Journal of Mathematical Inequalities
Volume 17, Number 4 (2023), 1551–1563
doi:10.7153/jmi-2023-17-102

APPROXIMATION OF FUNCTIONS IN A WEIGHTED LEBESGUE SPACE BY MEANS OF THE PICARD SINGULAR INTEGRAL

ABHAY PRATAP SINGH AND UADAY SINGH

(*Communicated by I. Raşa*)

Abstract. The integral transforms, particularly singular integrals, play an important role in approximation theory. In this paper, we study the approximation properties of the Picard singular integral in a weighted Lebesgue space and weighted Hölder space. We also show that many of the theorems in the literature dealing with approximation of functions by the Picard singular integral are the special cases of our results.

1. Introduction

For $1 \leqslant p < \infty$ and given positive weight function W, we denote by $L^{p,W}$, the set of all real valued functions f defined on $\mathbb{R}$ for which $|Wf|^p$ is Lebesgue integrable on $\mathbb{R}$. For $p = \infty$ and given weight function W, we denote by $L^{\infty,W}$, the set of all real valued functions f for which Wf is a uniformly continuous and bounded function on $\mathbb{R}$.

For $1 \leqslant p \leqslant \infty$, the norm of $f \in L^{p,W}$ is defined as

$$||f||_{p,W} = \begin{cases} \left(\int_{\mathbb{R}} |W(x)f(x)|^p dx\right)^{1/p}, & 1 \leqslant p < \infty; \\ \sup\limits_{x\in\mathbb{R}} W(x)|f(x)|, & p = \infty. \end{cases}$$

Throughout the paper, we consider those weight functions W which satisfy the following conditions.

1. $W(x)$ is an even function on $\mathbb{R}$ and non-increasing for $x > 0$.
2. $L(s) = \int_0^\infty e^{-st} \frac{1}{W(t)} dt < \infty$.
3. $\sup\limits_{x\in\mathbb{R}} \left(\frac{W(x-h)}{W(x)}\right) \leqslant \frac{1}{W(h)}$, $h \in \mathbb{R}$.

For $f \in L^{p,W}$, its modulus of smoothness of order 2 is defined by

$$\omega_2(f, L^{p,W}, t) = \sup_{|h|\leqslant t} ||\Delta_h^2 f(x)||_{p,W},$$

Mathematics subject classification (2020): 41A35, 41A36, 41A81.

Keywords and phrases: Modulus of smoothness, Picard singular integral, weighted Lebesgue spaces, weighted Hölder space.

Paper JMI-17-102

where $\Delta_h^2 f(x) = f(x+h) + f(x-h) - 2f(x)$.

We denote by Ω^2, the set of all real valued functions ω which satisfy the following conditions.

1. ω is continuous on $[0,\infty)$.

2. ω is increasing and $\omega(0) = 0$.

3. $\frac{\omega(t)}{t^3}$ is decreasing on $[0,\infty)$.

For a given $1 \leqslant p \leqslant \infty$ and $\omega \in \Omega^2$, we define the weighted Hölder space $H^{p,W,\omega}$ to be the set of all functions f in $L^{p,W}$ for which

$$||f||^*_{p,W,\omega} = \sup_{h>0} \left(\frac{||\Delta_h^2 f(.)||_{p,W}}{\omega(h)} \right) < \infty.$$

The norm in $H^{p,W,\omega}$ space is defined by

$$||f||_{p,W,\omega} = ||f||_{p,W} + ||f||^*_{p,W,\omega}. \tag{1}$$

We note that for $f \in H^{p,W,\omega}$,

$$\omega_2(f, L^{p,W}, t) \leqslant \omega(t) ||f||^*_{p,W,\omega}. \tag{2}$$

In particular, if $W(x) = e^{-q|x|}, q > 0,$ then $L^{p,W} = L^{p,q},\ H^{p,W,\omega} = H^{p,q,\omega}$ and the corresponding norms are defined by $\|.\|_{p,W} = \|.\|_{p,q},\ \|.\|_{p,W,\omega} = \|.\|_{p,q,\omega}$, [8].

The Picard singular integral $P_r(f,x)$ of a function $f \in L^{p,W}$ is defined by

$$P_r(f,x) = \frac{1}{2r} \int_{\mathbb{R}} f(x+t) e^{-\frac{|t|}{r}} dt, \ \ x \in \mathbb{R}, \ \ r > 0 \ \text{ and } \ r \to 0. \tag{3}$$

We obtain the following important upper estimates for the norm of $P_r(f,x)$.

