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INEQUALITIES FOR COMMUTATORS OF FRACTIONAL INTEGRALS AND SINGULAR INTEGRALS ON DIFFERENTIAL FORMS

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Abstract. In this paper, we define the commutators of fractional integral operators and Calderón-Zygmund singular integral operators on differential forms, and give the sufficient and necessary conditions for these commutators to be bounded on weighted Lebesgue spaces. As an application, the Caccioppoli-type inequalities with Orlicz norm for commutators of Calderón-Zygmund singular integral operators on differential forms are obtained.

1. Introduction

The purpose of this paper is to give the sufficient and necessary conditions for the L^p-boundedness of commutators of two types of singular integral operators acting on differential forms which include the fractional integral operators and Calderón-Zygmund singular integral operators for functions as special cases. Meanwhile, we establish some related norm inequalities for these commutators.

Given α, $0<\alpha<n$, the fractional integral operator I_α on differential forms is defined by

$$I_\alpha u(x)=\sum_I\left(\int_{\mathbb{R}^n}\frac{1}{|x-y|^{n-\alpha}}u_I(y)dy\right)dx_I, \tag{1.1}$$

where $u(x)$ is a differential l-form defined on $\mathbb{R}^n$ and the summation is over all ordered l-tuples $I=(i_1,i_2,\cdots,i_l)$, $1\leqslant i_1<\cdots<i_l\leqslant n$.

Similarly, the Calderón-Zygmund singular integral operator T on differential forms is defined by

$$Tu(x)=\sum_I\left(\int_{\mathbb{R}^n}\frac{\Omega(x-y)}{|x-y|^n}u_I(y)dy\right)dx_I, \tag{1.2}$$

where $\Omega(x)$ is defined on S^{n-1}, has mean 0, and is sufficiently smooth.

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If $b \in BMO(\mathbb{R}^n)$, the commutators of fractional integral operator and Calderón-Zygmund singular integral operator on differential forms are of the forms

$$\begin{aligned}[b,I_\alpha]u(x) &= b(x)I_\alpha u(x) - I_\alpha(bu)(x) \\ &= \sum_I \left(\int_{\mathbb{R}^n} (b(x)-b(y)) \frac{1}{|x-y|^{n-\alpha}} u_I(y)dy \right) dx_I \end{aligned} \tag{1.3}$$

and

$$\begin{aligned}[b,T]u(x) &= b(x)Tu(x) - T(bu)(x) \\ &= \sum_I \left(\int_{\mathbb{R}^n} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^n} u_I(y)dy \right) dx_I. \end{aligned} \tag{1.4}$$

As we know, differential forms are the generalizations of functions and are often used to describe various PDE systems, different geometric structures and the calculus of variations. People can find many important applications of differential forms in numerous fields, such as quasiconformal analysis, nonlinear elasticity and differential geometry, see [1, 8, 12, 11, 2, 14, 18], for example. When taking $u(x)$ as a 0-form, the commutators in (1.3) and (1.4) reduce to the corresponding operators in function spaces as follows

$$[b,I_\alpha]f(x) = \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{1}{|x-y|^{n-\alpha}} f(y)dy \tag{1.5}$$

and

$$[b,T]f(x) = \int_{\mathbb{R}^n} (b(x)-b(y)) \frac{\Omega(x-y)}{|x-y|^n} f(y)dy. \tag{1.6}$$

It is well known that these commutators are quite important operators in harmonic analysis and have the higher degree of singularity compared with the standard singular integral operators. In the past few decades, the research on commutators have attracted great attention of many scholars for their significant roles in many fields, especially in partial differential equations, see [5, 6, 10, 19] for details. The L^p-boundedness of commutators was first investigated by R. Coifman, R. Rochberg and G. Weiss in the study of certain fractorization theorems for generalized Hardy spaces in [7], where they showed a classical result that the linear commutator $[b,T]$ is bounded on $L^p(\mathbb{R}^n)$ for $1<p<\infty$. They also proved that the condition $b \in BMO(\mathbb{R}^n)$ is a necessary condition when $T=R_j$, the jth Riesz transform in $\mathbb{R}^n$, for $j=1,\ldots,n$. Since then, the study on the boundedness of commutators in various function spaces have rapidly developed and many articles have appeared.

In this paper, we are interested in the boundedness of commutators on weighted Lebesgue spaces. For the functional cases, D. Cruz-Uribe and A. Fiorenza [4] proved that the commutator of fractional integral operator is bounded from $L^p(w^p)$ into $L^q(w^q)$ if $w \in A_{p,q}$ and $b \in BMO(\mathbb{R}^n)$. The analogous result of A_p-weight for any Calderón-Zygmund singular integral operator was stated by C. Pérez in [16]. Recently, L. Chaffee and D. Cruz-Uribe [3] gave the necessary conditions for the boundedness of commutators on Banach function spaces which are satisfied by the commutators (1.5) and (1.6) and the weighted function spaces we mentioned above. Inspired by these works, we aim to give the weighted results for these commutators on differential forms. To be precise, we prove that $b \in BMO(\mathbb{R}^n)$ is a sufficient and necessary condition for the commutators of fractional integral operators and Calderón-Zygmund singular integral operators on differential forms to be bounded on weighted Lebesgue spaces. See Theorem 2.5 and Theorem 2.7, respectively. As an application of the estimate of boudedness, we establish the Caccioppoli-type inequality for commutator of Calderón-Zygmund singular integral operator on differential forms and extend it to the version of $A(\alpha,\beta,\gamma;E)$-weight. Finally, we generally derive the Caccioppoli-type inequality with Orlicz norm and present the related conclusions in Sect. 3. The results obtained in this paper will supplement and improve the study of the L^p-theory of the related operators and differential forms.

