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SOME INEQUALITIES FOR COVARIANCE WITH APPLICATIONS IN STATISTICS

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Abstract. In this paper, we derive upper bounds for the covariance of functions of random variables in different cases, which is the extension of the results established in [4]. Some applications in statistics are also provided. Comparing to other results, ours can be used in more general cases involved more complicated statistical dependency.

1. Introduction

Statistical dependency between two random variables X and Y is one common and key topic in probability theory, finance, and information science and so on. The covariance is the most popular measure of dependence between two random variables.

There are many results already in the literature to bound the covariance. The most well-known covariance bounds for $\mathrm{Cov}(X,Y)$ is

$$-\sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)} \leqslant \mathrm{Cov}(X,Y) \leqslant \sqrt{\mathrm{Var}(X)\,\mathrm{Var}(Y)}. \tag{1.1}$$

It is obvious the lower bounds and upper bounds in (1.1) only involve the variances of the related random variables, which turns out to be extremely feasible and powerful in many cases. Actually, many bounds on covariance are derived via the estimating of the corresponding variances involved, for example, the results in [4].

More precise results may be obtained once more information comes. For examples, many interesting bounds for the covariance between X and Y are attained when $X = f(Z)$ and $Y = g(Z)$ are some proper functions of some common random variable Z based on more powerful techniques, see [3, 5].

The covariance for functions of different random variables are also widely used in probability and statistical theory, industry, engineering and so on, see [2, 10, 11, 13]. For example, the covariance between two assets which are modeled by random variables with complicated copula are considered as the measure of dependency in many term structures models, more detailed information can be find in [12, 9].

Some inequalities for the covariance of functions of some common random variable involving functions with bounded derivatives are established in [4], but the similar

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results for functions of different random variables are still open. This motivates our work in this paper, we will estimate the covariance of functions based on different random variables and explore some applications in computation, sampling, statistics and so on.

This paper is organized as follows: our main results with tidy proofs are presented in Section 2, applications with examples will be illustrated in Section 3.

2. Main result

In this paper, inequalities for the covariance of two functions of different random variables are showed and our main result goes as follows:

THEOREM 2.1. *Assume that two functions $f:[a,b]\to\mathbb{R}$, $g:[c,d]\to\mathbb{R}$ are continuous and differentiable in (a,b) and (c,d) whose derivatives $f':(a,b)\to\mathbb{R}$ are bounded in (a,b) and $g':(c,d)\to\mathbb{R}$ are bounded in (c,d); if ξ,η are random variables with finite expected values $\mathbb{E}\xi,\mathbb{E}\eta$ and standard deviation $\sigma(\xi),\sigma(\eta)$. Then one has*

$$|\mathrm{Cov}(f(\xi),g(\eta))|\leqslant 2||f'||_\infty||g'||_\infty\sigma(\xi)\sigma(\eta),$$

where $a(c)$ is a real or $-\infty$; $b(d)$ is a real or $+\infty$ and

$$||f'||_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty;\ \ ||g'||_\infty=\sup_{t\in(c,d)}|g'(t)|<\infty.$$

Before the proof of our result, we first mention one result given in [4], in which the upper bound for the covariance of two functions of one common random variable were obtained. To make the paper readable, one proof which is based on the powerful technique named symmetrization argument which will be more probabilistic and feasible are also provided.

LEMMA 2.2. ([4]) *Assume that two functions $f,g\ :[a,b]\to\mathbb{R}$ are continuous and differentiable in (a,b) whose derivatives $f',g'\ :(a,b)\to\mathbb{R}$ are bounded in (a,b); if ξ is random variable which has finite expected value $\mathbb{E}\xi$ and variance* $\mathrm{Var}\,\xi$. *Then one has*

$$|\mathrm{Cov}(f(\xi),g(\xi))|\leqslant 2||f'||_\infty||g'||_\infty\,\mathrm{Var}\,\xi$$

where a is a real or $-\infty$; b is a real or $+\infty$ and

$$||f'||_\infty=\sup_{t\in(a,b)}|f'(t)|<\infty;\ \ ||g'||_\infty=\sup_{t\in(a,b)}|g'(t)|<\infty.$$

Proof of Lemma 2.2. We will use the same idea but one different technique used in the original proof in [4] to give upper bounds of the corresponding variances respectively.