Using properties of W, for $f \in L^{\infty,W}$, we have

$$\begin{aligned}
||P_r(f,x)||_{\infty,W} &= \sup_{x\in\mathbb{R}} \left(W(x) |P_r(f,x)| \right) \\
&\leqslant \frac{1}{r} \left(\int_0^\infty \sup_{x\in\mathbb{R}} \left(\frac{W(x)}{W(x+t)} \right) e^{\frac{-|t|}{r}} dt \right) ||f||_{\infty,W} \\
&\leqslant \frac{1}{r} \left(\int_0^\infty \frac{1}{W(t)} e^{\frac{-|t|}{r}} dt \right) ||f||_{\infty,W} \\
&\leqslant \frac{1}{r} L\left(\frac{1}{r}\right) ||f||_{\infty,W}.
\end{aligned}$$

For $f \in L^{p,W}$, $1 \leqslant p < \infty$, using the generalized Minkowski inequality and properties of W, we have

$$\begin{aligned}
||P_r(f,x)||_{p,W} &= \left(\int_{\mathbb{R}} |W(x)P_r(f,x)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}} \left|W(x)\frac{1}{2r}\int_{\mathbb{R}} f(x+t)e^{-\frac{|t|}{r}}dt\right|^p dx\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{2r}\left(\int_{\mathbb{R}}\int_{\mathbb{R}} \left|\frac{W(u-t)}{W(u)}W(u)f(u)e^{-\frac{|t|}{r}}\right|^p dtdu\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{r}\left(\int_0^\infty \frac{1}{W(t)}e^{\frac{-|t|}{r}}dt\right)\left(\int_{\mathbb{R}} |W(u)f(u)|^p du\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{r}L\left(\frac{1}{r}\right)||f||_{p,W}.
\end{aligned}$$

Thus for $1 \leqslant p \leqslant \infty$, we have

$$||P_r(f,x)||_{p,W} \leqslant \frac{1}{r}L\left(\frac{1}{r}\right)||f||_{p,W}. \tag{4}$$

Now for $f \in H^{p,W,\omega}$, $1 \leqslant p \leqslant \infty$, we have

$$\begin{aligned}
||P_r(f,x)||^*_{p,W,\omega} &= \sup_{h>0}\frac{||\Delta_h^2(P_r(f,x))||_{p,W}}{\omega(h)} \\
&= \sup_{h>0}\frac{\|P_r\left(\Delta_h^2 f(x)\right)\|_{p,W}}{\omega(h)} \\
&\leqslant \frac{1}{r}L\left(\frac{1}{r}\right)\sup_{h>0}\frac{||\Delta_h^2(f(x))||_{p,W}}{\omega(h)} \\
&= \frac{1}{r}L\left(\frac{1}{r}\right)||f||^*_{p,W,\omega}.
\end{aligned} \tag{5}$$

Combining (4), (5) and (1), we get

$$||P_r(f,x)||_{p,W,\omega} \leqslant \frac{1}{r}L\left(\frac{1}{r}\right)||f||_{p,W,\omega}. \tag{6}$$

From (4) and (6), it can be observed that for each $r > 0$ wherever $L\left(\frac{1}{r}\right)$ exists, the Picard singular integral given in (3) is well defined for all $f \in L^{p,W}$ and for all $f \in H^{p,W,\omega}$ as $L\left(\frac{1}{r}\right) > 0$.

The rate of approximation by using singular integrals for different function classes has been studied by various researchers. Mohapatra and Rodriguez [9] studied the rate of convergence of Picard, Cauchy and Gauss-Weierstrass singular integrals for Hölder continuous functions on $\mathbb{R}$ and $[-\pi,\pi]$. Gal [7, 6] investigated the degree of approximation of continuous functions by means of three singular integrals for periodic functions of single and double variable. Firlejy and Rempulska [5] proved the similar results for the generalized Hölder norm. Further estimates were studied by Rempulska and Walczak in [10] using modified Picard and Gauss-Weierstrass singular integrals. The study has further been extended to the weighted spaces [1, 2, 3, 8, 4, 11, 12] in different directions. More precisely, Lesniewicz et al. [8] has used the Lebesgue weighted space with $W(x) = e^{-q|x|}$, $q > 0$ and Bogalska et al. [4] has extended the results for the function of two variables. Yilmaz [11] has studied the problem for Gauss-Weierstrass singular integrals in weighted space with $W(x) = e^{-q|x|^2}$, $q > 0$. Some authors [1, 12, 2, 3] have suggested some slight modifications to the definitions of singular integrals and studied their convergence.

In this paper, we shall study the problem for a more general weighted function space $L^{p,W}$, $1 \leqslant p \leqslant \infty$ and $H^{p,W,\omega}$. We give some direct and inverse approximation theorems dealing with approximation properties of the Picard singular integral of $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$.

2. Direct approximation theorems

In this section, we prove the following direct approximation theorem for the functions belonging to class $L^{p,W}$, $1 \leqslant p \leqslant \infty$ and obtain the upper estimates for the deviation $||P_r(f,x) - f(x)||_{p,W}$. We shall denote by s_0 the maximum value of r for which $L\left(\frac{1}{r}\right)$ exist and use $\mathcal{L}(r) = \frac{1}{r}L\left(\frac{1}{r}\right) - \frac{2}{r^2}L'\left(\frac{1}{r}\right) + \frac{1}{r^3}L''\left(\frac{1}{r}\right)$, $r > 0$.

