $\int_{\varkappa}^{s}(\delta_{\varkappa}(y))d_{\mathfrak{q}}y=0$, we have

$$\begin{aligned}
\mathfrak{L}_1(\varkappa) = &\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)+\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\int_0^{\varkappa}(s-\varkappa)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&-\int_0^{\varkappa}\Big(\int_s^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\Big)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&+\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)-\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\int_0^{\varkappa}(\varkappa-s)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s. \qquad (33)
\end{aligned}$$

Proceeding similarly, we find that

$$\begin{aligned}
\mathfrak{L}_2(\varkappa) = &\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)+\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\int_{\varkappa}^{1}(s-\varkappa)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&+\int_{\varkappa}^{1}\Big(\int_{\varkappa}^{s}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\Big)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&+\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)-\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\int_{\varkappa}^{1}(s-\varkappa)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s. \qquad (34)
\end{aligned}$$

By combining (33) and (34), we get

$$\begin{aligned}
\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(\varkappa;w)-\mathfrak{f}(\varkappa) = &\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)+\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\int_0^{1}(s-\varkappa)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&+\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)-\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\Big)\int_0^{1}|s-\varkappa|(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&-\int_0^{\varkappa}\Big(\int_s^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\Big)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\\
&+\int_{\varkappa}^{1}\Big(\int_{\varkappa}^{s}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\Big)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
|\mathfrak{G}_{m,\mathfrak{q}}(\mathfrak{f})(\varkappa;w)-\mathfrak{f}(\varkappa)| \leqslant &\left|\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)+\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\right||\mathfrak{G}_{m,\mathfrak{q}}(s-\varkappa)(\varkappa;w)|\\
&+\left|\frac{1}{2}\Big(\mathfrak{D}_{\mathfrak{q}}^{+}\mathfrak{f}(\varkappa)-\mathfrak{D}_{\mathfrak{q}}^{-}\mathfrak{f}(\varkappa)\right|\mathfrak{G}_{m,\mathfrak{q}}(|s-\varkappa|)(\varkappa;w)
\end{aligned}$$

$$+\left|\int_0^{\varkappa}\left(\int_s^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\right)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\right|$$
$$+\left|\int_{\varkappa}^{1}\left(\int_{\varkappa}^{s}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\right)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s\right|. \quad (35)$$

Now, let

$$\mathfrak{A}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)=\int_0^{\varkappa}\left(\int_s^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\right)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s,$$

and

$$\mathfrak{B}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)=\int_{\varkappa}^{1}\left(\int_{\varkappa}^{s}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\right)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^{\mathfrak{q}}(\varkappa,w,s))d_{\mathfrak{q}}s.$$

Then, we just need to estimate the terms $\mathfrak{A}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)$ and $\mathfrak{B}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)$. From the definition of $\rho_m^{\mathfrak{q}}(\varkappa,w,s)$ given in Lemma 2, applying the $\mathfrak{q}$-integration by parts, we can write

$$\begin{aligned}\mathfrak{A}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)&=\int_0^{\varkappa}\left(\int_s^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(y)d_{\mathfrak{q}}y\right)\mathfrak{D}_{\mathfrak{q},s}\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s\\&=\int_0^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s)\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s.\end{aligned}$$

Thus,

$$\begin{aligned}|\mathfrak{A}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)|&\leqslant\int_0^{\varkappa}|\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s)|\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s\\&\leqslant\int_0^{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}|\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s)|\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s\\&\quad+\int_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}|\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s)|\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s.\end{aligned}$$

Since $\mathfrak{D}_q\mathfrak{f}_{\varkappa}(\varkappa)=0$ and $\rho_m^{\mathfrak{q}}(\varkappa,w,s)\leqslant 1$, we get

$$\begin{aligned}\int_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}|\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s)|\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s&=\int_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}|\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(s)-\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}(\varkappa)|\rho_m^{\mathfrak{q}}(\varkappa,w,s)d_{\mathfrak{q}}s\\&\leqslant\int_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}\left(\bigvee_s^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}\right)d_{\mathfrak{q}}t\\&\leqslant\left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}\right)\left(\int_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}d_{\mathfrak{q}}s\right)\\&=\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}\left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}^{\varkappa}\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}\right).\end{aligned}$$

