FRACTIONAL QUANTUM ANALOGUES
OF TRAPEZOID LIKE INEQUALITIES

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Abstract. We derive two new fractional quantum integral identities. Using these identities we obtain several new fractional quantum estimates of trapezoid like inequalities essentially using the class of preinvex functions.

1. Introduction and preliminaries

The theory of convexity has played a vital role in the development of the theory of inequalities. Many results in the theory of inequalities are direct consequences of the applications of convexity, for details, see [12, 13, 14]. This fact also leads us to a full-fledged area of study which is called inequalities involving convex functions. In this regard, Hermite-Hadamard's inequality (also known as trapezium-like inequality) which provides us with a necessary and sufficient condition for a function to be convex is one of the most studied results. It reads as:

Let $\Lambda : I \subset \mathbb{R} \to \mathbb{R}$ be a convex function, then

$$\Lambda\left(\frac{a+b}{2}\right) \leqslant \frac{1}{b-a}\int\limits_a^b \Lambda(x)\mathrm{d}x \leqslant \frac{\Lambda(a)+\Lambda(b)}{2}.$$

In recent years, several authors have obtained new analogues of the classical Hermite-Hadamard's inequality. Sarikaya et al. [15] used the concepts of fractional calculus and obtained the fractional analogues of Hermite-Hadamard's inequality. Using the concepts of quantum calculus introduced by Tariboon and Ntouyas [18] Alp et al. [2] obtained quantum analogue of Hermite-Hadamard's inequality.

Since Hermite-Hadamard's inequality can be obtained using the convexity property of the functions then it is a natural problem for research to check the refinements and variants of Hermite-Hadamard's inequality using generalizations of convexity. Noor [9] obtained Hermite-Hadamard-Noor type of inequality using the class

of preinvex functions which is a significant generalization of classical convexity. Iscan [5] obtained the fractional version of Hermite-Hadamard-Noor type of inequality and Noor et al. [11] obtained quantum analogue of similar inequality. For some recent investigations on quantum analogues of classical inequalities, see [1, 4, 8, 10, 16, 22].

Recently Kunt and Aljasem [7] obtained fractional quantum analogue of Hermite-Hadamard's inequality. They have also obtained some fractional quantum estimates of trapezoid like inequalities. This motivated us to check the fractional quantum analogue of trapezoid type inequalities using the preinvexity property of the functions. We hope that the ideas and techniques of the paper will inspire interested readers working in this field.

We now discuss some previously known concepts and results. First we recall the concept of q-derivative which was introduced and studied by [18, 20].

DEFINITION 1. ([18, 20]) For a continuous function $\Lambda : [a,b] \to \mathbb{R}$ the q-derivative of Λ at $x \in [a,b]$ is defined as:

$$_a D_q \Lambda(x) = \frac{\Lambda(x) - \Lambda(qx + (1-q)a)}{(1-q)(x-a)}, \ x \neq a. \tag{1.1}$$

q-definite integral is defined as:

DEFINITION 2. ([18, 20]) Let $\Lambda : [a,b] \to \mathbb{R}$ be a continuous function. Then the q-definite integral on $[a,b]$ is defined as:

$$\int_a^x \Lambda(t) \, _a d_q t = (1-q)(x-a) \sum_{n=0}^{\infty} q^n \Lambda(q^n x + (1-q^n)a), \tag{1.2}$$

for $x \in [a,b]$.

Interesting and more details of the following concepts can be found in [17, 19].

$$[m]_q = \frac{1-q^m}{1-q}, \ m \in \mathbb{R}. \tag{1.3}$$

The q-analogue of power function is defined as, if $\gamma \in \mathbb{R}$, then

$$(r-m)^{(\gamma)} = r^\gamma \prod_{n=0}^{\infty} \frac{r - q^n m}{r - q^{\gamma+n} m}, \ r \neq 0. \tag{1.4}$$

The q-gamma function is defined as:

$$\Gamma_q(t) = \frac{(1-q)^{(t-1)}}{(1-q)^{t-1}}, \ t \in \mathbb{R}/\{0, -1, -2, \dots\}. \tag{1.5}$$

For any $s,t > 0$, the q-beta function is defined as:

$$B_q(s,t) = \int_0^1 u^{(s-1)}(1 - qu)^{(t-1)} d_q u, \tag{1.6}$$

and

$$B_q(s,t) = \frac{\Gamma_q(s)\Gamma_q(t)}{\Gamma_q(s+t)}.$$

The q-pochhammer symbol is defined as:

$$(m;q)_0 = 1, \quad (m;q)_k = \prod_{n=0}^{k-1}(1 - q^n m), \tag{1.7}$$

here $k \in \mathbb{N} \cup \{\infty\}$.

THEOREM 1.1. ([3]) *Suppose* $\lambda, \mu \in \mathbb{R}$, *then*

$$\lim_{q \to 1^-} \frac{(q^\lambda x;q)}{(q^\mu x;q)} = (1-x)^{\mu-\lambda}, \tag{1.8}$$

uniformly on $\{x \in \mathbb{C} : |x| \leqslant 1\}$, *if* $\mu \geqslant \lambda$, $\lambda + \mu \geqslant 1$, *and uniformly on compact subset of* $\{x \in \mathbb{C} : |x| \leqslant 1, x \neq 1\}$ *for other choices of* μ *and* λ.

q-shifting operator as:

$$_a\Phi_q(m) = qm + (1-q)a. \tag{1.9}$$

For any positive integer k, one has:

$$_a\Phi_q^k(m) = {}_a\Phi_q^{k-1}({}_a\Phi_q(m)), \quad _a\Phi_q^0(m) = m. \tag{1.10}$$

The following properties for q-shifting operator are as follows:

THEOREM 1.2. ([17, 19]) *For any* $r, m \in \mathbb{R}$ *and for all positive integer* k, j, *one has:*

1. $_a\Phi_q^k(m) = {}_a\Phi_{q^k}(m)$;

2. $_a\Phi_q^k({}_a\Phi_q^j(m)) = {}_a\Phi_q^j({}_a\Phi_q^k(m)) = {}_a\Phi_q^{j+k}(m)$;

3. $_a\Phi_q(a) = a$;

4. $_a\Phi_q^k(m) - a = q^k(m-a)$;

5. $m - {}_a\Phi_q^k(m) = (1-q^k)(m-a)$;

6. $_a\Phi_q^k(m) = m_{\frac{a}{m}}\Phi_q^k(1), \ \text{for } m \neq 0$;

7. $_a\Phi_q(m) - {}_a\Phi_q^k(r) = q(m - {}_a\Phi_q^{k-1}(r))$.