THEOREM 2.1. *Let $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$. Then*

$$||P_r(f,x) - f(x)||_{p,W} \leqslant \frac{\omega_2(f, L^{p,W}, r)}{2}\mathcal{L}(r), \tag{7}$$

for every $r \in (0, s_0]$.

For proving this theorem, we need the following lemma.

LEMMA 2.2. *Let $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$. Then*

$$\omega_2(f, L^{p,W}, \lambda t) \leqslant (1+\lambda)^2 \frac{1}{W(\lambda t)} \omega_2(f, L^{p,W}, t) \ \textit{for any} \ \lambda, t > 0. \tag{8}$$

Proof. We have

$$\omega_2(f, L^{p,W}, nt) = \sup_{|h| \leqslant nt} ||\Delta_h^2 f||_{p,W} = \sup_{|m| \leqslant t} ||\Delta_{nm}^2 f||_{p,W}.$$

Now using the identity [[8], p. 234],

$$\Delta_{nm}^2 f(x) = \textstyle\sum_{k=1}^{n} k\Delta_m^2 f(x-(n-k)m) + \sum_{k=1}^{n-1}(n-k)\Delta_m^2 f(x+km),$$

we have,

$$\sup_{|m| \leqslant t} ||\Delta_{nm}^2 f||_{p,W} = \sup_{|m| \leqslant t} \left|\left| \sum_{k=1}^{n} k\Delta_m^2 f(x-(n-k)m) + \sum_{k=1}^{n-1}(n-k)\Delta_m^2 f(x+km) \right|\right|_{p,W}.$$

Therefore,

$$\omega_2(f,L^{p,W},nt) \leqslant \sup_{|m|\leqslant t}\left(\sum_{k=1}^{n} k\left|\left|\Delta_m^2 f(x-(n-k)m)\right|\right|_{p,W} + \sum_{k=1}^{n-1}(n-k)||\Delta_m^2 f(x+km)||_{p,W}\right). \tag{9}$$

Now using properties of W, for $1 \leqslant p < \infty$, we have

$$\begin{aligned}
||\Delta_m^2 f(x-(n-k)m)||_{p,W} &= \left(\int_{\mathbb{R}} |W(x)\Delta_m^2 f(x-(n-k)m)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}} |W(u+(n-k)m)\Delta_m^2 f(u)|^p du\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}} \left|\frac{W(u+(n-k)m)}{W(u)}\right| |W(u)\Delta_m^2 f(u)|^p du\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{W((n-k)m)}\left(\int_{\mathbb{R}} |W(u)\Delta_m^2 f(u)|^p du\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{W((n-k)m)}||\Delta_m^2 f||_{p,W},
\end{aligned} \tag{10}$$

and

$$||\Delta_m^2 f(x+mk)||_{p,W} \leqslant \frac{1}{W(mk)}||\Delta_m^2 f||_{p,W}. \tag{11}$$

Similarly for $p = \infty$, we have

$$\begin{aligned}
||\Delta_m^2 f(x-(n-k)m)||_{\infty,W} &= \sup_{x\in\mathbb{R}} W(x)\left|\Delta_m^2 f(x-(n-k)m)\right| \\
&= \sup_{u\in\mathbb{R}} W(u+(n-k)m)\left|\Delta_m^2 f(u)\right| \\
&= \sup_{u\in\mathbb{R}} \frac{W(u+(n-k)m)}{W(u)} W(u)\left|\Delta_m^2 f(u)\right| \\
&\leqslant \frac{1}{W((n-k)m)}||\Delta_m^2 f||_{\infty,W},
\end{aligned}$$

and

$$||\Delta_m^2 f(x+mk)||_{\infty,W} \leqslant \frac{1}{W(mk)}||\Delta_m^2 f||_{\infty,W}. \tag{12}$$

Collecting the estimates from (10) to (12) and using (9), we get

$$\begin{aligned}
\omega_2(f,L^{p,W},nt) &\leqslant \sup_{|m|\leqslant t}\left(\sum_{k=1}^{n} k\frac{1}{W((n-k)m)}||\Delta_m^2 f||_{p,W} + \sum_{k=1}^{n-1}(n-k)\frac{1}{W(mk)}||\Delta_m^2 f||_{p,W}\right) \\
&\leqslant \sup_{|m|\leqslant t}\left(\sum_{k=1}^{n-1}\frac{2k}{W((n-k)m)} + \frac{n}{W(0)}\right)\sup_{|m|\leqslant t}||\Delta_m^2 f||_{p,W} \\
&\leqslant \left(\frac{2}{W((n-1)t)}\frac{n(n-1)}{2} + \frac{n}{W((n-1)t)}\right)\omega_2(f,L^{p,W},t).
\end{aligned}$$

So finally, we get

$$\omega_2(f,L^{p,W},nt) \leqslant n^2 \frac{1}{W((n-1)t)} \omega_2(f,L^{p,W},t), \quad \text{for any natural number } n. \tag{13}$$