Let ξ' be an independent copy of ξ, i.e., ξ' and ξ are two independent random variables with identified probability distribution. Then under the conditions of the theorem above, we can apply the Lagrange mean theorem and get

$$\begin{aligned}
\mathrm{Var}(f(\xi')) &= \mathbb{E}_{\xi'}(f(\xi') - E_{\xi'}f(\xi'))^2 \\
&= \mathbb{E}_{\xi'}(f(\xi') - E_{\xi}f(\xi))^2 \\
&= \mathbb{E}_{\xi'}\mathbb{E}_{\xi}^2[f(\xi') - f(\xi)] \\
&= \mathbb{E}_{\xi'}\mathbb{E}_{\xi}^2[f'(\xi + \theta(\xi' - \xi))(\xi' - \xi)] \\
&\leqslant \mathbb{E}_{\xi'}||f'||_\infty^2\mathbb{E}_{\xi}^2(\xi' - \xi) \\
&= ||f'||_\infty^2\mathbb{E}_{\xi'}\mathbb{E}_{\xi}^2(\xi' - \xi) \\
&\leqslant ||f'||_\infty^2\mathbb{E}_{\xi'}\mathbb{E}_{\xi}(\xi' - \xi)^2 \\
&= ||f'||_\infty^2\mathbb{E}_{\xi'}[(\xi' - \mathbb{E}\xi)^2 + \mathrm{Var}(\xi)] \\
&= ||f'||_\infty^2\mathbb{E}_{\xi'}[(\xi' - \mathbb{E}\xi')^2 + \mathrm{Var}(\xi')] \\
&= 2||f'||_\infty^2\,\mathrm{Var}(\xi').
\end{aligned} \tag{2.1}$$

With the same argument we can get

$$\mathrm{Var}(g(\xi)) \leqslant 2||g'||_\infty^2\,\mathrm{Var}(\xi).$$

Then the inequality in the theorem above follows from

$$|\mathrm{Cov}(f(\xi), g(\xi))| \leqslant \sqrt{\mathrm{Var}(f(\xi))\,\mathrm{Var}(g(\xi))}. \quad \square$$

Proof of Theorem 2.1. With the same technique as in the proof of Theorem 2.2, we have

$$\mathrm{Var}(f(\xi)) \leqslant 2||f'||_\infty^2\,\mathrm{Var}(\xi), \quad \mathrm{Var}(g(\eta)) \leqslant 2||g'||_\infty^2\,\mathrm{Var}(\eta).$$

Then with the inequality

$$|\mathrm{Cov}(f(\xi), g(\eta))| \leqslant \sqrt{\mathrm{Var}(f(\xi))\,\mathrm{Var}(g(\eta))},$$

we complete our proof. $\square$

Moreover, we can give a more interesting result on the covariance of two random variables

$$V = f_1(X) + g_1(Y),\ W = f_2(X) + g_2(Y)$$

for some proper functions $f_1,\ f_2,\ g_1,\ g_2$ satisfying the same conditions with the ones in Theorem 2.1 and different variables X, Y. Similar result goes as follows:

COROLLARY 2.3.

$$\begin{aligned}
|\mathrm{Cov}(V, W)| \leqslant\ & 2||f_1'||_\infty||f_2'||_\infty\,\mathrm{Var}(X) + 2||g_1'||_\infty||g_2'||_\infty\,\mathrm{Var}(Y) \\
&+ 2||f_1'||_\infty||g_2'||_\infty\sigma(X)\sigma(Y) + 2||f_2'||_\infty||g_1'||_\infty\sigma(X)\sigma(Y)
\end{aligned} \tag{2.2}$$

Proof of Corollary 2.3.

$$\begin{aligned}|\mathrm{Cov}(V,W)| &= |\mathrm{Cov}(f_1(X)+g_1(Y),f_2(X)+g_2(Y))| \\ &= |\mathrm{Cov}(f_1(X),f_2(X))+\mathrm{Cov}(f_1(X),g_2(Y)) \\ &\quad +\mathrm{Cov}(g_1(Y),f_2(X))+\mathrm{Cov}(g_1(Y),g_2(Y))| \\ &\leqslant |\mathrm{Cov}(f_1(X),f_2(X))|+|\mathrm{Cov}(f_1(X),g_2(Y))| \\ &\quad +|\mathrm{Cov}(g_1(Y),f_2(X))|+|\mathrm{Cov}(g_1(Y),g_2(Y))|.\end{aligned} \tag{2.3}$$