Using Lemma 2 and considering $s=\varkappa-\frac{\varkappa}{u}$, we can derive

$$\begin{aligned}
\int_0^{\varkappa-\frac{\varkappa}{\sqrt{[m]_\mathfrak{q}}}} |\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)|\rho_m^\mathfrak{q}(\varkappa,w,s)d_\mathfrak{q}s &\leqslant \mu_{m,2}(\varkappa)\int_0^{\varkappa-\frac{\varkappa}{\sqrt{[m]_\mathfrak{q}}}} |\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)|\frac{d_\mathfrak{q}s}{(\varkappa-s)^2}\\
&\leqslant \mu_{m,2}(\varkappa)\int_0^{\varkappa-\frac{\varkappa}{\sqrt{[m]_\mathfrak{q}}}} \left(\bigvee_s^{\varkappa}\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa\right)\frac{d_\mathfrak{q}s}{(\varkappa-s)^2}\\
&= \mu_{m,2}(\varkappa)\int_1^{\sqrt{[m]_\mathfrak{q}}} \left(\bigvee_{\varkappa-\frac{\varkappa}{u}}^{\varkappa}\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa\right)\frac{\frac{\varkappa}{u^2}d_\mathfrak{q}u}{\left(\frac{\varkappa}{u}\right)^2}\\
&\leqslant \frac{\mu_{m,2}(\varkappa)}{\varkappa}\sum_{k=1}^{[\sqrt{[m]_\mathfrak{q}}]}\left(\bigvee_{\varkappa-\frac{\varkappa}{k}}^{\varkappa}\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa\right).
\end{aligned}$$

Therefore, on combining the above inequalities we reach to

$$|\mathfrak{A}_{m,\mathfrak{q}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa,\varkappa)| \leqslant \frac{\mu_{m,2}(\varkappa)}{\varkappa}\sum_{k=1}^{[\sqrt{[m]_\mathfrak{q}}]}\left(\bigvee_{\varkappa-\frac{\varkappa}{k}}^{\varkappa}\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa\right)+\frac{\varkappa}{\sqrt{[m]_\mathfrak{q}}}\left(\bigvee_{\varkappa-\frac{\varkappa}{\sqrt{[m]_\mathfrak{q}}}}^{\varkappa}\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa\right). \quad (36)$$

Again using $\mathfrak{q}$-integration by parts in $\mathfrak{B}_{m,\mathfrak{q}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa,\varkappa)$ and applying Lemma 2, we can acquire

$$\begin{aligned}
|\mathfrak{B}_{m,\mathfrak{q}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa,\varkappa)| &= \left|\int_\varkappa^1\left(\int_\varkappa^s\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(y)d_\mathfrak{q}y\right)(\mathfrak{D}_{\mathfrak{q},s}\mathfrak{L}_m^\mathfrak{q}(\varkappa,w,s))d_\mathfrak{q}s\right|\\
&\leqslant \int_\varkappa^1|\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)|(1-\rho_n^\mathfrak{q}(\varkappa,w,s))d_\mathfrak{q}s\\
&\leqslant \int_\varkappa^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}|\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)|d_\mathfrak{q}s+\mu_{m,2}(\varkappa)\int_{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}^1|\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)|\frac{d_\mathfrak{q}s}{(\varkappa-s)^2}\\
&= \int_\varkappa^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}|\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)-\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(\varkappa)|d_\mathfrak{q}s\\
&\quad+\mu_{m,2}(\varkappa)\int_{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}^1|\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(s)-\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa(\varkappa)|\frac{d_\mathfrak{q}s}{(\varkappa-s)^2}\\
&\leqslant \bigvee_\varkappa^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa)\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}+\mu_{m,2}(\varkappa)\int_{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}^1\bigvee_\varkappa^s(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa)\frac{d_\mathfrak{q}s}{(\varkappa-s)^2}.
\end{aligned}$$

