Journal of Mathematical Inequalities
Volume 16, Number 4 (2022), 1509–1524 doi:10.7153/jmi-2022-16-98

# MAXIMAL COMMUTATOR AND COMMUTATOR OF MAXIMAL FUNCTION ON TOTAL MORREY SPACES

VAGIF S. GULIYEV

(*Communicated by A. Meskhi*)

*Abstract.* In this paper we introduce a new variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. These spaces generalize the classical Morrey spaces so that $L^{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^{p,\lambda}(\mathbb{R}^n)$ and the modified Morrey spaces so that $L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. We give basic properties of the spaces $L^{p,\lambda,\lambda}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We also give necessary and sufficient conditions for the boundedness of the maximal commutator operator $M_b$ and commutator of maximal operator $[b,M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

## 1. Introduction

The classical Morrey spaces were introduced by Morrey [21] for the study of solutions of some quasi-linear elliptic partial differential equations. For more applications of Morrey spaces on partial differential equation, the reader is referred to [23, 24]. Some important results in harmonic analysis have been extended to some new type of Morrey spaces. Recently, the study of Morrey spaces had been extended to the Morrey-Lorentz spaces [20, 24], the Orlicz-Morrey spaces [22] and the Morrey spaces with variable exponents [2, 14, 15]. The study of these Morrey type spaces has applications on partial differential equations, for example, they are related to the viscosity solutions of some fully nonlinear elliptic equations [29]. In this paper we introduce a variant of Morrey spaces called total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$, $0<p<\infty$, $\lambda \in \mathbb{R}$ and $\mu \in \mathbb{R}$. Total Morrey spaces generalize the classical Morrey spaces so that $L^{p,\lambda,\lambda}(\mathbb{R}^n) \equiv L^{p,\lambda}(\mathbb{R}^n)$ and $L^{p,\lambda,0}(\mathbb{R}^n) = \widetilde{L}^{p,\lambda}(\mathbb{R}^n)$. We give basic properties of the spaces $L^{p,\lambda,\lambda}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We also give necessary and sufficient conditions for the boundedness of the maximal commutator operator $M_b$ and commutator of maximal operator $[b,M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We obtain some new characterizations for certain subclasses of $BMO(\mathbb{R}^n)$.

The study of maximal operators is one of the most important topics in harmonic analysis. These significant non-linear operators, whose behavior are very informative in particular in differentiation theory, provided the understanding and the inspiration

*Mathematics subject classification* (2020): Primary 42B20, 42B25, 42B35.
*Keywords and phrases*: Total Morrey spaces, maximal operator, commutator, *BMO* spaces.


Paper JMI-16-98

for the development of the general class of singular and potential operators (see, for instance [10, 28]). For $f \in L^1_{\rm loc}(\mathbb{R}^n)$, the maximal operator $M$ is defined by

$$Mf(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |f(y)|dy,$$

where $B(x,r)$ is the ball of radius $r$ centered at $x \in \mathbb{R}^n$, ${}^{\complement}B(x,r)$ is its complement and $|B(x,r)|$ denotes the Lebesgue measure of $B(x,r)$.

The commutator estimates play an important role in studying the regularity of solutions of elliptic, parabolic and ultraparabolic partial differential equations of second order, and their boundedness can be used to characterize certain function spaces (see, for instance [7, 10, 25, 26, 28]).

The maximal commutator generated by the operator $M$ and $b \in L^1_{\rm loc}(\mathbb{R}^n)$ is defined by

$$M_b f(x) = \sup_{r>0} |B(x,r)|^{-1} \int_{B(x,r)} |b(x)-b(y)||f(y)|dy.$$

The commutator generated by the operator $M$ and a suitable function $b$ is defined by

$$[b,M]f(x) = b(x)Mf(x) - M(bf)(x).$$

Obviously, the operators $M_b$ and $[b,M]$ essentially differ from each other since $M_b$ is positive and sublinear and $[b,M]$ is neither positive nor sublinear. The operators $M$, $[b,M]$ and $M_b$ play an important role in real and harmonic analysis and applications (see, for instance [1, 4, 5, 9, 18, 19, 27, 28, 30]). The nonlinear commutator of Hardy-Littlewood maximal function $[b,M]$ can be used in studying the product of a function in $H^1$ and a function in $BMO$ [6].

The boundedness of the Hardy-Littlewood maximal operator $M$ on $L^p(\mathbb{R}^n)$ is one of the most fundamental results in harmonic analysis. It has been extended to a range of other function spaces, and to many variations of the standard maximal operator. In particular, one can study commutators of $M$ with $BMO$ functions $b$. These turn out to be $L^p$ bounded for $1<p<\infty$ if and only if $b \in BMO$ and $b^- \equiv -\min\{b,0\} \in L^\infty(\mathbb{R}^n)$ [5]. This is useful, for instance, when studying the product of an $H^1$ function with a $BMO$ function. Note that, the boundedness of the operator $M_b$ on $L^p$ spaces was proved by Garcia-Cuerva et al. [9].

The structure of the paper is as follows. In Section 2 we introduce the total Morrey spaces $L^{p,\lambda,\mu}(\mathbb{R}^n)$. We give basic properties of the spaces $L^{p,\lambda,\lambda}(\mathbb{R}^n)$ and study some embeddings into the Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. some definitions and auxiliary results and study some embeddings into the total Morrey space $L^{p,\lambda,\mu}(\mathbb{R}^n)$. In Section 3 we find necessary and sufficient conditions for the boundedness of the maximal commutator $M_b$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces. In Section 4 we find necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b,M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

By $A \lesssim B$ we mean that $A \leqslant CB$ with some positive constant $C$ independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that $A$ and $B$ are equivalent.

## 2. Definition and basic properties of total Morrey spaces

DEFINITION 1. Let $0<p<\infty$, $\lambda\in\mathbb{R}$, $\mu\in\mathbb{R}$, $[t]_1=\min\{1,t\}$, $t>0$. We denote by $L^{p,\lambda}(\mathbb{R}^n)$ the classical Morrey space, by $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ the modified Morrey space [16], and by $L^{p,\lambda,\mu}(\mathbb{R}^n)$ the total Morrey space the set of all classes of locally integrable functions $f$ with the finite norms

$$\|f\|_{L^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0} t^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,t))},$$

$$\|f\|_{\widetilde{L}^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0} [t]_1^{-\frac{\lambda}{p}}\|f\|_{L^p(B(x,t))},$$

$$\|f\|_{L^{p,\lambda,\mu}}=\sup_{x\in\mathbb{R}^n,t>0} [t]_1^{-\frac{\lambda}{p}}[1/t]_1^{\frac{\mu}{p}}\|f\|_{L^p(B(x,t))},$$

respectively.