The power of q-shifting operator is defined as, if $\gamma \in \mathbb{R}$, then

$$_a(r-m)_q^{(\gamma)} = (r-a)^\gamma \prod_{n=0}^{\infty} \frac{r - {}_a\Phi_q^n(m)}{r - {}_a\Phi_q^{\gamma+n}(m)}. \tag{1.11}$$

THEOREM 1.3. ([17, 19]) *For any* $\gamma, r, m \in \mathbb{R}$, $r \neq a$ *and* $k \in \mathbb{N}$, *one has:*

1. $_a(r-m)_q^{(k)} = (r-a)^k \left(\frac{m-a}{r-a};q\right)_k$;

2. $_a(r-m)_q^{(\gamma)} = (r-a)^\gamma \prod_{n=0}^{\infty} \frac{1 - \frac{m-a}{r-a}q^n}{1 - \frac{m-a}{r-a}q^{n+\gamma}} = (r-a)^\gamma \frac{\left(\frac{m-a}{r-a};q\right)_\infty}{\left(\frac{m-a}{r-a}q^\gamma;q\right)_\infty}$;

3. $_a(r - {_a}\Phi_q^k(r))_q^{(\gamma)} = (r-a)^\gamma \dfrac{(q^k;q)_\infty}{(q^{\gamma+k};q)_\infty}.$

DEFINITION 3. ([17, 19]) Let $\alpha \geqslant 0$ and Λ be a continuous function on $[a,b]$. Then the Riemann-Liouville type fractional quantum integral is given by $(_aJ_q^0\Lambda)(t) = \Lambda(t)$ and

$$(_aJ_q^\alpha\Lambda)(x) = (_aJ_q^\alpha\Lambda(t))(x) \tag{1.12}$$

$$= \frac{1}{\Gamma_q(\alpha)} \int_a^x {_a}(x - {_a}\Phi_q(t))_q^{(\alpha-1)}\Lambda(t)\,_ad_qt$$

$$= \frac{(1-q)(x-a)}{\Gamma_q(\alpha)} \sum_{n=0}^\infty q^n\,_a(x - {_a}\Phi_q^{n+1}(x))_q^{(\alpha-1)}\Lambda(_a\Phi_q^n(x)), \tag{1.13}$$

where $\alpha > 0$ and $x \in [a,b]$.

2. Main results

In this section, we will discuss our main results.

2.1. Key Lemmas

LEMMA 2.1. Let $\Lambda : [a, a+\zeta(b,a)] \to \mathbb{R}$ be a continuous function and $\alpha > 0$. If $_aD_q\Lambda$ is q-integrable on $(a, a+\zeta(b,a))$, then

$$\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) - \frac{([\alpha+1]_q - 1)\Lambda(a) + \Lambda(a+\zeta(b,a))}{[\alpha+1]_q}$$

$$= \frac{\zeta(b,a)}{[\alpha+1]_q} \int_0^1 \left([\alpha+1]_q\,_0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1\right) {_aD_q}\Lambda(a+t\zeta(b,a))\,_0d_qt. \tag{2.14}$$

Proof. Let

$$\frac{\zeta(b,a)}{[\alpha+1]_q} \int_0^1 \left([\alpha+1]_q\,_0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1\right) {_aD_q}\Lambda(a+t\zeta(b,a))\,_0d_qt$$

$$= \zeta(b,a) \int_0^1 {_0}(1 - {_0}\Phi_q(t))_q^{(\alpha)}\,_aD_q\Lambda(a+t\zeta(b,a))\,_0d_qt$$

$$- \frac{\zeta(b,a)}{[\alpha+1]_q} \int_0^1 {_aD_q}\Lambda(a+t\zeta(b,a))\,_0d_qt$$

$$= \mathscr{S}_1 - \mathscr{S}_2. \tag{2.15}$$

Now

$$\mathscr{S}_1 = \zeta(b,a) \int_0^1 {}_0(1 - {}_0\Phi_q(t))_q^{(\alpha)} {}_aD_q\Lambda(a + t\zeta(b,a))_0 d_q t$$

$$= \zeta(b,a) \int_0^1 {}_0(1 - {}_0\Phi_q(t))_q^{(\alpha)} \frac{\Lambda(a + t\zeta(b,a)) - \Lambda(a + qt\zeta(b,a))}{(1-q)\zeta(b,a)t} {}_0 d_q t$$

$$= \frac{1}{1-q} \int_0^1 {}_0(1 - {}_0\Phi_q(t))_q^{(\alpha)} \frac{\Lambda(a + t\zeta(b,a))}{t} {}_0 d_q t$$

$$- \frac{1}{1-q} \int_0^1 {}_0(1 - {}_0\Phi_q(t))_q^{(\alpha)} \frac{\Lambda(a + qt\zeta(b,a))}{t} {}_0 d_q t$$

$$= \sum_{n=0}^{\infty} q^n {}_0(1 - {}_0\Phi_q^{n+1}(1))_q^{(\alpha)} \frac{\Lambda\left(a + {}_0\Phi_q^n(1)\zeta(b,a)\right)}{{}_0\Phi_q^n(1)}$$

$$- \sum_{n=0}^{\infty} q^n {}_0(1 - {}_0\Phi_q^{n+1}(1))_q^{(\alpha)} \frac{\Lambda\left(a + q {}_0\Phi_q^n(1)\zeta(b,a)\right)}{{}_0\Phi_q^n(1)}$$

$$= \begin{bmatrix} \sum\limits_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n+1};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ - \sum\limits_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n+1};q)_\infty} \Lambda\left(a + q^{n+1} \zeta(b,a)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{n=0}^{\infty} (1 - q^{\alpha+n}) \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ - \sum\limits_{n=0}^{\infty} (1 - q^{n+1}) \frac{(q^{n+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty} \Lambda\left(a + q^{n+1} \zeta(b,a)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{n=0}^{\infty} \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ - \sum\limits_{n=0}^{\infty} \frac{(q^{n+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty} \Lambda\left(a + q^{n+1} \zeta(b,a)\right) \end{bmatrix}$$