Now considering $s=\varkappa+\frac{1-\varkappa}{u}$, we obtain

$$\begin{aligned}
|\mathfrak{B}_{m,\mathfrak{q}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa,\varkappa)| &\leqslant \bigvee_\varkappa^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa)\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}+\mu_{m,2}(\varkappa)\int_{\sqrt{[m]_\mathfrak{q}}}^1\bigvee_\varkappa^{\varkappa+\frac{1-\varkappa}{u}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa)\frac{\left(\frac{\varkappa-1}{u^2}\right)d_\mathfrak{q}u}{\left(\frac{1-\varkappa}{u}\right)^2}\\
&= \bigvee_\varkappa^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa)\frac{1-\varkappa}{\sqrt{[m]_\mathfrak{q}}}+\frac{\mu_{m,2}(\varkappa)}{1-\varkappa}\int_1^{\sqrt{[m]_\mathfrak{q}}}\bigvee_\varkappa^{\varkappa+\frac{1-\varkappa}{u}}(\mathfrak{D}_\mathfrak{q}\mathfrak{f}_\varkappa)d_\mathfrak{q}u.
\end{aligned}$$

Therefore, we can deduce

$$|\mathfrak{B}_{m,\mathfrak{q}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa},\varkappa)| \leqslant \bigvee_{\varkappa}^{\varkappa+\frac{1-\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa})\frac{1-\varkappa}{\sqrt{[m]_{\mathfrak{q}}}}+\frac{\mu_{m,2}}{1-\varkappa}\sum_{k=1}^{[\sqrt{[m]_{\mathfrak{q}}}]}\bigvee_{\varkappa}^{\varkappa+\frac{1-\varkappa}{k}}(\mathfrak{D}_{\mathfrak{q}}\mathfrak{f}_{\varkappa}). \tag{37}$$

Now from (36), (37) and (35), we obtain the desired result. □

REFERENCES

[1] E. W. CHENEY AND A. SHARMA, *Bernstein power series*, Canad. J. Math. 16, 241–252 (1964).
[2] R. A. DEVORE AND G. G. LORENTZ, *Constructive Approximation*, Springer Verlag, Berlin, 1993.
[3] E. DUMAN AND O. DUMAN, *Statistical approximation properties of high order operators constructed with the Chan-Chyan-Srivastava polynomials*, Appl. Math. Comput. 218 (5), 1927–1933 (2011).
[4] O. DUMAN AND C. ORHAN, *Statistical approximation by positive linear operators*, Stud. Math. 161, 187–197 (2004).
[5] O. DUMAN AND C. ORHAN, *Rates of A-statistical convergence of positive linear operators*, Appl. Math. Lett., 18, 1339–1344 (2005).
[6] O. DUMAN, M. K. KHAN AND C. ORHAN, *A-statistical convergence of approximating operators*, Math. Inequal. Appl., 6 (4), 689–699 (2003).
[7] H. FAST, *Sur la convergence statistique*, Colloq. Math. 2, 241–244 (1951).
[8] A. GADJIEV AND C. ORHAN, *Some approximation theorems via statistical convergence*, Rocky Mt. J. Math. 32, 129–138 (2002).
[9] W. HAHN, *Uber orthogonal polynome, die q-differenzengleichungen genugen*, Math. Nachr. 2, 4–34 (1949).
[10] F. H. JACKSON, *Basic double hypergeometric functions (II)*, Quart. J. Math. Oxford, Ser. 15 (1), 49–61 (1944).
[11] F. H. JACKSON, *On q-definite integrals*, Quart. J. Pure Appl. Math., 41, 193–203 (1910).
[12] V. KAC, P. CHEUNG, *Quantum Calculus*, Universitext. Springer-Verlag, New York, 2002.
[13] V. KARAKAYA AND T. A. CHISHTI, *Weighted statistical convergence*, Iran. J. Sci. Technol., Trans. A. Sci. 33, 219–223 (2009).
[14] H. KARSLI, *Some properties of q-Bernstein-Durrmeyer operators*, Tbilisi Math. J., 12 (4) 189–204, (2019).
[15] H. KARSLI, *On approximation to discrete q-derivatives of functions via q-Bernstein-Schurer operators*, Math. Found. Comput., 4 (1), 15–30 (2021).
[16] G. KIROV, AND I. POPOVA, *A generalization of linear positive operators*, Math Balkanica, 7, 149–162 (1993).
[17] P. P. KOROVKIN, *Linear Operators and Approximation Theory*, Hindustan Publishing Corporation, Delhi, India, 1960.
[18] D. S. MOAK, *The q-analogue of the Laguerre polynomials*, J. Math. Anal. Appl. 81, 20–47 (1981).
[19] S. A. MOHIUDDINE, *Statistical weighted A-summability with application to Korovkin's type approximation theorem*, J. Inequal. Appl. 2016, 101 (2016).
[20] M. MURSALEEN, *λ-statistical convergence*, Math. Slovaca, 50 (1), 111–115 (2000).
[21] M. MURSALEEN, K. J. ANSARI AND A. KHAN, *Approximation properties and error estimation of q-Bernstein shifted operators*, Numer. Algor. 84, 207–227 (2020).
[22] M. MURSALEEN, V. KARAKAYA, M. ERTÜRK AND F. GÜRSOY, *Weighted statistical convergence and its application to Korovkin type approximation theorem*, Appl. Math. Comput. 218, 9132–9137 (2012).
[23] M. MURSALEEN, MD. NASIRUZZAMAN AND A. A. H. AL-ABIED, *Dunkl Generalization of q-Parametric Szász-Mirakjan Operators*, Int. J. Anal. Appl., 13 (2), 206–215 (2017).
[24] MD. NASIRUZZAMAN, K. J. ANSARI AND M. MURSALEEN, *On the parametric approximation results of Phillips operators involving the q-Appell polynomials*, Iran. J. Sci. Technol. Trans. Sci. (2021), https://doi.org/10.1007/s40995-021-01219-9.