With Theorem 2.1, we can get the desired result. □

REMARK 2.4. If X and Y are two independent random variables, then the result can be simplified into

$$|\mathrm{Cov}(V,W)| \leqslant 2||f_1'||_\infty||f_2'||_\infty \mathrm{Var}(X)+2||g_1'||_\infty||g_2'||_\infty \mathrm{Var}(Y)$$

3. Applications

3.1. Application from theoretical aspects

In fact, our results above can give good estimation for the covariance of some cumbersome random variables once more information comes. We can have very neat results once the specific random variables are given. In the following part, we will show some results for classical random variables regarding to Theorem 2.1.

THEOREM 3.1. *Let $X \sim U(a,b)$, $Y \sim U(c,d)$, then*

$$|\mathrm{Cov}(f(X),g(Y))| \leqslant \frac{(b-a)(d-c)}{6}||f'||_\infty||g'||_\infty.$$

REMARK 3.2. Our result is much more powerful and feasible, for example when $f(x)=\sin x$, $g(x)=\cos x$, we can give very good estimation for the covariance without complicated calculations. Moreover, we can always obtain this neat bound for the covariance of two $U(0,1)$ random variables, nevertheless the statistical dependency between them.

THEOREM 3.3. *Let f, $g : (-\infty,+\infty)\to\mathbb{R}$ be continuous in $(-\infty,+\infty)$ and differential in $(-\infty,+\infty)$ whose derivatives f', $g' : (-\infty,+\infty)\to\mathbb{R}$ are bounded in $(-\infty,\infty)$. Then for any μ_1,μ_2 and $\sigma_1,\sigma_2>0$, $-1\leqslant\rho\leqslant 1$, we have*

$$\begin{aligned}&\Big|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x)g(y)h(x,y)dxdy-\int_{-\infty}^{+\infty} f(x)\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}dx \\ &\qquad\times\int_{-\infty}^{+\infty} g(y)\exp\left\{-\frac{(x-\mu_2)^2}{2\sigma_2^2}\right\}dy\Big| \\ &\qquad\leqslant 4\pi||f'||_\infty||g'||_\infty\sigma_1^2\sigma_2^2\end{aligned}$$

in which

$$h(x,y)=\frac{1}{\sqrt{1-\rho^2}}\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}.$$

Proof of Theorem 3.3. Let (X,Y) be 2-dimensional Normal distribution $N(\mu_1,\mu_2,\sigma_1^2,\sigma_2^2,\rho)$, i.e., the probability distribution function

$$\begin{aligned} p(x,y) &= \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \\ &\quad\times\exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_1)^2}{\sigma_1^2}-\frac{2\rho(x-\mu_1)(y-\mu_2)}{\sigma_1\sigma_2}+\frac{(y-\mu_2)^2}{\sigma_2^2}\right]\right\}. \end{aligned}$$

From the properties of the normal distribution, the marginal distribution of (X,Y) is also Normal distribution $N(\mu_1,\sigma_1^2)$ and $N(\mu_2,\sigma_2^2)$ with probability distribution function

$$p(x)=\frac{1}{\sqrt{2\pi}\sigma_1}\exp\left\{-\frac{(x-\mu_1)^2}{2\sigma_1^2}\right\}$$

and

$$p(y)=\frac{1}{\sqrt{2\pi}\sigma_2}\exp\left\{-\frac{(y-\mu_2)^2}{2\sigma_2^2}\right\}$$

respectively.

Then we have

$$\begin{aligned} &|\mathrm{Cov}(f(X),g(Y))| \\ &=|\mathbb{E}(f(X)g(Y))-\mathbb{E}(f(X))\mathbb{E}(g(Y))| \\ &=\left|\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty}f(x)g(x)p(x,y)dxdy-\int_{-\infty}^{+\infty}f(x)p(x)dx\int_{-\infty}^{+\infty}g(y)p(y)dy\right| \\ &\leqslant 2||f'||_\infty||g'||_\infty\sigma_1\sigma_2. \end{aligned} \tag{3.1}$$

Substituting the corresponding items and rearrange them, we complete the proof. □

3.2. More examples

In the last part of our paper, we will give several examples in which our result can find powerful applications.