DEFINITION 2. Let $0<p<\infty$, $\lambda\in\mathbb{R}$ and $\mu\in\mathbb{R}$. We define the weak Morrey space $WL^{p,\lambda}(\mathbb{R}^n)$, the weak modified Morrey space $W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ [16] and the weak total Morrey space $WL^{p,\lambda,\mu}(\mathbb{R}^n)$ as the set of all locally integrable functions $f$ with finite norms

$$\|f\|_{WL^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0} t^{-\frac{\lambda}{p}}\|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{W\widetilde{L}^{p,\lambda}}=\sup_{x\in\mathbb{R}^n,t>0} [t]_1^{-\frac{\lambda}{p}}\|f\|_{WL^p(B(x,t))},$$

$$\|f\|_{WL^{p,\lambda,\mu}}=\sup_{x\in\mathbb{R}^n,t>0} [t]_1^{-\frac{\lambda}{p}}[1/t]_1^{\frac{\mu}{p}}\|f\|_{WL^p(B(x,t))},$$

respectively.

Note that

$$L^{p,0,0}(\mathbb{R}^n)=\widetilde{L}^{p,0}(\mathbb{R}^n)=L^{p,0}(\mathbb{R}^n)=L^p(\mathbb{R}^n),$$
$$WL^{p,0,0}(\mathbb{R}^n)=W\widetilde{L}^{p,0}(\mathbb{R}^n)=WL^{p,0}(\mathbb{R}^n)=WL^p(\mathbb{R}^n),$$
$$L^{p,\lambda,\lambda}(\mathbb{R}^n)=L^{p,\lambda}(\mathbb{R}^n),\quad L^{p,\lambda,0}(\mathbb{R}^n)=\widetilde{L}^{p,\lambda}(\mathbb{R}^n),$$
$$\|f\|_{WL^{p,\lambda,\mu}}\leqslant\|f\|_{L^{p,\lambda,\mu}} \text{ and therefore } L^{p,\lambda,\mu}\subset WL^{p,\lambda,\mu}(\mathbb{R}^n)$$

and

$$L^{p,\lambda,\mu}(\mathbb{R}^n)\subset_{\succ} L^{p,\lambda}(\mathbb{R}^n) \text{ and } \|f\|_{L^{p,\lambda}}\leqslant\|f\|_{L^{p,\lambda,\mu}}, \tag{1}$$

$$L^{p,\lambda,\mu}(\mathbb{R}^n)\subset_{\succ} L^{p,\mu}(\mathbb{R}^n) \text{ and } \|f\|_{L^{p,\mu}}\leqslant\|f\|_{L^{p,\lambda,\mu}} \tag{2}$$

$$\widetilde{L}^{p,\lambda}(\mathbb{R}^n)\subset_{\succ} L^{p}(\mathbb{R}^n) \text{ and } \|f\|_{L^{p}}\leqslant\|f\|_{\widetilde{L}^{p,\lambda}} \tag{3}$$

and if $\lambda<0$ or $\lambda>n$, then $L^{p,\lambda}(\mathbb{R}^n)=\widetilde{L}^{p,\lambda}(\mathbb{R}^n)=WL^{p,\lambda}(\mathbb{R}^n)=W\widetilde{L}^{p,\lambda}(\mathbb{R}^n)=\Theta$, where $\Theta\equiv\Theta(\mathbb{R}^n)$ is the set of all functions equivalent to $0$ on $\mathbb{R}^n$.

LEMMA 1. *If* $0 < p < \infty$, $0 \leqslant \lambda \leqslant n$ *and* $0 \leqslant \mu \leqslant n$, *then for* $1 \leqslant p < \infty$, $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *is a Banach space and for* $0 < p < 1$, $L^{p,\lambda,\mu}(\mathbb{R}^n)$ *is a quasi-Banach space.*

*Proof.* If $1 \leqslant p < \infty$, then $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is a normed space. If $0 < p < 1$, recall that for $f,g \in L^p(B(x,r))$

$$\|f+g\|_{L^p(B(x,r))} \leqslant 2^{\frac{1}{p}-1}\Big(\|f\|_{L^p(B(x,r))} + \|g\|_{L^p(B(x,r))}\Big),$$

where $2^{\frac{1}{p}-1}$ is a sharp constant. Hence for $0 < p < 1$, $f,g \in L^{p,\lambda,\mu}(\mathbb{R}^n)$

$$\|f+g\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} \leqslant 2^{\frac{1}{p}-1}\Big(\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} + \|g\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}\Big). \tag{4}$$

Here $2^{\frac{1}{p}-1} > 1$ is also a sharp constant. (The equality is attained, say, for $f = \chi_{B(0,1)\cap\mathbb{R}^n_+}$, $g = \chi_{B(0,1)\cap\mathbb{R}^n_-}$, where $\mathbb{R}^n_+ = \{x \in \mathbb{R}^n : x_n > 0\}$, $\mathbb{R}^n_- = \{x \in \mathbb{R}^n : x_n < 0\}$). Hence for $0 < p < 1$, $L^{p,\lambda,\mu}(\mathbb{R}^n)$ is a quasi-normed space (and not a normed space).

Let $f_k \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, $k \in \mathbb{N}$, and

$$\lim_{k,m\to\infty} \|f_k - f_m\|_{L^{p,\lambda,\mu}} = 0.$$

Then for any $\varepsilon > 0$ there exists $k_0 \in \mathbb{N}$ such that for all $k,m \in \mathbb{N}$, $k,m \geqslant k_0$ and for all $x \in \mathbb{R}^n$, $r > 0$

$$[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,\|f_k - f_m\|_{L^p(B(x,r))} \leqslant \|f_k - f_m\|_{L^{p,\lambda,\mu}} < \varepsilon. \tag{5}$$

Hence for all $x \in \mathbb{R}^n$ and $r > 0$

$$\lim_{k,m\to\infty} \|f_k - f_m\|_{L^p(B(x,r))} = 0.$$

Due to the completeness of $L^p_{\mathrm{loc}}(\mathbb{R}^n)$ there exists a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ such that for all $x \in \mathbb{R}^n$ and $r > 0$

$$\lim_{k\to\infty} \|f_k - f\|_{L^p(B(x,r))} = 0.$$

Passing in (5) to the limit as $m \to \infty$ we get that for all $x \in \mathbb{R}^n$ and $r > 0$

$$[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,\|f_k - f\|_{L^p(B(x,r))} \leqslant \varepsilon,$$

hence

$$\|f_k - f\|_{L^{p,\lambda,\mu}} \leqslant \varepsilon,$$

which means that $\lim_{k\to\infty} \|f_k - f\|_{L^{p,\lambda,\mu}} = 0$. So the space $L^{p,\lambda,\mu}$ is complete. □

LEMMA 2. *If* $0 < p < \infty$, $0 \leqslant \lambda \leqslant n$ *and* $0 \leqslant \mu \leqslant n$, *then*

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$$

*and*

$$\|f\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}.$$

*Proof.* Let $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$. Then from (1) and (2) we have that $f \in L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\} \leqslant \|f\|_{L^{p,\lambda,\mu}}$.