$$= \begin{bmatrix} \sum\limits_{n=0}^{\infty} q^{\alpha+n} \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ - \sum\limits_{n=0}^{\infty} q^{n+2} \frac{(q^{n+2};q)_\infty}{(q^{\alpha+n+1};q)_\infty} \Lambda\left(a + q^{n+1} \zeta(b,a)\right) \end{bmatrix}$$

$$= \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty} \Lambda(a + \zeta(b,a)) - \Lambda(a) - \begin{bmatrix} \sum\limits_{n=0}^{\infty} q^{\alpha+n} \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ - \sum\limits_{n=1}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \end{bmatrix}$$

$$= \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty} \Lambda(a + \zeta(b,a)) - \Lambda(a) - \begin{bmatrix} \sum\limits_{n=0}^{\infty} q^{\alpha+n} \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ - \sum\limits_{n=0}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda\left(a + q^n \zeta(b,a)\right) \\ + \frac{(q^1;q)_\infty}{(q^\alpha;q)_\infty} \Lambda(a + \zeta(b,a)) \end{bmatrix}$$

$$= -\Lambda(a) + (1 - q^n) \sum_{n=0}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda(a + q^n \zeta(b,a))$$

$$= -\Lambda(a) + [\alpha]_q (1 - q) \sum_{n=0}^{\infty} q^n \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda(a + q^n \zeta(b,a))$$

$$= -\Lambda(a) + \frac{[\alpha]_q \Gamma_q(\alpha)}{\zeta^\alpha(b,a)} \left(\frac{(1-q)\zeta(b,a)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n \zeta^{\alpha-1}(b,a) \right.$$

$$\left. \times \frac{(q^{n+1};q)_\infty}{(q^{\alpha+n};q)_\infty} \Lambda(a + q^n \zeta(b,a)) \right)$$

$$= -\Lambda(a) + \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} \left(\frac{(1-q)\zeta(b,a)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n \zeta^{\alpha-1}(b,a) \right.$$

$$\left. \times \frac{(q^{n+1};q)_\infty}{(q^{(\alpha+1)+(n+1)};q)_\infty} \Lambda(a + q^n \zeta(b,a)) \right)$$

$$= -\Lambda(a) + \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} \left(\frac{(1-q)\zeta(b,a)}{\Gamma_q(\alpha)} \sum_{n=0}^{\infty} q^n {}_a(a + \zeta(b,a)) \right.$$

$$\left. - {}_a\Phi_q^{n+1}(a + \zeta(b,a))_q^{(\alpha-1)} \Lambda\left({}_a\Phi_q^n(a + \zeta(b,a)) \right) \right)$$

$$= -\Lambda(a) + \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} \left(\frac{1}{\Gamma_q(\alpha)} \int_{a}^{a+\zeta(b,a)} {}_a(a + \zeta(b,a) - {}_a\Phi_q(t))_q^{(\alpha-1)} \Lambda(t) \, {}_a d_q t \right)$$

$$= -\Lambda(a) + \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} ({}_aJ_q^\alpha \Lambda)(a + \zeta(b,a)). \tag{2.16}$$

Similarly

$$\mathscr{S}_2 = \frac{\zeta(b,a)}{[\alpha+1]_q} \int_0^1 {}_aD_q\Lambda(a + t\zeta(b,a)) \, {}_0d_q t$$

$$= \frac{\zeta(b,a)}{[\alpha+1]_q} \int_0^1 \frac{\Lambda(a + t\zeta(b,a)) - \Lambda(a + qt\zeta(b,a))}{(1-q)\zeta(b,a)t} \, {}_0d_q t$$

$$= \frac{1}{(1-q)[\alpha+1]_q} \int_0^1 \frac{\Lambda(a + t\zeta(b,a))}{t} \, {}_0d_q t - \frac{1}{(1-q)[\alpha+1]_q} \int_0^1 \frac{\Lambda(a + qt\zeta(b,a))}{t} \, {}_0d_q t$$

$$= \frac{1}{[\alpha+1]_q} \left[\sum_{n=0}^{\infty} \Lambda(a + q^n \zeta(b,a)) - \sum_{n=0}^{\infty} \Lambda(a + q^{n+1} \zeta(b,a)) \right]$$

$$= \frac{\Lambda(a + \zeta(b,a)) - \Lambda(a)}{[\alpha+1]_q}. \tag{2.17}$$

Using (2.16) and (2.17) in (2.15) completes the proof. $\quad\square$

LEMMA 2.2. *Let* $\Lambda : [a, a + \zeta(b,a)] \to \mathbb{R}$ *be a continuous function and* $\alpha > 0$. *If* $_aD_q\Lambda$ *is* q*-integrable on* $(a, a + \zeta(b,a))$, *then*

$$
\Lambda\left(\frac{([\alpha+1]_q - 1)a + (a + \zeta(b,a))}{[\alpha+1]_q}\right) - \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a + \zeta(b,a))
$$

$$
= \zeta(b,a)\left[\begin{array}{l} \displaystyle\int_0^{\frac{1}{[\alpha+1]_q}}\left(1 - {_0(1 - \Phi_q(t))_q^{(\alpha)}}\right) {_aD_q\Lambda(a + t\zeta(b,a))_0d_qt} \\[4mm] + \displaystyle\int_{\frac{1}{[\alpha+1]_q}}^1 - {_0(1 - \Phi_q(t))_q^{(\alpha)}} {_aD_q\Lambda(a + t\zeta(b,a))_0d_qt} \end{array}\right]. \tag{2.18}
$$

Proof. Let

$$
S_3 = \int_0^{\frac{1}{[\alpha+1]_q}} {_aD_q\Lambda(a + t\zeta(b,a))_0d_qt}
$$

$$
= \int_0^{\frac{1}{[\alpha+1]_q}} \frac{\Lambda(a + t\zeta(b,a)) - \Lambda(a + qt\zeta(b,a))}{(1-q)\zeta(b,a)t} {_0d_qt}
$$

$$
= \frac{1}{(1-q)\zeta(b,a)}\int_0^{\frac{1}{[\alpha+1]_q}} \frac{\Lambda(a + t\zeta(b,a))}{t}{_0d_qt} - \frac{1}{(1-q)\zeta(b,a)}\int_0^{\frac{1}{[\alpha+1]_q}} \frac{\Lambda(a + qt\zeta(b,a))}{t}{_0d_qt}
$$