Let n be the largest integer not exceeding $\lambda > 0$, i.e, $n \leqslant \lambda < n+1$. Then by using (13), we get our result. □

Proof of Theorem 2.1. We have

$$P_r(f,x) - f(x) = \frac{1}{2r} \int_0^\infty \Delta_t^2(f(x)) e^{-\frac{t}{r}} dt.$$

Therefore, using Lemma 2.2 and derivatives of $L\left(\frac{1}{r}\right)$, we have

$$\begin{aligned}
||P_r(f,x) - f(x)||_{p,W} &\leqslant \frac{1}{2r} \int_0^\infty e^{-\frac{t}{r}} ||\Delta_t^2(f(x))||_{p,W} dt \\
&\leqslant \frac{1}{2r} \int_0^\infty \omega_2(f,L^{p,W},t) e^{-\frac{t}{r}} dt \\
&\leqslant \omega_2(f,L^{p,W},r) \frac{1}{2r} \int_0^\infty \left(1+\frac{t}{r}\right)^2 \frac{1}{W(t)} e^{-\frac{t}{r}} dt \\
&= \omega_2(f,L^{p,W},r) \frac{1}{2r} \int_0^\infty \left(1+\frac{t^2}{r^2}+\frac{2t}{r}\right) \frac{1}{W(t)} e^{-\frac{t}{r}} dt \\
&\leqslant \omega_2(f,L^{p,W},r) \frac{1}{2r} \left(L\left(\frac{1}{r}\right) + \frac{1}{r^2} L''\left(\frac{1}{r}\right) - \frac{2}{r} L'\left(\frac{1}{r}\right)\right) \\
&\leqslant \frac{\omega_2(f,L^{p,W},r)}{2} \mathcal{L}(r).
\end{aligned}$$

Hence proof is completed. □

The following corollaries can be derived from Theorem 2.1.

COROLLARY 2.3. *Let* $f \in H^{p,W,\omega}$, $1 \leqslant p \leqslant \infty$ *and* $\omega \in \Omega^2$. *Then*

$$||P_r(f,x) - f(x)||_{p,W} \leqslant \frac{||f||^*_{p,W,\omega}}{2} \mathcal{L}(r) \omega(r), \tag{14}$$

for every $r \in (0, s_0]$.

Proof. We can easily prove this result by using (7) of Theorem 2.1 and (2). □

COROLLARY 2.4. *Let* $f \in L^{p,q}$, $1 \leqslant p \leqslant \infty$. *Then*

$$||P_r(f,x) - f(x)||_{p,q} \leqslant \frac{5}{2} (1 - s_0 q)^{-3} \omega_2(f,L^{p,q},r), \tag{15}$$

where $0 < r \leqslant s_0 < \frac{1}{q}$.

This corollary is Theorem 1 of [[8], p. 238].

Proof. For $W(t) = e^{-q|t|}$, $q > 0$, $L\left(\frac{1}{r}\right)$ exists for $0 < r < \frac{1}{q}$.

$$L\left(\frac{1}{r}\right) = \frac{r}{1-rq}, L'\left(\frac{1}{r}\right) = -\left(\frac{r}{1-rq}\right)^2, L''\left(\frac{1}{r}\right) = 2\left(\frac{r}{1-rq}\right)^3.$$

and

$$\mathcal{L}(r) = \frac{5}{2}(1-rq)^{-3} \leqslant \frac{5}{2}(1-s_0q)^{-3} \text{ for } 0 < r \leqslant s_0 < \frac{1}{q}. \tag{16}$$

Using (16) in Theorem 2.1, we get (15). □

COROLLARY 2.5. *Let $f \in H^{p,q,\omega}$, $1 \leqslant p \leqslant \infty$, and $\omega \in \Omega^2$. Then*

$$||P_r(f,x) - f(x)||_{p,q} \leqslant \tfrac{5}{2}(1-s_0q)^{-3}\omega(r)||f||^*_{p,q,\omega},$$

for all $r \in (0,1] \cap (0,s_0]$, where $0 < s_0 < \frac{1}{q}$.

This corollary is Corollary 1 of [[8], p. 238].

Proof. By using (14) and (16), we get the result. □

Now we shall give an analogue of Theorem 2.1 in the weighted Hölder space specified by moduli of smothness.

For $\omega, \mu \in \Omega^2$, let us assume that $\phi(t) = \frac{\omega(t)}{\mu(t)}$, $t > 0$ is an increasing function so that $H^{p,W,\omega} \subset H^{p,W,\mu}$. With this notation we state our next theorem as follows.

THEOREM 2.6. *Let $f \in H^{p,W,\omega}$, $1 \leqslant p \leqslant \infty$. Then there exists a function $M(r) >$ 0 such that*

$$||P_r(f,x) - f(x)||_{p,W,\mu} \leqslant M(r)||f||^*_{p,W,\omega}\phi(r), \tag{17}$$

for every $r \in (0,s_0] \cap (0,1]$.