Now let $f \in L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$. Then

$$\begin{aligned}
\|f\|_{L^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda}\, [1/t]_1^{\mu} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \\
&= \max\left\{ \sup_{x\in\mathbb{R}^n, 0<t\leqslant 1} \left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p}, \sup_{x\in\mathbb{R}^n, t>1} \left(t^{-\mu} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \right\} \\
&\leqslant \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\}.
\end{aligned}$$

Therefore, $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and the embedding $L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n) \subset\!\!\!\succ L^{p,\lambda,\mu}(\mathbb{R}^n)$ is valid.

Thus $L^{p,\lambda,\mu}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^{p,\mu}(\mathbb{R}^n)$ and $\max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^{p,\mu}}\right\} = \|f\|_{L^{p,\lambda,\mu}}$. □

COROLLARY 1. *If* $0 < p < \infty$, $0 \leqslant \lambda \leqslant n$, *then*

$$\widetilde{L}^{p,\lambda}(\mathbb{R}^n) = L^{p,\lambda}(\mathbb{R}^n) \cap L^p(\mathbb{R}^n)$$

*and*

$$\|f\|_{\widetilde{L}^{p,\lambda}} = \max\left\{\|f\|_{L^{p,\lambda}}, \|f\|_{L^p}\right\}.$$

LEMMA 3. *If* $0 < p < \infty$, $0 \leqslant \lambda \leqslant n$ *and* $0 \leqslant \mu \leqslant n$, *then*

$$WL^{p,\lambda,\mu}(\mathbb{R}^n) = WL^{p,\lambda}(\mathbb{R}^n) \cap WL^{p,\mu}(\mathbb{R}^n)$$

*and*

$$\|f\|_{WL^{p,\lambda,\mu}(\mathbb{R}^n)} = \max\left\{\|f\|_{WL^{p,\lambda}}, \|f\|_{WL^{p,\mu}}\right\}.$$

REMARK 1. If $0 < p < \infty$, and $\lambda < 0$ or $\lambda > n$ or $\mu < 0$ or $\mu > n$, then

$$L^{p,\lambda,\mu}(\mathbb{R}^n) = WL^{p,\lambda,\mu}(\mathbb{R}^n) = \Theta(\mathbb{R}^n).$$

EXAMPLE 1. If $\alpha > -\frac{n}{p}$, $0 < p < \infty$ and $0 \leqslant \lambda \leqslant \mu \leqslant n$, then

$$|\cdot|^\alpha \in L^{p,\lambda,\mu}(\mathbb{R}^n) \Leftrightarrow -\frac{n-\lambda}{p} \leqslant \alpha \leqslant -\frac{n-\mu}{p},$$

$$|\cdot|^{\alpha}\chi_{B(0,1)} \in L^{p,\lambda,\mu}(\mathbb{R}^n) \Leftrightarrow \alpha \geqslant -\frac{n-\lambda}{p},$$

and

$$|\cdot|^{\alpha}\chi_{{}^{c}B(0,1)} \in L^{p,\lambda,\mu}(\mathbb{R}^n) \Leftrightarrow \alpha \leqslant -\frac{n-\mu}{p}.$$

Indeed,

$$\begin{aligned}
\||\cdot|^{\alpha}\|_{L^{p,\lambda,\mu}} &= \sup_{r>0,x\in\mathbb{R}} [r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,\||y|^{\alpha}\|_{L^p(B(x,r))} \\
&= \sup_{r>0}[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,\||y|^{\alpha}\|_{L^p(B(0,r))} \\
&= \sup_{r>0}[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\left(\int_{B(0,r)}|y|^{\alpha p}dy\right)^{\frac{1}{p}} \\
&= \sup_{r>0}[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\left(\sigma_n\int_0^r t^{\alpha p+n-1}dt\right)^{\frac{1}{p}} \\
&= \left(\frac{\sigma_n}{\alpha p+n}\right)^{\frac{1}{p}}\sup_{r>0}[r]_1^{-\frac{\lambda}{p}}\,[1/r]_1^{\frac{\mu}{p}}\,r^{\alpha+\frac{n}{p}} \\
&= \left(\frac{\sigma_n}{\alpha p+n}\right)^{\frac{1}{p}}\max\left\{\sup_{0<t\leqslant 1}t^{\alpha+\frac{n-\lambda}{p}},\ \sup_{t>1}t^{\alpha+\frac{n-\mu}{p}}\right\} \\
&= \left(\frac{\sigma_n}{\alpha p+n}\right)^{\frac{1}{p}},
\end{aligned}$$

where $v_n$ is the volume of the unit ball in $\mathbb{R}^n$ and $\sigma_n = nv_n$ is the surface area of the unit sphere in $\mathbb{R}^n$.