$$
= \frac{1}{\zeta(b,a)[\alpha+1]_q}\sum_{n=0}^\infty q^n \frac{\Lambda\left(a + \frac{q^n}{[\alpha+1]_q}\zeta(b,a)\right)}{\frac{q^n}{[\alpha+1]_q}}
$$

$$
- \frac{1}{\zeta(b,a)[\alpha+1]_q}\sum_{n=0}^\infty q^n \frac{\Lambda\left(a + \frac{q^{n+1}}{[\alpha+1]_q}\zeta(b,a)\right)}{\frac{q^n}{[\alpha+1]_q}}
$$

$$
= \frac{1}{\zeta(b,a)}\left[\sum_{n=0}^\infty \Lambda\left(a + \frac{q^n}{[\alpha+1]_q}\zeta(b,a)\right) - \sum_{n=0}^\infty \Lambda\left(a + \frac{q^{n+1}}{[\alpha+1]_q}\zeta(b,a)\right)\right]
$$

$$
= \frac{1}{\zeta(b,a)}\left[\Lambda\left(\frac{([\alpha+1]_q - 1)a + (a + \zeta(b,a))}{[\alpha+1]_q}\right) - \Lambda(a)\right]. \tag{2.19}
$$

Similarly

$$
\mathscr{S}_1 = \zeta(b,a)\int_0^1 {_0(1 - {_0\Phi_q(t))_q^{(\alpha)}} {_aD_q\Lambda(a + t\zeta(b,a))_0d_qt}}
$$

$$
= -\Lambda(a) + \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a + \zeta(b,a)). \tag{2.20}
$$

Using (2.19) and (2.20) in the following integral, we have

$$\zeta(b,a)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}}\left(1-{}_0(1-\Phi_q(t))_q^{(\alpha)}\right){}_aD_q\Lambda(a+t\zeta(b,a))\,{}_0d_qt\\+\int_{\frac{1}{[\alpha+1]_q}}^1 -{}_0(1-\Phi_q(t))_q^{(\alpha)}{}_aD_q\Lambda(a+t\zeta(b,a))\,{}_0d_qt\end{array}\right]$$

$$=\zeta(b,a)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}}{}_aD_q\Lambda(a+t\zeta(b,a))\,{}_0d_qt\\-\int_0^1 {}_0(1-\Phi_q(t))_q^{(\alpha)}{}_aD_q\Lambda(a+t\zeta(b,a))\,{}_0d_qt\end{array}\right]$$

$$=\zeta(b,a)\left[\begin{array}{l}\frac{1}{\zeta(b,a)}\left[\Lambda\left(\frac{([\alpha+1]_q-1)a+(a+\zeta(b,a))}{[\alpha+1]_q}\right)-\Lambda(a)\right]\\-\frac{1}{\zeta(b,a)}\left[-\Lambda(a)+\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}({}_aJ_q^\alpha\Lambda)(a+\zeta(b,a))\right]\end{array}\right]$$

$$=\Lambda\left(\frac{([\alpha+1]_q-1)a+(a+\zeta(b,a))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}({}_aJ_q^\alpha\Lambda)(a+\zeta(b,a)).$$

This completes the proof. $\square$

2.2. Fractional quantum estimates

Before we move to our next section of the paper, let us recall the definitions of invex set and preinvex function.

DEFINITION 4. ([6]) A non-empty set $\mathscr{K}\subseteq\mathbb{R}$ is said to be invex with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$ if

$$a+\mu\zeta(b,a)\in\mathscr{K}$$

for all $a,b\in\mathscr{K}$ and $\mu\in[0,1]$.

DEFINITION 5. ([21]) Let $\mathscr{K}\subseteq\mathbb{R}$ be an invex set with respect to the bivariate function $\zeta:\mathbb{R}\times\mathbb{R}\to\mathbb{R}$. Then the real-valued function $\Lambda:\mathscr{K}\to\mathbb{R}$ is said to be preinvex with respect to ζ if

$$\Lambda(a+\mu\zeta(b,a))\leqslant(1-t)\Lambda(a)+t\Lambda(b)$$

for all $a,b\in\mathscr{K}$ and $t\in[0,1]$.

THEOREM 2.3. *Let* $\Lambda:[a,a+\zeta(b,a)]\to\mathbb{R}$ *be a continuous function and* $\alpha>0$ *and* ${}_aD_q\Lambda$ *is* q*-integrable on* $(a,a+\zeta(b,a))$. *If* $|{}_aD_q\Lambda|$ *is preinvex on* $[a,a+\zeta(b,a)]$, *then*

$$\left|\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}({}_aJ_q^\alpha\Lambda)(a+\zeta(b,a))-\frac{([\alpha+1]_q-1)\Lambda(a)+\Lambda(a+\zeta(b,a))}{[\alpha+1]_q}\right|$$

$$\leqslant\frac{\zeta(b,a)}{[\alpha+1]_q}(A_1|{}_aD_q\Lambda(a)|+A_2|{}_aD_q\Lambda(b)|),\tag{2.21}$$

where

$$A_1 = \int\limits_0^1 \left| [\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right| (1-t) \, _0 d_q t,$$

and

$$A_2 = \int\limits_0^1 \left| [\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right| t \, _0 d_q t.$$

Proof. Using Lemma 2.14, property of modulus and preinvexity of $|{_a}D_q\Lambda|$, we have

$$\left| \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} ({_a}J_q^\alpha \Lambda)(a + \zeta(b,a)) - \frac{([\alpha+1]_q - 1)\Lambda(a) + \Lambda(a + \zeta(b,a))}{[\alpha+1]_q} \right|$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q} \int\limits_0^1 \left| \left([\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right) {_a}D_q\Lambda(a + t\zeta(b,a)) \right| {_0}d_q t$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q} \int\limits_0^1 \left| [\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right| \left[(1-t)|{_a}D_q\Lambda(a)| + t|{_a}D_q\Lambda(b)| \right] {_0}d_q t$$

$$= \frac{\zeta(b,a)}{[\alpha+1]_q} \left[\begin{array}{l} |{_a}D_q\Lambda(a)| \int\limits_0^1 \left| [\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right| (1-t) \, _0 d_q t \\ + |{_a}D_q\Lambda(b)| \int\limits_0^1 \left| [\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right| t \, _0 d_q t \end{array} \right]$$

$$= \frac{\zeta(b,a)}{[\alpha+1]_q} (|{_a}D_q\Lambda(a)|A_1 + |{_a}D_q\Lambda(b)|A_2),$$