[25] MD. NASIRUZZAMAN, A. MUKHEIMER AND M. MURSALEEN, *Approximation results on Dunkl generalization of Phillips operators via q-calculus*, Adv. Differ. Equ., 2019, 244 (2019), https://doi.org/10.1186/s13662-019-2178-1.
[26] J. A. OSIKIEWICZ, *Summability of Matrix Submethods and Spliced Sequences*, Ph. D. Thesis, August, (1997).
[27] M. A. ÖZARSLAN, *q-Laguerre type linear positive operators*, Studia Sci. Math. Hungar., 44 (1), 65–80, (2007).
[28] O. SHISHA AND B. MOND, *The degree of convergence of sequences of linear positive operators*, Proc. Natl. Acad. Sci. USA 60 (4) 1196–1200 (1968).
[29] H. M. SRIVASTAVA, M. MURSALEEN AND MD. NASIRUZZAMAN, *Approximation by a class of q-Beta operators of the second Kind via the Dunkl-type generalization on weighted spaces*, Complex Anal. Oper. Theory, 13 (3), 1537–1556 (2019).
[30] U. STADTMULLER AND A. TALI, *On certain families of generalized Nörlund methods and power series methods*, J. Math. Anal. Appl., 238, 44–66 (1999).
[31] H. STEINHAUS, *Sur la convergence ordinaire et la convergence asymptotique*, Colloq. Math. 2, 73–74 (1951).
[32] E. TAS, *Some results concerning Mastroianni operators by power series method*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat. 63 (1), 187–195 (2016).
[33] E. TAS AND T. YURDAKADIM, *Approximation by positive linear operators in modular spaces by power series method*, Positivity 21 (4), 1293–1306 (2017).
[34] J. THOMAE, *Beiträge zur Theorie der durch die Heinische Reihe: Darstellbaren Functionen*, J. Reine Angew. Math., 70, 258–281 (1869).
[35] MN. ZAMAN, A. M. ALOTAIBI AND M. MURSALEEN, *Approximation by Phillips operators via q-Dunkl generalization based on a new parameter*, J. King Saud Univ. Sci. 33 (4) (2021): 101413, doi:10.1016/j.jksus.2021.101413.
[36] A. ZYGMUND, *Trigonometric Series*, Cambridge Univ. Press, Cambridge, 1979.

(Received June 3, 2021)

P. N. Agrawal
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, India
e-mail: pnappfma@gmail.com

Behar Baxhaku
Department of Mathematics
University of Prishtina
Prishtina, Kosovo
e-mail: behar.baxhaku@uni-pr.edu

Sompal Singh
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, India
e-mail: ssingh@ma.iitr.ac.in