For the proof of Theorem 2.6, we need the following lemma.

LEMMA 2.7. *Let $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$. Then*

$$||f(x+h)||_{p,W} \leqslant \tfrac{1}{W(h)}||f||_{p,W}, \quad h \in \mathbb{R}.$$

Proof. For $1 \leqslant p < \infty$, we have

$$\begin{aligned}
||f(x+h)||_{p,W} &= \left(\int_{\mathbb{R}} |W(x)f(x+h)|^p dx\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}} |W(u-h)f(u)|^p du\right)^{\frac{1}{p}} \\
&= \left(\int_{\mathbb{R}} \left|\frac{W(u-h)}{W(u)}\right| |W(u)f(u)|^p du\right)^{\frac{1}{p}} \\
&\leqslant \frac{1}{W(h)}||f||_{p,W},
\end{aligned}$$

in view of the properties of W.

Similarly, for $p=\infty$, we have

$$\begin{aligned}||f(x+h)||_{\infty,W} &= \sup_{x\in\mathbb{R}} W(x)\,|f(x+h)| \\ &= \sup_{u\in\mathbb{R}} W(u-h)\,|f(u)| \\ &= \sup_{u\in\mathbb{R}} \left|\frac{W(u-h)}{W(u)}\right| W(u)\,|f(u)| \\ &\leqslant \frac{1}{W(h)}||f||_{\infty,W}.\end{aligned}$$

Thus proof is completed. □

Proof of Theorem 2.6. Let r be a fixed point in $(0,s_0]\cap(0,1]$. Then, as in [[8], p. 241], by using definition of the weighted Hölder spaces and (1), we can write

$$\begin{aligned}||P_r(f,x)-f(x)||_{p,W,\mu} \leqslant ||P_r(f,x)-f(x)||_{p,W} &+ \sup_{h\in(0,r]} \frac{||\Delta_h^2\left(P_r(f,x)-f(x)\right)||_{p,W}}{\mu(h)} \\ &+ \sup_{h\in(r,1]} \frac{||\Delta_h^2\left(P_r(f,x)-f(x)\right)||_{p,W}}{\mu(h)}.\end{aligned} \tag{18}$$

Now we calculate estimates for (18) by using Corollary 2.3 and Lemma 2.7. We have

$$||P_r(f,x)-f(x)||_{p,W} \leqslant \frac{||f||^*_{p,W,\omega}}{2}\mathcal{L}(r)\mu(r)\phi(r), \tag{19}$$

$$\sup_{h\in(0,r]} \frac{||\Delta_h^2\left(P_r(f,x)-f(x)\right)||_{p,W}}{\mu(h)} \leqslant \left(\frac{1}{r}L\left(\frac{1}{r}\right)+1\right)\phi(r)||f||^*_{p,W,\omega}, \tag{20}$$

and

$$\begin{aligned}&\sup_{h\in(r,1]} \frac{||\Delta_h^2\left(P_r(f,x)-f(x)\right)||_{p,W}}{\mu(h)} \\ &= \sup_{h\in(r,1]} \frac{\|P_r(f,x+h)-f(x+h)\|_{p,W}+\|P_r(f,x-h)-f(x-h)\|_{p,W}-2\|P_r(f,x)-f(x)\|_{p,W}}{\mu(h)} \\ &\leqslant \frac{1}{W(1)}\sup_{h\in(r,1]} \frac{\|P_r(f,x)-f(x)\|_{p,W}}{\mu(h)} \\ &\leqslant \frac{1}{W(1)}\mathcal{L}(r)\phi(r)||f||^*_{p,W,\omega}.\end{aligned} \tag{21}$$

Collecting all the estimates from (19) to (21) and using (18), we obtain (17), where $M(r)=\max\left\{\frac{1}{2}\mathcal{L}(r)\mu(r), \frac{1}{r}L\left(\frac{1}{r}\right)+1, \frac{1}{W(1)}\mathcal{L}(r)\right\}$. □

COROLLARY 2.8. *Let* $f \in H^{p,q,\omega}$, $1 \leqslant p \leqslant \infty$. *Then for* $s_0 \in \left(0, \frac{1}{q}\right)$, *there exists a constant* $M = M(p,q,s_0,\mu) > 0$ *such that*

$$||P_r(f,x) - f(x)||_{p,q,\mu} \leqslant M||f||^{*}_{p,q,\omega}\phi(r), \tag{22}$$

for all $r \in (0,s_0] \cap (0,1]$.

This corollary is Theorem 3 of [[8], p. 241].