which completes the proof. □

THEOREM 2.4. *Let* $\Lambda : [a, a + \zeta(b,a)] \to \mathbb{R}$ *be a continuous function and* $\alpha > 0$ *and* ${_a}D_q\Lambda$ *is q-integrable on* $(a, a + \zeta(b,a))$. *If* $|{_a}D_q\Lambda|^r$ *is preinvex on* $[a, a + \zeta(b,a)]$ *for* $r > 1$ *and* $p^{-1} + r^{-1} = 1$, *then the following inequality holds:*

$$\left| \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} ({_a}J_q^\alpha \Lambda)(a + \zeta(b,a)) - \frac{([\alpha+1]_q - 1)\Lambda(a) + \Lambda(a + \zeta(b,a))}{[\alpha+1]_q} \right|$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q} A_3^{\frac{1}{p}} \left(\frac{q|{_a}D_q\Lambda(a)|^r + |{_a}D_q\Lambda(b)^r}{1+q} \right)^{\frac{1}{r}}, \tag{2.22}$$

where

$$A_3 = \int\limits_0^1 \left| [\alpha+1]_q \, _0(1 - {_0}\Phi_q(t))_q^{(\alpha)} - 1 \right|^p {_0}d_q t.$$

Proof. Using Lemma 2.14, Hölder's integral inequality and preinvexity of $|_aD_q\Lambda|^r$, we have

$$\left|\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) - \frac{([\alpha+1]_q-1)\Lambda(a)+\Lambda(a+\zeta(b,a))}{[\alpha+1]_q}\right|$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right||_aD_q\Lambda(a+t\zeta(b,a))|_0d_qt$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|^p{}_0d_qt\right)^{\frac{1}{p}}$$

$$\times\left(\int_0^1|_aD_q\Lambda(a+t\zeta(b,a))|^r{}_0d_qt\right)^{\frac{1}{r}}$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|^p{}_0d_qt\right)^{\frac{1}{p}}$$

$$\times\left(|_aD_q\Lambda(a)|^r\int_0^1(1-t)_0d_qt+|_aD_q\Lambda(b)|^r\int_0^1t_0d_qt\right)^{\frac{1}{r}}$$

$$= \frac{\zeta(b,a)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|^p{}_0d_qt\right)^{\frac{1}{p}}$$

$$\times\left(\frac{q|_aD_q\Lambda(a)|^r+|_aD_q\Lambda(b)|^r}{1+q}\right)^{\frac{1}{r}},$$

which completes the proof. $\square$

THEOREM 2.5. *Let* $\Lambda:[a,a+\zeta(b,a)]\to\mathbb{R}$ *be a continuous function and* $\alpha>0$ *and* $_aD_q\Lambda$ *is* q-*integrable on* $(a,a+\zeta(b,a))$. *If* $|_aD_q\Lambda|^r$ *is preinvex on* $[a,a+\zeta(b,a)]$ *for* $r\geqslant 1$, *then the following inequality holds:*

$$\left|\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) - \frac{([\alpha+1]_q-1)\Lambda(a)+\Lambda(a+\zeta(b,a))}{[\alpha+1]_q}\right|$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}A_4^{1-\frac{1}{r}}\left(A_1|_aD_q\Lambda(a)|^r+A_2|_aD_q\Lambda(b)^r\right)^{\frac{1}{r}}, \tag{2.23}$$

where A_1,A_2 *are given in Theorem* 2.3 *and* A_4 *is given as:*

$$A_4=\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|_0d_qt.$$

Proof. Using Lemma 2.14, power mean integral inequality and preinvexity of $|_aD_q\Lambda|^r$, we have

$$\left|\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) - \frac{([\alpha+1]_q-1)\Lambda(a)+\Lambda(a+\zeta(b,a))}{[\alpha+1]_q}\right|$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right||_aD_q\Lambda(a+t\zeta(b,a))|\,_0d_qt$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|\,_0d_qt\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right||_aD_q\Lambda(a+t\zeta(b,a))|^r\,_0d_qt\right)^{\frac{1}{r}}$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|\,_0d_qt\right)^{1-\frac{1}{r}}$$

$$\times\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|\left[|_aD_q\Lambda(a)|^r(1-t)+|_aD_q\Lambda(b)|^rt\right]\,_0d_qt\right)^{\frac{1}{r}}$$

$$\leqslant \frac{\zeta(b,a)}{[\alpha+1]_q}\left(\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|\,_0d_qt\right)^{1-\frac{1}{r}}$$

$$\times\left[\begin{array}{l}|_aD_q\Lambda(a)|^r\int_0^1\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|(1-t)\,_0d_qt\\[2mm]+|_aD_q\Lambda(b)|^r\left|[\alpha+1]_{q\,0}(1-{}_0\Phi_q(t))_q^{(\alpha)}-1\right|t\,_0d_qt\end{array}\right]^{\frac{1}{r}}$$

$$= \frac{\zeta(b,a)}{[\alpha+1]_q}A_4^{1-\frac{1}{r}}\left(A_1|_aD_q\Lambda(a)|^r+A_3|_aD_q\Lambda(b)|^r\right)^{\frac{1}{r}},$$

which completes the proof. $\square$

THEOREM 2.6. *Let* $\Lambda:[a,a+\zeta(b,a)]\to\mathbb{R}$ *be a continuous function and* $\alpha>0$ *and* $_aD_q\Lambda$ *is* q*-integrable on* $(a,a+\zeta(b,a))$. *If* $|_aD_q\Lambda|$ *is preinvex on* $[a,a+\zeta(b,a)]$, *then*

$$\left|\Lambda\left(\frac{([\alpha+1]_q-1)a+(a+\zeta(b,a))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a+\zeta(b,a))\right|$$

$$\leqslant \zeta(b,a)\left[(A_5+A_7)|_aD_q\Lambda(a)|+(A_6+A_8)|_aD_q\Lambda(b)|\right], \qquad (2.24)$$

where

$$A_5=\int_0^{\frac{1}{[\alpha+1]_q}}\left|1-{}_0(1-\Phi_q(t))_q^{(\alpha)}\right|(1-t)\,_0d_qt$$

$$A_6 = \int_0^{\frac{1}{[\alpha+1]_q}} \left|1 - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| t \, {}_0d_qt$$

$$A_7 = \int_{\frac{1}{[\alpha+1]_q}}^1 \left| - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| (1-t) \, {}_0d_qt$$

$$A_8 = \int_{\frac{1}{[\alpha+1]_q}}^1 \left| - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| t \, {}_0d_qt.$$