LEMMA 4. *If* $0 < p < \infty$, $0 \leqslant \lambda_2 \leqslant \lambda_1 \leqslant n$ *and* $0 \leqslant \mu_1 \leqslant \mu_2 \leqslant n$, *then*

$$L^{p,\lambda_1,\mu_1}(\mathbb{R}^n) \subset_{\succ} L^{p,\lambda_2,\mu_2}(\mathbb{R}^n)$$

*and*

$$\|f\|_{L^{p,\lambda_2,\mu_2}} \leqslant \|f\|_{L^{p,\lambda_1,\mu_1}}.$$

*Proof.* Let $f \in L^{p,\lambda,\mu}$, $0 < p < \infty$, $0 \leqslant \lambda_2 \leqslant \lambda_1 \leqslant n$, $0 \leqslant \mu_1 \leqslant \mu_2 \leqslant n$. Then

$$\begin{aligned}
\|f\|_{L^{p,\lambda_2,\mu_2}} = \max\Bigg\{ &\sup_{x\in\mathbb{R}^n,\,0<t\leqslant 1}\left(t^{\lambda_1-\lambda_2}t^{-\lambda_1}\int_{B(x,t)}|f(y)|^p dy\right)^{1/p}, \\
&\sup_{x\in\mathbb{R}^n,\,t\geqslant 1}\left(t^{\mu_1-\mu_2}t^{-\mu_1}\int_{B(x,t)}|f(y)|^p dy\right)^{1/p}\Bigg\} \leqslant \|f\|_{L^{p,\lambda_1,\mu_1}}. \quad \square
\end{aligned}$$

LEMMA 5. *If* $0 < p < \infty$, $0 \leqslant \lambda \leqslant n$ *and* $0 \leqslant \mu \leqslant n$, *then*

$$L^{p,n,\mu}(\mathbb{R}^n) \subset_{\succ} L^{\infty}(\mathbb{R}^n) \subset_{\succ} L^{p,\lambda,n}(\mathbb{R}^n)$$

*and*

$$\|f\|_{L^{p,\lambda,n}} \leqslant v_n^{1/p}\|f\|_{L^\infty} \leqslant \|f\|_{L^{p,n,\mu}}.$$

*Proof.* Let $f \in L^\infty(\mathbb{R}^n)$. Then for all $x \in \mathbb{R}^n$ and $0 < t \leqslant 1$

$$\left(t^{-\lambda} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \leqslant v_n^{1/p} \|f\|_{L^\infty}, \quad 0 \leqslant \lambda \leqslant n$$

and for all $x \in \mathbb{R}^n$ and $t \geqslant 1$

$$\left(t^{-n} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \leqslant v_n^{1/p} \|f\|_{L^\infty}.$$

Therefore $f \in L^{p,\lambda,n}(\mathbb{R}^n)$ and

$$\|f\|_{L^{p,\lambda,n}} \leqslant v_n^{1/p}\|f\|_{L^\infty}.$$

Let $f \in L^{p,n,\mu}(\mathbb{R}^n)$. By the Lebesgue's Theorem we have (see [28])

$$\lim_{t\to 0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|^p dy = |f(x)|^p.$$

Then

$$\begin{aligned} |f(x)| &= \left(\lim_{t\to 0} |B(x,t)|^{-1} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \\ &\leqslant v_n^{-1/p} \sup_{x\in\mathbb{R}^n,\, 0<t\leqslant 1} \left(t^{-n} \int_{B(x,t)} |f(y)|^p dy\right)^{1/p} \leqslant v_n^{-1/p}\|f\|_{L^{p,n,\mu}}. \end{aligned}$$

Therefore $f \in L^\infty(\mathbb{R}^n)$ and

$$\|f\|_{L^\infty} \leqslant v_n^{-1/p}\|f\|_{L^{p,n,\mu}}. \quad \square$$

COROLLARY 2. *If* $0 < p < \infty$, *then*

$$L^{p,n}(\mathbb{R}^n) = \widetilde{L}^{p,n}(\mathbb{R}^n) = L^\infty(\mathbb{R}^n)$$

*and*

$$\|f\|_{L^{p,n}} = \|f\|_{\widetilde{L}^{p,n}} = v_n^{1/p}\|f\|_{L^\infty}.$$

LEMMA 6. *If* $0 \leqslant \lambda < n$, $0 \leqslant \mu < n$, $0 \leqslant \alpha < n-\lambda$ *and* $0 \leqslant \beta < n-\mu$, *then for* $\frac{n-\lambda}{\alpha} \leqslant p \leqslant \frac{n-\mu}{\beta}$

$$L^{p,\lambda,\mu}(\mathbb{R}^n) \subset_{\succ} L^{1,n-\alpha,n-\beta}(\mathbb{R}^n)$$

*and for* $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ *the following inequality*

$$\|f\|_{L^{1,n-\alpha,n-\beta}} \leqslant v_n^{1/p'}\|f\|_{L^{p,\lambda,\mu}}$$

*is valid.*

*Proof.* Let $0<\alpha<n$, $0\leqslant\lambda<n$, $f\in L^{p,\lambda,\mu}(\mathbb{R}^n)$ and $\frac{n-\lambda}{\alpha}\leqslant p\leqslant\frac{n-\mu}{\beta}$. By the Hölder's inequality we have

$$\|f\|_{L^{1,n-\alpha,n-\beta}}=\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{\alpha-n}\,[1/t]_1^{n-\beta}\int_{B(x,t)}|f(y)|dy$$
$$\leqslant v_n^{1/p'}\sup_{x\in\mathbb{R}^n,t>0}\left([t]_1\,t^{-1}\right)^{-n/p'}[t]_1^{\alpha-\frac{n-\lambda}{p}}\,[1/t]_1^{n-\beta-\frac{\mu}{p}}\left([t]_1^{-\lambda}\,[1/t]_1^{\mu}\int_{B(x,t)}|f(y)|^p dy\right)^{1/p}$$
$$\leqslant v_n^{1/p'}\|f\|_{L^{p,\lambda,\mu}}\sup_{t>0}\left([t]_1\,t^{-1}\right)^{\frac{n-\mu}{p}-\beta}\,[t]_1^{\alpha-\frac{n-\lambda}{p}}.$$

Note that

$$\sup_{t>0}\left([t]_1\,t^{-1}\right)^{\frac{n-\mu}{p}-\beta}[t]_1^{\alpha-\frac{n-\lambda}{p}}=\max\left\{\sup_{0<t\leqslant1}t^{\alpha-\frac{n-\lambda}{p}},\sup_{t>1}t^{\beta-\frac{n-\mu}{p}}\right\}<\infty$$
$$\Longleftrightarrow\frac{n-\lambda}{\alpha}\leqslant p\leqslant\frac{n-\mu}{\beta}.$$

Therefore $f\in L^{1,n-\alpha,n-\beta}(\mathbb{R}^n)$ and

$$\|f\|_{L^{1,n-\alpha,n-\beta}}\leqslant v_n^{1/p'}\|f\|_{L^{p,\lambda,\mu}}.\quad\square$$