Proof. Using the definition of $\|.\|_{p,q,W}$, we have

$$||P_r(f,x) - f(x)||_{p,q,\mu} = ||P_r(f,x) - f(x)||_{p,W,\mu}.$$

Using Corollary 2.5 and Lemma 2.7, we can easily find the estimates

$$||P_r(f,x) - f(x)||_{p,W} = ||P_r(f,x) - f(x)||_{p,q} \leqslant \frac{5}{2}\left(1 - s_0q\right)^{-3}\mu(s_0)||f||^{*}_{p,q,\omega}\phi(r), \tag{23}$$

$$\begin{aligned} \sup_{h\in(0,r]} \frac{||\Delta_h^2\left(P_r(f,x) - f(x)\right)||_{p,W}}{\mu(h)} &= \sup_{h\in(0,r]} \frac{||\Delta_h^2\left(P_r(f,x) - f(x)\right)||_{p,q}}{\mu(h)} \\ &\leqslant \left(\frac{1}{1-s_0q} + 1\right)||f||^{*}_{p,q,\omega}\phi(r), \end{aligned} \tag{24}$$

and

$$\begin{aligned} \sup_{h\in(r,1]} \frac{||\Delta_h^2\left(P_r(f,x) - f(x)\right)||_{p,W}}{\mu(h)} &= \sup_{h\in(r,1]} \frac{||\Delta_h^2\left(P_r(f,x) - f(x)\right)||_{p,q}}{\mu(h)} \\ &\leqslant 10e^q\left(\frac{1}{1-s_0q}\right)||f||^{*}_{p,q,\omega}\phi(r). \end{aligned} \tag{25}$$

Putting the values of the estimates (23), (24) and (25) in (18), we get (22). □

3. Inverse approximation theorem

Now we shall prove the results that are related to inverse approximation by means of the Picard singular integral $P_r(f,x)$.

THEOREM 3.1. *Let* $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$, *satisfy the following condition*

$$||P_r(f,x) - f(x)||_{p,W} \leqslant \omega(r), \text{ for every } r > 0, \tag{26}$$

where ω *is a given function belonging to* Ω^2. *Then there exists a constant* $c > 0$ *such that*

$$\omega_2(f, L^{p,W}, t) \leqslant ct^2 \int_t^1 \frac{\omega(x)}{x^4}dx,$$

for all $t \in \left(0, \frac{1}{2}\right) \cap (0, s_0]$.

We need the following lemmas for proving this result.

LEMMA 3.2. *Let $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$. Then all the derivatives of $P_r(f,x)$ belong to $L^{p,W}$ and satisfy the following relation*

$$||P_r^{(n)}(f,x)||_{p,W} \leqslant r^{-n-1} L\left(\frac{1}{r}\right) ||f||_{p,W}, \tag{27}$$

for every $n \in \mathbb{N}$ and $r \in (0, s_0]$.

Proof. Using successive derivatives of $P_r(f,x)$ and induction, we get

$$P_r^{(n)}(f,x) = r^{-n} P_r(f,x). \tag{28}$$

Taking $||.||_{p,W}$ norm in both sides of (28) and using (4), we obtain (27). Hence proof is completed. □

LEMMA 3.3. *Let $f \in L^{p,W}$, $1 \leqslant p \leqslant \infty$. Then*

$$||\Delta_h^2 P_r(f,x)||_{p,W} \leqslant r^{-3} L\left(\frac{1}{r}\right) h^2 \frac{1}{W(h)} ||f||_{p,W}, \tag{29}$$

for every $r \in (0, s_0]$.

Proof. For $1 \leqslant p < \infty$, we have

$$\begin{aligned}
||\Delta_h^2 P_r(f,x)||_{p,W} &= \left(\int_{\mathbb{R}} \left| W(x) \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} P_r''(f, x+t_1+t_2) dt_1 dt_2 \right|^p dx \right)^{\frac{1}{p}} \\
&\leqslant \left(\int_{\mathbb{R}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \left| \frac{W(u-t_1-t_2)}{W(u)} W(u) P_r''(f,u) \right|^p dt_1 dt_2 du \right)^{\frac{1}{p}}.
\end{aligned}$$

Using the generalized Minkowski inequality and properties of W, we get

$$\begin{aligned}
||\Delta_h^2 P_r(f,x)||_{p,W} &\leqslant \left(\int_{\mathbb{R}} \left| W(u) P_r''(f,u) \right|^p du \right)^{\frac{1}{p}} \left(\int_{\frac{-h}{2}}^{\frac{h}{2}} \int_{\frac{-h}{2}}^{\frac{h}{2}} \frac{1}{W(t_1+t_2)} dt_1 dt_2 \right) \\
&= ||P_r''(f,x)||_{p,W} \left(\int_0^{\frac{h}{2}} \int_0^{\frac{h}{2}} \frac{1}{W(t_1+t_2)} dt_1 dt_2 \right) \\
&\leqslant \frac{1}{W(h)} h^2 ||P_r''(f,x)||_{p,W} \\
&\leqslant r^{-3} L\left(\frac{1}{r}\right) h^2 \frac{1}{W(h)} ||f||_{p,W},
\end{aligned}$$

in view of (27) for $n=2$.