Proof. Using Lemma 2.18 and the preinvexity of $|{}_aD_q\Lambda|$, we have

$$\left| \Lambda\left(\frac{([\alpha+1]_q - 1)a + (a + \zeta(b,a))}{[\alpha+1]_q} \right) - \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} ({}_aJ_q^\alpha\Lambda)(a + \zeta(b,a)) \right|$$

$$\leqslant \zeta(b,a) \left[\begin{array}{l} \int_0^{\frac{1}{[\alpha+1]_q}} \left|1 - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| |{}_aD_q\Lambda(a + t\zeta(b,a))| \, {}_0d_qt \\ + \int_{\frac{1}{[\alpha+1]_q}}^1 \left| - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| |{}_aD_q\Lambda(a + t\zeta(b,a))| \, {}_0d_qt \end{array} \right]$$

$$\leqslant \zeta(b,a) \left[\begin{array}{l} \int_0^{\frac{1}{[\alpha+1]_q}} \left|1 - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| [(1-t)|{}_aD_q\Lambda(a)| + t|{}_aD_q\Lambda(b)] \, {}_0d_qt \\ + \int_{\frac{1}{[\alpha+1]_q}}^1 \left| - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| [(1-t)|{}_aD_q\Lambda(a) + t|{}_aD_q\Lambda(b)|] \, {}_0d_qt \end{array} \right]$$

$$= \zeta(b,a) \left[\begin{array}{l} |{}_aD_q\Lambda(a)| \left[\begin{array}{l} \int_0^{\frac{1}{[\alpha+1]_q}} \left|1 - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| (1-t) \, {}_0d_qt \\ + \int_{\frac{1}{[\alpha+1]_q}}^1 \left| - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| (1-t) \, {}_0d_qt \end{array} \right] \\ + |{}_aD_q\Lambda(b)| \left[\begin{array}{l} \int_0^{\frac{1}{[\alpha+1]_q}} \left|1 - {}_0(1 - \Phi_q(t)_q^{(\alpha)})\right| t \, {}_0d_qt \\ + \int_{\frac{1}{[\alpha+1]_q}}^1 \left| - {}_0(1 - \Phi_q(t))_q^{(\alpha)}\right| t \, {}_0d_qt \end{array} \right] \end{array} \right].$$

This completes the proof. $\square$

THEOREM 2.7. *Let $\Lambda : [a, a + \zeta(b,a)] \to \mathbb{R}$ be a continuous function and $\alpha > 0$ and ${}_aD_q\Lambda$ is q-integrable on $(a, a + \zeta(b,a))$. If $|{}_aD_q\Lambda|^r$ is preinvex on $[a, a + \zeta(b,a)]$, then the following inequality holds for $p^{-1} + r^{-1} = 1$:*

$$\left| \Lambda\left(\frac{([\alpha+1]_q-1)a+(a+\zeta(b,a))}{[\alpha+1]_q} \right) - \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}({}_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) \right|$$

$$\leqslant \zeta(b,a) \begin{bmatrix} A_9^{\frac{1}{p}}\left(|{}_aD_q\Lambda(a)|^r \left(\frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2} \right) \right. \\[2mm] \left. + |{}_aD_q\Lambda(b)|^r \left(\frac{1}{(1+q)([\alpha+1]_q)^2} \right) \right)^{\frac{1}{r}} \\[2mm] + A_{10}^{\frac{1}{p}}\left(|{}_aD_q\Lambda(a)|^r \left(\frac{q}{1+q} - \frac{(1+q)[\alpha+1]_q-1}{(1+q)([\alpha+1]_q)^2} \right) \right. \\[2mm] \left. + |{}_aD_q\Lambda(b)|^r \left(\frac{1}{1+q} - \frac{1}{(1+q)([\alpha+1]_q)^2} \right) \right)^{\frac{1}{r}} \end{bmatrix}, \tag{2.25}$$

where

$$A_9 = \int_0^{\frac{1}{[\alpha+1]_q}} \left| 1 - {}_0(1-\Phi_q(t))_q^{(\alpha)} \right| {}_0d_qt$$

$$A_{10} = \int_{\frac{1}{[\alpha+1]_q}}^1 | - {}_0(1-\Phi_q(t))_q^{(\alpha)}| {}_0d_qt.$$

Proof. Using Lemma 2.18, Hölder's inequality and the preinvexity of $|{}_aD_q\Lambda|^r$, we have

$$\left| \Lambda\left(\frac{([\alpha+1]_q-1)a+(a+\zeta(b,a))}{[\alpha+1]_q} \right) - \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}({}_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) \right|$$

$$\leqslant \zeta(b,a) \begin{bmatrix} \int_0^{\frac{1}{[\alpha+1]_q}} |1 - {}_0(1-\Phi_q(t))_q^{(\alpha)}| |{}_aD_q\Lambda(a+t\zeta(b,a))| {}_0d_qt \\[2mm] + \int_{\frac{1}{[\alpha+1]_q}}^1 | - {}_0(1-\Phi_q(t))_q^{(\alpha)}| |{}_aD_q\Lambda(a+t\zeta(b,a))| {}_0d_qt \end{bmatrix}$$

$$\leqslant \zeta(b,a) \begin{bmatrix} \left(\int_0^{\frac{1}{[\alpha+1]_q}} |1 - {}_0(1-\Phi_q(t))_q^{(\alpha)}|^p {}_0d_qt \right)^{\frac{1}{p}} \\[2mm] \times \left(\int_{\frac{1}{[\alpha+1]_q}}^1 |{}_aD_q\Lambda(a+t\zeta(b,a))|^r {}_0d_qt \right)^{\frac{1}{r}} \\[2mm] + \left(\int_{\frac{1}{[\alpha+1]_q}}^1 | - {}_0(1-\Phi_q(t))_q^{(\alpha)}|^p {}_0d_qt \right)^{\frac{1}{p}} \\[2mm] \times \left(\int_{\frac{1}{[\alpha+1]_q}}^1 |{}_aD_q\Lambda(a+t\zeta(b,a))|^r {}_0d_qt \right)^{\frac{1}{r}} \end{bmatrix}$$