3.2.1. Applications in computation

We always consider independent random variables in statistical theory. However, the random variables always depend on each other in some case. To simplified the model, weak dependency is acceptable for random variables. Our result can give estimation for the covariance without complicated computations.

EXAMPLE 3.4. Let $X \sim U(0,1)$ and

$$X_1 = \begin{cases} \frac{1}{2}, & X \leqslant \frac{1}{16} \\ 0, & \text{else} \end{cases}, \qquad X_2 = \begin{cases} \frac{1}{2}, & X \leqslant \frac{15}{16} \\ 0, & \text{else} \end{cases}.$$

One can get $\mathrm{Cov}(X_1,X_2) = \frac{1}{32^2}$ by spending much time to calculate. With our result, we can get

$$\mathrm{Cov}(X_1,X_2) \leqslant 2\sigma_{X_1}\sigma_{X_2} = \frac{30}{32^2}.$$

The estimation is not so tight though, it is enough to consider these two random variables as "weakly" dependent ones.

3.2.2. Applications in sampling

Constructing a bivariate distribution with specific marginals and correlation has been an extremely difficult problems in 1930s. Besides specifying the univariate marginals, it is additionally required to appropriately define the dependence structure among the random variables involved. The Copulas, which are distribution functions on $[0,1]^d$ with uniform univariate marginals, are introduced as a technique to solve this issue, see [7, 6] for more information. Our results can provide some approximate algorithm for sampling from some complicated structures.

Let ξ and η be two random variables with copula C such that $\xi \sim F$, $\eta \sim G$, i.e., $H(x,y) = \mathbb{P}(\xi \leqslant x, \eta \leqslant y) = C(F(x),G(y))$, where $C(u,v)$, $(u,v) \in [0,1]^2$ is the connecting copula. The main challenging task here is to sample from (ξ,η).

EXAMPLE 3.5. Let $\mu = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ and $R = \begin{bmatrix} 1 & 0.65 \\ 0.65 & 1 \end{bmatrix}$. We want to draw a sample from $(\sin\frac{X}{10000}, Y)$ with Gaussian copula

$$C(x,y) = \Phi_R(\Phi^{-1}(x), \Phi^{-1}(y)), \quad (x,y) \in [0,1]^2$$

where Φ^{-1} is the inverse cumulative distribution function of standard normal, Φ_R is the joint distribution function of the two-dimensional normal distribution $N(\mu,R)$, $X \sim U(0,1)$ and $Y \sim Gamma(5,6)$.

We can easily check that

$$\left|\mathrm{Cov}\left(\sin\left(\frac{X}{10000}\right), Y\right)\right| \leqslant 2 \cdot \frac{1}{10000} \cdot \sqrt{\frac{1}{12}} \cdot \sqrt{\frac{2}{81}} \sim 0.$$

which means, the two variables are "approximate" independent, although are connected by Gaussian copula. In one word, we can just sample two random independently from the marginal distributions accordingly.

3.2.3. Applications in identifying singular distribution

It is always difficult to estimate the covariance matrix for high dimensional random variable(data). Actually, the covariance matrix is singular or approximately singular in many cases. Our results above will provide a useful tool to the pre-process of the data via dimension reduction of the rank of the covariance matrix.

We will give one example about singular distributions used in many areas, see [8, 13, 1]. A *singular* distribution is a distribution in p-space that is concentrated on a lower dimensional set; that is, the probability associated with any set not intersecting the given set is 0. Our inequality can give upper bound for the covariance and give a quick answer to identify the singular distribution.

EXAMPLE 3.6. Let $\xi = \left(U_1, \sin\frac{U_2}{10000}, \exp\frac{U_3}{10000}\right)$, $\eta = \left(U_4, \cos\frac{U_5}{10000}, U_6^{10}\right)$, in which $U_i \sim U(0,1)$, $i = 1,2,\cdots,6$ and are connected with complicated copulas. Instead of complicated calculation of the covariances, we can get

$$\left|\mathrm{Cov}\left(\sin\frac{U_2}{10000}, \cos\frac{U_5}{10000}\right)\right| \leqslant 2\cdot\frac{1}{10000}\cdot\frac{1}{10000}\cdot\frac{1}{10000^2}\cdot\frac{1}{12} \sim 0.$$

In other words, the covariance matrix is approximately singular, leading to an approximately singular distribution.

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