Similary, for $p=\infty$, we have

$$\begin{aligned}||\Delta_h^2 P_r(f,x)||_{\infty,W} &= \sup_{x\in\mathbb{R}}\left(W(x)|\Delta_h^2 P_r(f,x)|\right)\\ &= \sup_{x\in\mathbb{R}}\left(W(x)\left|\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-h}{2}}^{\frac{h}{2}} P_r''(f,x+t_1+t_2)dt_1dt_2\right|\right)\\ &\leqslant \left(\sup_{u\in\mathbb{R}} W(u)\left|P_r''(f,u)\right|\right)\left(\int_{\frac{-h}{2}}^{\frac{h}{2}}\int_{\frac{-h}{2}}^{\frac{h}{2}}\sup_{u\in\mathbb{R}}\left(\frac{W(u-t_1-t_2)}{W(u)}\right)dt_1dt_2\right)\\ &\leqslant ||P_r''(f,x)||_{\infty,W}\int_0^{\frac{h}{2}}\int_0^{\frac{h}{2}}\frac{1}{W(t_1+t_2)}dt_1dt_2\\ &\leqslant \frac{1}{W(h)}h^2||P_r''(f,x)||_{\infty,W}\\ &\leqslant r^{-3}L\left(\frac{1}{r}\right)h^2\frac{1}{W(h)}||f||_{\infty,W}.\end{aligned}$$

Hence proof is completed. □

Proof of Theorem 3.1. We can find two natural numbers m and n such that $0<\frac{1}{2^n}<\frac{1}{2^m}\leqslant s_0$. For all $x,h\in\mathbb{R}$ and $f\in L^{p,W}$, we can write

$$\Delta_h^2 f(x)=\Delta_h^2 P_{2^{-m}}(f,x)+\sum_{i=m}^{n-1}\Delta_h^2\left(P_{2^{-i-1}}(f,x)-P_{2^{-i}}(f,x)\right)+\Delta_h^2\left(f(x)-P_{2^{-n}}(f,x)\right). \tag{30}$$

Using Lemma 3.3, we get

$$||\Delta_h^2 P_{2^{-m}}(f,x)||_{p,W}\leqslant L\left(\tfrac{1}{2^{-m}}\right)2^{3m}h^2\tfrac{1}{W(h)}||f||_{p,W}.$$

Now

$$\Delta_h^2\left(P_{2^{-i-1}}(f,x)-P_{2^{-i}}(f,x)\right)=\Delta_h^2 P_{2^{-i-1}}\left(f-P_{2^{-i}}(f),x\right)+\Delta_h^2 P_{2^{-i}}\left(P_{2^{-i-1}}(f)-f,x\right),$$

so that

$$\begin{aligned}||\Delta_h^2\left(P_{2^{-i-1}}(f,x)-P_{2^{-i}}(f,x)\right)||_{p,W} \leqslant{}& ||\Delta_h^2 P_{2^{-i-1}}\left(f-P_{2^{-i}}(f),x\right)||_{p,W}\\ &+||\Delta_h^2 P_{2^{-i}}\left(P_{2^{-i-1}}(f)-f,x\right)||_{p,W}.\end{aligned} \tag{31}$$

Using Lemma 3.3, we get

$$\begin{aligned}||\Delta_h^2\left(P_{2^{-i-1}}(f,x)-P_{2^{-i}}(f,x)\right)||_{p,W}\leqslant{}& 2^{3i+3}L\left(\frac{1}{2^{-i-1}}\right)h^2\frac{1}{W(h)}||f-P_{2^{-i}}(f)||_{p,W}\\ &+2^{3i}h^2L\left(\frac{1}{2^{-i}}\right)||P_{2^{-i-1}}(f)-f||_{p,W}\frac{1}{W(h)}\\ \leqslant{}& \left(L\left(\frac{1}{2^{-i}}\right)+L\left(\frac{1}{2^{-i-1}}\right)\right)\frac{1}{W(h)}h^2 2^{3i+3}\omega(2^{-i}).\end{aligned} \tag{32}$$

Now from Lemma 2.7 and (26), we have

$$
\begin{aligned}
&||\Delta_h^2(f(x)-P_{2^{-n}}(f,x))||_{p,W} \\
&\leqslant \|f(x+h)-P_{2^{-n}}(f,x+h)\|_{p,W}+\|f(x-h)-P_{2^{-n}}(f,x-h)\|_{p,W} \\
&\quad -2\|f(x)-P_{2^{-n}}(f,x)\|_{p,W} \\
&\leqslant \frac{1}{W(h)}\|f(x)-P_{2^{-n}}(f,x)\|_{p,W}+\frac{1}{W(-h)}\|f(x)-P_{2^{-n}}(f,x)\|_{p,W} \\
&\quad +2\omega(2^{-n}) \\
&\leqslant \frac{1}{W(h)}\omega(2^{-n})+\frac{1}{W(-h)}\omega(2^{-n})+2\omega(2^{-n}) \\
&\leqslant 2\left(\frac{1}{W(h)}+1\right)\omega(2^{-n}).
\end{aligned} \tag{33}
$$

Using (30) to (33), we get

$$
||\Delta_h^2(f(x)||_{p,W} \leqslant \frac{1}{W(h)}\left(2^{3m+3}Kh^2||f||_{p,W}+h^2K\sum_{i=m}^{n-1}2^{3i+3}\omega(2^{-i})+K'\omega(2^{-n})\right), \tag{34}
$$

where $K=\max\limits_{m\leqslant i\leqslant n-1}\left\{L\left(\frac{1}{2^{-m}}\right),L\left(\frac{1}{2^{-i}}\right)+L\left(\frac{1}{2^{-i-1}}\right)\right\}$ and $K'=1+W(0)$.