From Lemma 6 we get the following

COROLLARY 3. *If* $0\leqslant\mu\leqslant\lambda<n$, $0\leqslant\alpha<n-\lambda$, *then for* $\frac{n-\lambda}{\alpha}\leqslant p\leqslant\frac{n-\mu}{\alpha}$

$$L^{p,\lambda,\mu}(\mathbb{R}^n)\subset_{\succ}L^{1,n-\alpha}(\mathbb{R}^n)$$

*and for* $f\in L^{p,\lambda,\mu}(\mathbb{R}^n)$ *the following inequality*

$$\|f\|_{L^{1,n-\alpha}}\leqslant v_n^{1/p'}\|f\|_{L^{p,\lambda,\mu}}$$

*is valid.*

COROLLARY 4. *If* $0\leqslant\lambda<n$ *and* $0\leqslant\alpha<n-\lambda$, *then for* $p=\frac{n-\lambda}{\alpha}$

$$L^{p,\lambda}(\mathbb{R}^n)\subset L^{1,n-\alpha}(\mathbb{R}^n)\quad\textit{and}\quad\|f\|_{L^{1,n-\alpha}}\leqslant v_n^{1/p'}\|f\|_{L^{p,\lambda}}.$$

COROLLARY 5. *If* $0\leqslant\lambda<n$ *and* $0\leqslant\alpha<n-\lambda$, *then for* $\frac{n-\lambda}{\alpha}\leqslant p\leqslant\frac{n}{\alpha}$

$$\widetilde{L}^{p,\lambda}(\mathbb{R}^n)\subset L^{1,n-\alpha}(\mathbb{R}^n)\quad\textit{and}\quad\|f\|_{L^{1,n-\alpha}}\leqslant v_n^{1/p'}\|f\|_{\widetilde{L}^{p,\lambda}}.$$

## 3. $L^{p,\lambda,\mu}$-boundedness of the maximal commutator operator $M_b$

In this section we find necessary and sufficient conditions for the boundedness of the maximal commutator $M_b$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

The following local estimate is valid (see also [11, 12, 13]).

LEMMA 7. ([3]) *Let* $1 \leqslant p < \infty$ *and* $B(x,r)$ *be any ball in* $\mathbb{R}^n$. *If* $p > 1$, *then the inequality*

$$\|Mf\|_{L^p(B(x,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L^p(B(x,t))} \tag{6}$$

*holds for all* $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$.

*Moreover if* $p = 1$, *then the inequality*

$$\|Mf\|_{WL^1(B(x,r))} \lesssim r^{\frac{n}{p}} \sup_{t>2r} t^{-\frac{n}{p}} \|f\|_{L^1(B(x,t))} \tag{7}$$

*holds for all* $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.

THEOREM 1. *1. If* $f \in L^{1,\lambda,\mu}(\mathbb{R}^n)$, $0 \leqslant \lambda < n$ *and* $0 \leqslant \mu < n$, *then* $Mf \in WL^{1,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{WL^{1,\lambda,\mu}} \leqslant C_{1,\lambda,\mu} \|f\|_{L^{1,\lambda,\mu}}, \tag{8}$$

*where* $C_{1,\lambda,\mu}$ *is independent of* $f$.

*2. If* $f \in L^{p,\lambda,\mu}(\mathbb{R}^n)$, $1 < p < \infty$, $0 \leqslant \lambda < n$ *and* $0 \leqslant \mu < n$, *then* $Mf \in L^{p,\lambda,\mu}(\mathbb{R}^n)$ *and*

$$\|Mf\|_{L^{p,\lambda,\mu}} \leqslant C_{p,\lambda,\mu} \|f\|_{L^{p,\lambda,\mu}}, \tag{9}$$

*where* $C_{p,\lambda,\mu}$ *depends only on* $p, \lambda, \mu$ *and* $n$.

*Proof.* Let $p = 1$. From the inequality (7) we get

$$\begin{aligned}
\|Mf\|_{WL^{1,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda} [1/t]_1^{\mu} \|Mf\|_{WL^1(B(x,t))} \\
&\lesssim \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda} [1/t]_1^{\mu} t^n \sup_{\tau>2t} \tau^{-n} \|f\|_{L^1(B(x,\tau))} \\
&\lesssim \|f\|_{L^{1,\lambda,\mu}} \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{-\lambda} [1/t]_1^{\mu} t^n \sup_{\tau>t} \tau^{-n} [\tau]_1^{\lambda} [1/\tau]_1^{-\mu} \\
&= \|f\|_{L^{1,\lambda,\mu}} \sup_{x\in\mathbb{R}^n, t>0} [t]_1^{n-\lambda} [1/t]_1^{\mu-n} \sup_{\tau>t} [\tau]_1^{\lambda-n} [1/\tau]_1^{n-\mu} \\
&= \|f\|_{L^{1,\lambda,\mu}}
\end{aligned}$$

which implies that the operator $M$ is bounded from $L^{1,\lambda,\mu}(\mathbb{R}^n)$ to $WL^{1,\lambda,\mu}(\mathbb{R}^n)$.

Let $1<p<\infty$. From the inequality (6) we get

$$\begin{aligned}
\|Mf\|_{L^{p,\lambda,\mu}} &= \sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\frac{\lambda}{p}}[1/t]_1^{\frac{\mu}{p}}\|Mf\|_{L^p(B(x,t))}\\
&\lesssim \sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\frac{\lambda}{p}}[1/t]_1^{\frac{\mu}{p}}\, t^{\frac{n}{den}}\sup_{\tau>2t}\tau^{-\frac{n}{p}}\|f\|_{L^p(B(x,\tau))}\\
&\lesssim \|f\|_{L^{p,\lambda,\mu}}\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{-\frac{\lambda}{p}}[1/t]_1^{\frac{\mu}{p}}\, t^{\frac{n}{p}}\sup_{\tau>t}\tau^{-\frac{n}{p}}[\tau]_1^{\frac{\lambda}{p}}[1/\tau]_1^{-\frac{\mu}{p}}\\
&= \|f\|_{L^{p,\lambda,\mu}}\sup_{x\in\mathbb{R}^n,t>0}[t]_1^{\frac{n-\lambda}{p}}[1/t]_1^{\frac{\mu-n}{p}}\sup_{\tau>t}[\tau]_1^{\frac{\lambda-n}{p}}[1/\tau]_1^{\frac{n-\mu}{p}}\\
&= \|f\|_{L^{p,\lambda,\mu}}
\end{aligned}$$

which implies that the operator $M$ is bounded in $L^{p,\lambda,\mu}(\mathbb{R}^n)$. □