$$
\leqslant \zeta(b,a)
\left[
\begin{array}{l}
\left(\displaystyle\int_{0}^{\frac{1}{[\alpha+1]_q}} |1 - {}_0(1-\Phi_q(t))_q^{(\alpha)}|^p \,{}_0\mathrm{d}_q t \right)^{\frac{1}{p}} \\[2em]
\times \left[|{}_aD_q\Lambda(b)|^r \displaystyle\int_{0}^{\frac{1}{[\alpha+1]_q}} (1-t)\,{}_0\mathrm{d}_q t + |{}_aD_q\Lambda(b)|^r \displaystyle\int_{0}^{\frac{1}{[\alpha+1]_q}} t\,{}_0\mathrm{d}_q t \right]^{\frac{1}{r}} \\[2em]
+ \left(\displaystyle\int_{\frac{1}{[\alpha+1]_q}}^{1} |-{}_0(1-\Phi_q(t))_q^{(\alpha)}|^p\,{}_0\mathrm{d}_q t \right)^{\frac{1}{p}} \\[2em]
\times \left[|{}_aD_q\Lambda(a)|^r \displaystyle\int_{\frac{1}{[\alpha+1]_q}}^{1} (1-t)\,{}_0\mathrm{d}_q t + |{}_aD_q\Lambda(b)|^r \displaystyle\int_{\frac{1}{[\alpha+1]_q}}^{1} t\,{}_0\mathrm{d}_q t \right]^{\frac{1}{r}}
\end{array}
\right]
$$

$$
= \zeta(b,a)
\left[
\begin{array}{l}
\left(\displaystyle\int_{0}^{\frac{1}{[\alpha+1]_q}} |1 - {}_0(1-\Phi_q(t))_q^{(\alpha)}|^p\,{}_0\mathrm{d}_q t \right)^{\frac{1}{p}} \\[2em]
\times \left(|{}_aD_q\Lambda(a)|^r \left(\frac{(1+q)[\alpha+1]_q - 1}{(1+q)([\alpha+1]_q)^2} \right) + |{}_aD_q\Lambda(b)|^r \left(\frac{1}{(1+q)([\alpha+1]_q)^2} \right) \right)^{\frac{1}{r}} \\[2em]
+ \left(\displaystyle\int_{\frac{1}{[\alpha+1]_q}}^{1} |-{}_0(1-\Phi_q(t))_q^{(\alpha)}|^p\,{}_0\mathrm{d}_q t \right)^{\frac{1}{p}} \\[2em]
\times \left(|{}_aD_q\Lambda(a)|^r \left(\frac{q}{1+q} - \frac{(1+q)[\alpha+1]_q - 1}{(1+q)([\alpha+1]_q)^2} \right) \right. \\[1.5em]
\left. + |{}_aD_q\Lambda(b)|^r \left(\frac{1}{1+q} - \frac{1}{(1+q)([\alpha+1]_q)^2} \right) \right)^{\frac{1}{r}}
\end{array}
\right].
$$

This completes the proof. $\quad\square$

THEOREM 2.8. *Let* $\Lambda : [a, a+\zeta(b,a)] \to \mathbb{R}$ *be a continuous function and* $\alpha > 0$ *and* ${}_aD_q\Lambda$ *is* q-*integrable on* $(a, a+\zeta(b,a))$. *If* $|{}_aD_q\Lambda|^r, r \geqslant 1$ *is preinvex on* $[a, a+\zeta(b,a)]$, *then*

$$
\left| \Lambda \left(\frac{([\alpha+1]_q - 1)a + (a+\zeta(b,a))}{[\alpha+1]_q} \right) - \frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)} ({}_aJ_q^\alpha\Lambda)(a+\zeta(b,a)) \right|
$$
$$
= \zeta(b,a)
\left[
\begin{array}{l}
A_9^{1-\frac{1}{r}} \left[A_5 |{}_aD_q\Lambda(b)|^r + A_6 |{}_aD_q\Lambda(b)|^r \right]^{\frac{1}{r}} \\
+ A_{10}^{1-\frac{1}{r}} \left[A_7 |{}_aD_q\Lambda(a)|^r + A_8 |{}_aD_q\Lambda(b)|^r \right]^{\frac{1}{r}},
\end{array}
\right]
$$

(2.26)

where

$$
A_9 = \int_{0}^{\frac{1}{[\alpha+1]_q}} \left| 1 - {}_0(1-\Phi_q(t))_q^{(\alpha)} \right| {}_0\mathrm{d}_q t
$$

$$
A_{10} = \int_{\frac{1}{[\alpha+1]_q}}^{1} |-{}_0(1-\Phi_q(t))_q^{(\alpha)}| {}_0\mathrm{d}_q t.
$$

Proof. Using Lemma 2.18, power mean integral inequality and the preinvexity of $|_aD_q\Lambda|^r$, we have

$$\left|\Lambda\left(\frac{([\alpha+1]_q-1)a+(a+\zeta(b,a))}{[\alpha+1]_q}\right)-\frac{\Gamma_q(\alpha+1)}{\zeta^\alpha(b,a)}(_aJ_q^\alpha\Lambda)(a+\zeta(b,a))\right|$$

$$\leqslant \zeta(b,a)\left[\begin{array}{l}\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t))_q^{(\alpha)}||_aD_q\Lambda(a+t\zeta(b,a))|_0d_qt\\[2ex]+\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}||_aD_q\Lambda(a+t\zeta(b,a))|_0d_qt\end{array}\right]$$

$$\leqslant \zeta(b,a)\left[\begin{array}{l}\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t))_q^{(\alpha)}|_0d_qt\right)^{1-\frac{1}{r}}\\[2ex]\times\left(\int_{\frac{1}{[\alpha+1]_q}}^1|1-_0(1-\Phi_q(t)_q^{(\alpha)})||_aD_q\Lambda(a+t\zeta(b,a))|^r_0d_qt\right)^{\frac{1}{r}}\\[2ex]+\left(\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}|_0d_qt\right)^{1-\frac{1}{r}}\\[2ex]\times\left(\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}||_aD_q\Lambda(a+t\zeta(b,a))|^r_0d_qt\right)^{\frac{1}{r}}\end{array}\right]$$

$$\leqslant \zeta(b,a)\left[\begin{array}{l}\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t))_q^{(\alpha)}|_0d_qt\right)^{1-\frac{1}{r}}\\[2ex]\times\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t)_q^{(\alpha)})|\left[(1-t)|_aD_q\Lambda(b)|^r+t|_aD_q\Lambda(b)|^r\right]_0d_qt\right)^{\frac{1}{r}}\\[2ex]+\left(\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}|_0d_qt\right)^{1-\frac{1}{r}}\\[2ex]\times\left(\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}|\left[(1-t)|_aD_q\Lambda(a)|^r+t|_aD_q\Lambda(b)|^r\right]_0d_qt\right)^{\frac{1}{r}}\end{array}\right]$$