If $t\in\left(0,\frac{1}{2}\right)\cap(0,s_0]$, $|h|\leqslant t$, $m<n$ and n be a natural number such that $2^{-n}\leqslant t<2^{-n+1}$, then from (34), we obtain

$$
\omega_2(f,L^{p,W},t)\leqslant d\left(t^2+t^2\sum_{i=m}^{n-1}2^{3i+3}\omega(2^{-i})+\omega(t)\right), \tag{35}
$$

where $d=\max\limits_{2^{-n}\leqslant t<2^{-n+1}}\frac{1}{W(t)}\left\{2^{3m+3}K||f||_{p,W},K,K'\right\}$.

Since $\frac{\omega(x)}{x^3}$ is non-increasing for $x>0$, we can write

$$
\sum_{i=m}^{n-1}2^{3i}\omega(2^{-i})\leqslant\frac{1}{ln2}\int_t^1\frac{\omega(x)}{x^4}dx. \tag{36}
$$

It is easy to find constants c_1 and c_2 such that

$$
c_1t^2\leqslant\omega(t)\leqslant c_2t^2\int_t^1\frac{\omega(x)}{x^4}dx, \tag{37}
$$

for all $t\in\left(0,\frac{1}{2}\right)\cap(0,s_0]$ and $\omega\in\Omega^2$.

Collecting (35), (36) and (37), we obtain our result, where $c=d\left(1+\frac{8}{ln2}+c_2\right)$. □

REMARK. Theorem 2 of [[8], p. 239] can also be derived similarly by using $W(x)=e^{-q|x|}$, $q>0$ and $L\left(\frac{1}{r}\right)=\frac{r}{1-rq}$.

Declarations

Conflict of interest. The authors declare that they have no conflict of interest.

Acknowledgement. Authors are grateful to the reviewer for his valuable comments for the improvement of the paper.

REFERENCES

[1] O. AGRATINI, A. ARAL, E. DENIZ, *On two classes of approximation process of integral type*, Positivity, **3** (21), 1189–1199, 2017.

[2] A. ARAL, B. YILMAZ, E. DENIZ, *Weighted approximation by modified Picard operators*, Positivity, **24**, 427–439, 2020.

[3] C. BARDARO, I. MANTELLINI, G. UYSAL AND B. YILMAZ, *A class of integral operators that fix exponential functions*, Mediterr. J. Math. **18**: 179, 2021.

[4] K. BOGALSKA, E. GOJTKA, M. GURDEK, L. REMPULSKA, *The Picard and the Gauss-Weierstrass singular integrals of function of two variables*, Le Mathematiche, 71–84, 1997.

[5] B. FIRLEJY, L. REMPULSKA, *On some singular integrals in Hölder spaces*, Math. Nachr. 93–100, 1994.

[6] S. G. GAL, *Degree of approximation of continuous functions by some singular integrals*, Revue d'Analyse Numërique et de Théorie de l' approximation, **27** (2), 251–261, 1998.

[7] S. G. GAL, *Remark on the degree of approximation of continuous functions by singular integrals*, Math. Nachr. 197–199, 1993.

[8] A. LESNIEWICZ, L. REMPULSKA AND J. WASIAK, *Approximation properties of the Picard singular integral in exponential weighted spaces*, Matematiques. **40** (2), 233–242, 1996.

[9] R. N. MOHAPATRA AND R. S. RODRIGUEZ, *On the rate of convergence of singular integrals for Hölder continuous functions*, Math. Nachr. **149** (1), 117–124, 1990.

[10] L. REMPULSKA AND Z. WALCZAK, *On modified Picard and Gauss-Weierstrass singular integrals*, Ukrain. Math. J. **57** (11), 1844–1852, 2005.

[11] B. YILMAZ, *On some approximation properties of the Gauss-Weierstrass operators*, Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat, **68** (2), 2154–2160, 2019.

[12] M. M YILMAZ, G. UYSAL, *Convergence of singular operators in weighted Lebesgue spaces*, Eur. J. Pure Appl. Math. **10** (2), 335–347, 2017.

(Received January 8, 2023)

Abhay Pratap Singh
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, India
e-mail: apratapsingh@ma.iitr.ac.in

Uaday Singh
Department of Mathematics
Indian Institute of Technology Roorkee
Roorkee-247667, India
e-mail: uadayfma@iitr.ac.in