From Theorem 1 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

COROLLARY 6. ([8]) *1. If $f\in L^{1,\lambda}(\mathbb{R}^n)$ and $0\leqslant\lambda<n$, then $Mf\in WL^{1,\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{WL^{1,\lambda}}\leqslant C_{1,\lambda}\|f\|_{L^{1,\lambda}},$$

*where $C_{1,\lambda}$ is independent of $f$.*

*2. If $f\in L^{p,\lambda}(\mathbb{R}^n)$, $1<p<\infty$ and $0\leqslant\lambda<n$, then $Mf\in L^{p,\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{L^{p,\lambda}}\leqslant C_{p,\lambda}\|f\|_{L^{p,\lambda}},$$

*where $C_{p,\lambda}$ depends only on $p$, $\lambda$ and $n$.*

COROLLARY 7. ([4]) *1. If $f\in\widetilde{L}^{1,\lambda}(\mathbb{R}^n)$ and $0\leqslant\lambda<n$, then $Mf\in W\widetilde{L}^{1,\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{W\widetilde{L}^{1,\lambda}}\leqslant C_{1,\lambda}\|f\|_{\widetilde{L}^{1,\lambda}},$$

*where $C_{1,\lambda}$ is independent of $f$.*

*2. If $f\in\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$, $1<p<\infty$ and $0\leqslant\lambda<n$, then $Mf\in\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$ and*

$$\|Mf\|_{\widetilde{L}^{p,\lambda}}\leqslant C_{p,\lambda}\|f\|_{\widetilde{L}^{p,\lambda}},$$

*where $C_{p,\lambda}$ depends only on $p$, $\lambda$ and $n$.*

DEFINITION 3. We define the space $BMO(\mathbb{R}^n)$ as the set of all locally integrable functions $f$ with finite norm

$$\|f\|_*=\sup_{x\in\mathbb{R}^n,t>0}|B(x,t)|^{-1}\int_{B(x,t)}|f(y)-f_{B(x,t)}|dy<\infty,$$

where $f_{B(x,t)}=|B(x,t)|^{-1}\int_{B(x,t)}f(y)dy$.

THEOREM 2. ([1, Corollary 1.11]) *If $b \in BMO(\mathbb{R}^n)$, then there exists a positive constant $C$ such that*

$$M_b f(x) \leqslant C\|b\|_* M^2 f(x) \tag{10}$$

*for almost every $x \in \mathbb{R}^n$ and for all $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$.*

THEOREM 3. *Let $1 < p < \infty$, $0 \leqslant \lambda \leqslant n$ and $0 \leqslant \mu \leqslant n$. The following assertions are equivalent:*

$(i)$ $b \in BMO(\mathbb{R}^n)$.

$(ii)$ *The operator $M_b$ is bounded on $L^{p,\lambda,\mu}(\mathbb{R}^n)$.*

*Proof.* $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 1 and 2, we get

$$\begin{aligned}\|M_b f\|_{L^{p,\lambda,\mu}} &\lesssim \|b\|_* \|M^2 f\|_{L^{p,\lambda,\mu}} \\ &\lesssim \|b\|_* \|Mf\|_{L^{p,\lambda,\mu}} \\ &\lesssim \|b\|_* \|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $M_b$ is bounded on $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. We consider $f = \chi_B$. It is easy to compute that

$$\begin{aligned}\|\chi_B\|_{L^{p,\lambda,\mu}} &\approx \sup_{y\in\mathbb{R}^n, t>0} \left([t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{B(y,t)} \chi_B(z)dz\right)^{\frac{1}{p}} \\ &= \sup_{y\in\mathbb{R}^n, t>0} \left(|B(y,t)\cap B|\ [t]_1^{-\lambda}\ [1/t]_1^{\mu}\right)^{\frac{1}{p}} \\ &= \sup_{B(y,t)\subseteq B} \left(|B(y,t)|\ [t]_1^{-\lambda}\ [1/t]_1^{\mu}\right)^{\frac{1}{p}} \\ &= r^{\frac{n}{p}}\ [r]_1^{-\frac{\lambda}{p}}\ [1/r]_1^{\frac{\mu}{p}}.\end{aligned} \tag{11}$$

On the other hand, since

$$M_b(\chi_B)(x) \gtrsim \frac{1}{|B|}\int_B |b(z)-b_B|dz \quad \text{for all} \quad x \in B,$$

we have

$$\begin{aligned}\|M_b(\chi_B)\|_{L^{p,\lambda,\mu}} &\approx \sup_{B(y,t)} \left(\ [t]_1^{-\lambda}\ [1/t]_1^{\mu} \int_{B(y,t)} |M_b(\chi_B)(z)|^p dz\right)^{\frac{1}{p}} \\ &\gtrsim r^{\frac{n}{p}}\ [r]_1^{-\frac{\lambda}{p}}\ [1/r]_1^{\frac{\mu}{p}} \frac{1}{|B|}\int_B |b(z)-b_B|dz.\end{aligned} \tag{12}$$

Since by assumption

$$\|M_b(\chi_B)\|_{L^{p,\lambda,\mu}} \lesssim \|\chi_B\|_{L^{p,\lambda,\mu}},$$

by (11) and (12), we get that

$$\frac{1}{|B|}\int_B |b(z)-b_B|dz \lesssim 1. \quad \square$$

From Theorem 3 in the case $\lambda=\mu$ or $\mu=0$ we get the following corollaries.

COROLLARY 8. ([1]) *Let $1<p<\infty$ and $0\leqslant\lambda\leqslant n$. The following assertions are equivalent:*
$(i)$ $b\in BMO(\mathbb{R}^n)$.
$(ii)$ *The operator $M_b$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

COROLLARY 9. ([4]) *Let $1<p<\infty$ and $0\leqslant\lambda\leqslant n$. The following assertions are equivalent:*
$(i)$ $b\in BMO(\mathbb{R}^n)$.
$(ii)$ *The operator $M_b$ is bounded on $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.*

## 4. $L^{p,\lambda,\mu}$-boundedness of the commutator of maximal operator $[b,M]$

In this section we find necessary and sufficient conditions for the boundedness of the commutator of maximal operator $[b,M]$ on $L^{p,\lambda,\mu}(\mathbb{R}^n)$ spaces.

For a function $b$ defined on $\mathbb{R}^n$, we denote

$$b^-(x) := \begin{cases} 0\,, & \text{if } b(x)\geqslant 0 \\ |b(x)|, & \text{if } b(x)<0 \end{cases}$$

and $b^+(x):=|b(x)|-b^-(x)$. Obviously, $b^+(x)-b^-(x)=b(x)$.