$$= \zeta(b,a)\left[\begin{array}{l}\left(\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t))_q^{(\alpha)}|_0d_qt\right)^{1-\frac{1}{r}}\\[2ex]\times\left[|_aD_q\Lambda(b)|^r\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t))_q^{(\alpha)}|(1-t)_0d_qt\right.\\[2ex]\left.+|_aD_q\Lambda(b)|^r\int_0^{\frac{1}{[\alpha+1]_q}}|1-_0(1-\Phi_q(t)_q^{(\alpha)})|t_0d_qt\right]^{\frac{1}{r}}\\[2ex]+\left(\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}|_0d_qt\right)^{1-\frac{1}{r}}\\[2ex]\times\left[|_aD_q\Lambda(a)|^r\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}|(1-t)_0d_qt\right.\\[2ex]\left.+|_aD_q\Lambda(b)|^r\int_{\frac{1}{[\alpha+1]_q}}^1|-_0(1-\Phi_q(t))_q^{(\alpha)}|t_0d_qt\right]^{\frac{1}{r}}\end{array}\right].$$

This completes the proof.   □

3. Declarations

REFERENCES

[1] N. AKHTAR, M. U. AWAN, M. Z. JAVED, M. TH. RASSIAS, M. V. MIHAI, M. A. NOOR, K. I. NOOR, *Ostrowski type inequalities involving harmonically convex functions and applications*, Symmetry, **13** (2): 201, (2021).

[2] N. ALP, M. Z. SARIKAYA, M. KUNT AND İ. İŞCAN, q-*Hermite Hadamard inequalities and quantum estimates for midpoint type inequalities via convex and quasi-convex functions*, J. King Saud Univ. Sci., 2018, **30** (2), 193–203.

[3] M. H. ANNABY, Z. S. MANSOUR, q-*Fractional Calculus and Equations*, Springer, Heidelberg, (2012).

[4] S. ERDEN, S. IFTIKHAR, M. R. DELAVAR P. KUMAM, P. THOUNTHONG, W. KUMAM, *On generalizations of some inequalities for convex functions via quantum integrals*, RACSAM, (2020), 114:110, https://doi.org/10.1007/s13398-020-00841-3.

[5] I. ISCAN, *Hermite-Hadamard's inequalities for preinvex function via fractional integrals and related fractional inequalities*, American J. Math. Anal., **1** (3), 33–38, (2013).

[6] A. B.-ISRAEL AND B. MOND, *What is invexity?*, J. Austral. Math. Soc., 1986, **28B**(1), 1–9.

[7] M. KUNT, M. ALJASEM, *Fractional quantum Hermite-Hadamard type inequalities*, Konuralp J. Math., **8** (1), 122–136, (2020).

[8] W.-J. LIU AND H.-F. ZHUANG, *Some quantum estimates of Hermite-Hadamard inequalities for convex functions*, J. Appl. Anal. Comput., 2017, **7** (2), 501–522.

[9] M. A. NOOR, *Hermite-Hadamard integral inequalities for* log-*preinvex functions*, J. Math. Anal. Approx. Theory, 2, 126–131, (2007).

[10] M. A. NOOR, K. I. NOOR, M. U. AWAN, *Some Quantum estimates for Hermite-Hadamard inequalities*, Appl. Math. Comput., 2015, **251**, 675–679.

[11] M. A. NOOR, K. I. NOOR, M. U. AWAN, *Some quantum integral inequalities via preinvex functions*, Appl. Math.Comput., 2015, **269**, 242–251.

[12] M. A. NOOR, K. I. NOOR, B. MOHSEN, M. TH. RASSIAS, A. RAIGORODSKII, *General preinvex functions and variational-like inequalities*, in: Approximation and Computation in Science and Engineering, Springer Optimization and Its Applications, in: Nicholas J. Daras & Themistocles M. Rassias (ed.), Approximation and Computation in Science and Engineering, pages 643–666, Springer.

[13] M. A. NOOR, K. I. NOOR, M. TH. RASSIAS, *Characterizations of higher order strongly generalized convex functions*, Springer Optimization and Its Applications, in: Themistocles M. Rassias (ed.), Nonlinear Analysis, Differential Equations, and Applications, pages 341–364, Springer.

[14] M. A. NOOR, K. I. NOOR, M. TH. RASSIAS, *New Trends in General Variational Inequalities*, Acta Appl. Mathematicae, **170** (1), 981–1064, (2020).

[15] M. Z. SARIKAYA, E. SET, H. YALDIZ, N. BASAK, *Hermite-Hadamard's inequalities for fractional integrals and related fractional inequalities*, Math. Comput. Model., **57** (9–10), 2403–2407, (2013).

[16] W. SUDSUTAD, S. K. NTOUYAS AND J. TARIBOON, *Quantum integral inequalities for convex functions*, J. Math. Inequal., 2015, **9** (3), 781–793.

[17] W. SUDSUTAD, S. K. NTOUYAS, J. TARIBOON, *Integral inequalities via fractional quantum calculus*, J. Inequal. Appl., **81** (2016) 1–15.

[18] J. TARIBOON AND S. K. NTOUYAS, *Quantum integral inequalities on finite intervals*, J. Inequal. Appl., 2014, **2014**, Article 121, 13 pages.

[19] J. TARIBOON, S. K. NTOUYAS, P. AGARWAL, *New concepts of fractional quantum calculus and applications to impulsive fractional q-difference equations*, Adv. Diff. Equ. **18** (2015) 1–19.

[20] J. TARIBOON, S. K. NTOUYAS, *Quantum calculus on finite intervals and applications to impulsive difference equations*, Adv. Difference Equ. **282** (2013), 1–19.

[21] T. WEIR AND B. MOND, *Pre-invex functions in multiple objective optimization*, J. Math. Anal. Appl., 1988, **136** (1), 29–38.

[22] Y. ZHANG, T.-S. DU, H. WANG AND Y.-J. SHEN, *Different types of quantum integral inequalities via (α, m)-convexity*, J. Inequal. Appl., 2018, **2018**, Article 264, 24 pages.

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