The following relations between $[b,M]$ and $M_b$ are valid:

Let $b$ be any non-negative locally integrable function. Then for all $f\in L^1_{\rm loc}(\mathbb{R}^n)$ and $x\in\mathbb{R}^n$ the following inequality is valid

$$\begin{aligned}\big|[b,M]f(x)\big| &= \big|b(x)Mf(x)-M(bf)(x)\big| \\ &= \big|M(b(x)f)(x)-M(bf)(x)\big| \\ &\leqslant M(|b(x)-b|f)(x)=M_bf(x).\end{aligned}$$

If $b$ is any locally integrable function on $\mathbb{R}^n$, then

$$|[b,M]f(x)|\leqslant M_bf(x)+2b^-(x)Mf(x), \qquad x\in\mathbb{R}^n \tag{13}$$

holds for all $f\in L^1_{\rm loc}(\mathbb{R}^n)$ (see, for example [17, 30]).

Obviously, the $M_b$ and $[b,M]$ operators are essentially different from each other because $M_b$ is positive and sublinear and $[b,M]$ is neither positive nor sublinear.

Applying Theorem 3, we obtain the following result.

THEOREM 4. *Let $1 < p < \infty$, $0 \leqslant \lambda \leqslant n$ and $0 \leqslant \mu \leqslant n$. Suppose that $b$ is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*
(i) $b \in BMO(\mathbb{R}^n)$ *such that* $b^- \in L^\infty(\mathbb{R}^n)$.
(ii) *The operator* $[b,M]$ *is bounded on* $L^{p,\lambda,\mu}(\mathbb{R}^n)$.

*Proof.* $(i) \Rightarrow (ii)$. Suppose that $b \in BMO(\mathbb{R}^n)$. Combining Theorems 1 and 3, and inequality (13), we get

$$\begin{aligned}\|[b,M]f\|_{L^{p,\lambda,\mu}} &\leqslant \|M_b f + 2b^- Mf\|_{L^{p,\lambda,\mu}} \\ &\leqslant \|M_b f\|_{L^{p,\lambda,\mu}} + \|b^-\|_{L^\infty} \|Mf\|_{L^{p,\lambda,\mu}} \\ &\lesssim \left(\|b\|_* + \|b^-\|_{L^\infty}\right) \|f\|_{L^{p,\lambda,\mu}}.\end{aligned}$$

$(ii) \Rightarrow (i)$. Assume that $[b,M]$ is bounded on $L^{p,\lambda,\mu}(\mathbb{R}^n)$. Let $B = B(x,r)$ be a fixed ball. Denote by $M_b f$ the local maximal function of $f$:

$$M_b f(x) := \sup_{B' \ni x:\, B' \subset B} \frac{1}{|B'|} \int_{B'} |f(y)|\, dy, \; x \in \mathbb{R}^n.$$

Since

$$M(b\chi_B)\chi_B = M_b(b) \quad \text{and} \quad M(\chi_B)\chi_B = \chi_B,$$

we have

$$\begin{aligned}|M_b(b) - b\chi_B| &= |M(b\chi_B)\chi_B - bM(\chi_B)\chi_B| \\ &\leqslant |M(b\chi_B) - bM(\chi_B)| \\ &= |[b,M]\chi_B|.\end{aligned}$$

Hence

$$\|M_b(b) - b\chi_B\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} \leqslant \|[b,M]\chi_B\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)}.$$

Thus from (11) we get

$$\begin{aligned}\frac{1}{|B|}\int_B |b - M_b(b)| &\leqslant \left(\frac{1}{|B|}\int_B |b - M_b(b)|^p\right)^{\frac{1}{p}} \\ &\leqslant |B|^{-\frac{1}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|b\chi_B - M_b(b)\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{n}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|[b,M]\chi_B\|_{L^{p,\lambda,\mu}(\mathbb{R}^n)} \\ &\lesssim r^{-\frac{n}{p}} [r]_1^{\frac{\lambda}{p}} [1/r]_1^{-\frac{\mu}{p}} \|\chi_B\|_{L^{p,\lambda,\mu}} \approx 1.\end{aligned}$$

Denote by

$$E := \{x \in B : b(x) \leqslant b_B\}, \quad F := \{x \in B : b(x) > b_B\}.$$

Since

$$\int_E |b(t) - b_B|\, dt = \int_F |b(t) - b_B|\, dt,$$

in view of the inequality $b(x) \leqslant b_B \leqslant M_b(b)$, $x \in E$, we get

$$\frac{1}{|B|}\int_B |b-b_B| = \frac{2}{|B|}\int_E |b-b_B| \leqslant \frac{2}{|B|}\int_E |b-M_b(b)| \leqslant \frac{2}{|B|}\int_B |b-M_b(b)| \lesssim c.$$

Consequently, $b \in BMO(\mathbb{R}^n)$.

In order to show that $b^- \in L^\infty(\mathbb{R}^n)$, note that $M_b(b) \geqslant |b|$. Hence

$$0 \leqslant b^- = |b| - b^+ \leqslant M_b(b) - b^+ + b^- = M_b(b) - b.$$

Thus

$$(b^-)_B \leqslant c,$$

and by the Lebesgue Differentiation theorem we get that

$$b^-(x) \leqslant c \quad \text{for a.e. } x \in \mathbb{R}^n. \quad \square$$

From Theorem 4 in the case $\lambda = \mu$ or $\mu = 0$ we get the following corollaries.

COROLLARY 10. ([1]) *Let $1 < p < \infty$ and $0 \leqslant \lambda \leqslant n$. Suppose that $b$ is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

*(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L^\infty(\mathbb{R}^n)$.*

*(ii) The operator $[b,M]$ is bounded on $L^{p,\lambda}(\mathbb{R}^n)$.*

COROLLARY 11. ([4]) *Let $1 < p < \infty$ and $0 \leqslant \lambda \leqslant n$. Suppose that $b$ is a real valued locally integrable function in $\mathbb{R}^n$. Then the following assertions are equivalent:*

*(i) $b \in BMO(\mathbb{R}^n)$ such that $b^- \in L^\infty(\mathbb{R}^n)$.*

*(ii) The operator $[b,M]$ is bounded on $\widetilde{L}^{p,\lambda}(\mathbb{R}^n)$.*

*Acknowledgements.* The author thanks the referee(s) for careful reading the paper and useful comments. The research of V. Guliyev was supported by the grant of Cooperation Program 2532 TUBITAK – RFBR (RUSSIAN foundation for basic research) (Agreement number no. 119N455) and by the RUDN University Strategic Academic Leadership Program.

## REFERENCES


[1] M. AGCAYAZI, A. GOGATISHVILI, K. KOCA, R. MUSTAFAYEV, *A note on maximal commutators and commutators of maximal functions*, J. Math. Soc. Japan, **67** (2015), no. 2, 581–593.

[2] A. ALMEIDA, J. J. HASANOV, S. G. SAMKO, *Maximal and potential operators in variable exponent Morrey spaces*, Georgian Math. J., **15** (2008), no. 2, 195–208.

[3] A. AKBULUT, V. S. GULIYEV, R. CH. MUSTAFAYEV, *On the boundedness of the maximal operator and singular integral operators in generalized Morrey spaces*, Math. Bohem., **137** (2012), no. 1, 27–43.

[4] C. AYKOL, H. ARMUTCU, M. N. OMAROVA, *Maximal commutator and commutator of maximal function on modified Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **36** (2016), no. 1, Mathematics, 29–35.

[5] J. BASTERO, M. MILMAN, F. J. RUIZ, *Commutators for the maximal and sharp functions*, Proc. Amer. Math. Soc., **128** (2000), no. 11, 3329–3334 (electronic).

[6] A. BONAMI, T. IWANIEC, P. JONES, M. ZINSMEISTER, *On the product of functions in BMO and $H_1$*, Ann. Inst. Fourier Grenoble, **57** (2007), no. 5, 1405–1439.

[7] R. R. COIFMAN, R. ROCHBERG, G. WEISS, *Factorization theorems for Hardy spaces in several variables*, Ann. of Math., **103** (1976), no. 3, 611–635.

[8] F. CHIARENZA, M. FRASCA, *Morrey spaces and Hardy–Littlewood maximal function*, Rend. Math., **7** (1987), 273–279.

[9] J. GARCIA-CUERVA, E. HARBOURE, C. SEGOVIA, J. L. TORREA, *Weighted norm inequalities for commutators of strongly singular integrals*, Indiana Univ. Math. J., **40** (1991), 1397–1420.

[10] L. GRAFAKOS, *Modern Fourier analysis*, 2nd ed., Graduate Texts in Mathematics, vol. 250, Springer, New York, 2009.

[11] V. S. GULIYEV, *Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$*, (in Russian), Doctor of Sciences, Moscow, Mat. Inst. Steklova (1994) 1–329.

[12] V. S. GULIYEV, *Function spaces, Integral Operators and Two Weighted Inequalities on Homogeneous Groups*, Some Applications, (Russian), ELM, Baku, 1996.

[13] V. S. GULIYEV, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl., 2009, Art. ID 503948, 20 pp.

[14] V. S. GULIYEV, J. J. HASANOV, S. G. SAMKO, *Boundedness of maximal, potential type and singular integral operators in the generalized variable exponent Morrey type spaces*, J. Math. Sci. (N.Y.), **170** (2010), 423–443.

[15] V. S. GULIYEV, J. J. HASANOV, S. G. SAMKO, *Boundedness of the maximal, potential and singular operators in the generalized variable exponent Morrey spaces*, Math. Scand., **107** (2010), 285–304.

[16] V. S. GULIYEV, J. J. HASANOV, Y. ZEREN, *Necessary and sufficient conditions for the boundedness of the Riesz potential in modified Morrey spaces*, J. Math. Inequal., **5** (2011), no. 4, 491–506.

[17] V. S. GULIYEV, *Some characterizations of BMO spaces via commutators in Orlicz spaces on stratified Lie groups*, Results Math., **77** (2022), no. 1, Paper No. 42, 18 pp.

[18] G. HU, D. YANG, *Maximal commutators of BMO functions and singular integral operators with non-smooth kernels on spaces of homogeneous type*, J. Math. Anal. Appl., **354** (2009), 249–262.

[19] S. JANSON, *Mean oscillation and commutators of singular integral operators*, Ark. Mat., **16** (1978) 263–270.

[20] G. MINGIONE, *Gradient estimates below the duality exponent*, Math. Ann., **346** (2010), no. 3, 571–627.

[21] C. B. MORREY, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43** (1938), 126–166.

[22] E. NAKAI, *Orlicz-Morrey spaces and Hardy-Littlewood maximal function*, Studia Math., **188** (2008), 193–221.

[23] P. A. OLSEN, *Fractional integration, Morrey spaces and a Schrödinger equation*, Comm. Partial Differential Equations, **20** (1995), 2005–2055.

[24] M. A. RAGUSA, *Regularity for weak solutions to the Dirichlet problem in Morrey space*, Rev. Mat. Univ. Parma, **5** (1994), no. 3, 355–369.

[25] Y. SAWANO, G. DI FAZIO, D. I. HAKIM, *Morrey Spaces-Introduction and Applications to Integral Operators and PDE's*, Vol. I, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 479 pp. (2020).

[26] Y. SAWANO, G. DI FAZIO, D. I. HAKIM, *Morrey Spaces-Introduction and Applications to Integral Operators and PDE's*, Vol. II, Monographs and Research Notes in Mathematics, CRC Press, Boca Raton, FL, 409 pp. (2020).

[27] C. SEGOVIA, J. L. TORREA, *Weighted inequalities for commutators of fractional and singular integrals*, Publ. Mat., **35** (1991) 209–235.

[28] E. M. STEIN, *Harmonic analysis: Real Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, New Jersey (1993).

[29] J. ZHANG, S. ZHENG, *Weighted Lorentz and Lorentz-Morrey estimates to viscosity solutions of fully nonlinear elliptic equations*, Complex Var. Elliptic Equ., **63** (2017), no. 9, 1271–1289.

[30] P. ZHANG, J. WU, J. SUN, *Commutators of some maximal functions with Lipschitz function on Orlicz spaces*, Mediterr. J. Math., **15** (6) (2018) Paper No. 216, 13 pp.

(Received June 15, 2022)

*Vagif S. Guliyev*
*Institute of Applied Mathematics*
*Baku State University*
*Baku, Azerbaijan*
*and*
*Institute of Mathematics and Mechanics of NAS of Azerbaijan*
*Baku, Azerbaijan*
*and*
*Peoples Friendship University of Russia (RUDN University)*
*6 Miklukho-Maklaya St, Moscow, 117198, Russian Federation*
*e-mail:* vagif@guliyev.com