Throughout of this paper, let $\Omega \subset \mathbb{R}^n$ be a bounded domain, $n \geqslant 2$, B and σB be the balls with the same center and $\operatorname{diam}(\sigma B) = \sigma \operatorname{diam}(B)$. We use $|E|$ to denote the Lebesgue measure of a set $E \subset \mathbb{R}^n$. Let $\Lambda^l = \Lambda^l(\mathbb{R}^n)$, $l = 1,2,\cdots,n$, be the set of all l-forms $u(x) = \sum_I u_I(x)dx_I = \sum u_{i_1\cdots i_l}(x)dx_{i_1} \wedge \cdots \wedge dx_{i_l}$ with summation over all ordered l-tuples $I = (i_1,i_2,\cdots,i_l)$, $1 \leqslant i_1 < \cdots < i_l \leqslant n$. $D'(\Omega,\Lambda^l)$ is the space of all differential l-forms on Ω, namely, the coefficients of the l-forms are differentiable on Ω. The operator $\star : \Lambda^l(\mathbb{R}^n) \to \Lambda^{n-l}(\mathbb{R}^n)$ is the Hodge-star operator as usual and the linear operator $d : D'(\Omega,\Lambda^l) \to D'(\Omega,\Lambda^{l+1})$, $0 \leqslant l \leqslant n-1$ is the exterior differential operator. The Hodge codifferential operator $d^\star : D'(\Omega,\Lambda^{l+1}) \to D'(\Omega,\Lambda^l)$, the formal adjoint of d, is defined by $d^\star = (-1)^{nl+1} \star d \star$, see [17, 15] for more introduction. We shall denote by $L^p(\Omega,\Lambda^l)$ the space of differential l-forms with coefficients in $L^p(\Omega,\mathbb{R}^n)$ and with norm $\|u\|_{p,\Omega} = \left(\int_\Omega \left(\sum_I |u_I(x)|^2\right)^{\frac{p}{2}} dx\right)^{\frac{1}{p}}$. A nonnegative function w is called a weight if $w \in L^1_{loc}(\mathbb{R}^n)$ and $w > 0$ $a.e$. Also the norm of $u \in L^p(\Omega,\Lambda^l,w)$ is defined by $\|u\|_{p,\Omega,w} = \left(\int_\Omega |u|^p w(x)dx\right)^{1/p}$. The non-homogeneous A-harmonic equation is of the form

$$d^\star A(x,du) = B(x,du), \tag{1.7}$$

where $A : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^l(\mathbb{R}^n)$ and $B : \Omega \times \Lambda^l(\mathbb{R}^n) \to \Lambda^{l-1}(\mathbb{R}^n)$ satisfy the following conditions:

$$|A(x,\xi)| \leqslant a|\xi|^{p-1},\ A(x,\xi)\cdot\xi \geqslant |\xi|^p \ \text{ and } \ |B(x,\xi)| \leqslant b|\xi|^{p-1}$$

for almost every $x \in \Omega$ and all $\xi \in \Lambda^l(\mathbb{R}^n)$. Here $p > 1$ is a constant related to the equation (1.7), and $a,b > 0$. A solution of (1.7) is an element of the Sobolev space $W^{1,p}_{loc}(\Omega,\Lambda^{l-1})$ such that

$$\int_\Omega A(x,du)\cdot d\varphi + B(x,du)\cdot\varphi = 0$$

for all $\varphi \in W^{1,p}_{loc}(\Omega,\Lambda^{l-1})$ with compact support.

2. Boundedness for commutators of fractional integrals and singular integrals

In this section, we give the sufficient and necessary conditions for the L^p-boundedness of two types of commutators generated by fractional integral operators and Calderón-Zygmund singular integral operators applied to differential forms. We also derive some strong-type estimates for these commutators. To state our results, we will need the following two lemmas appearing in [4] and [16], respectively.

LEMMA 2.1. *Given α, $0<\alpha<n$ and $1<p<\frac{n}{\alpha}$, define q by $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let w be a weight satisfying $A_{p,q}$ condition*

$$\Big(\frac{1}{|Q|}\int_Q w(x)^q dx\Big)^{\frac{1}{q}}\Big(\frac{1}{|Q|}\int_Q w(x)^{-p'}dx\Big)^{\frac{1}{p'}}\leqslant C_1<\infty, \tag{2.1}$$

for all cubes Q, where C_1 is a constant and $\frac{1}{p}+\frac{1}{p'}=1$. Then, given any function $b\in BMO(\mathbb{R}^n)$, $[b,I_\alpha]$ satisfies the following inequality

$$\Big(\int_{\mathbb{R}^n}|[b,I_\alpha]f(x)|^q w(x)^q dx\Big)^{\frac{1}{q}}\leqslant C_2\|b\|_{BMO}\Big(\int_{\mathbb{R}^n}|f(x)|^p w(x)^p dx\Big)^{\frac{1}{p}} \tag{2.2}$$

for some constant C_2.

LEMMA 2.2. *Let w be a weight satisfying A_p condition: for all cubes Q, if there is a constant C such that*

$$\Big(\frac{1}{|Q|}\int_Q w(x)dx\Big)\Big(\frac{1}{|Q|}\int_Q w(x)^{1-p'}dx\Big)^{p-1}\leqslant C<\infty, \tag{2.3}$$

where $1<p<\infty$ and $\frac{1}{p}+\frac{1}{p'}=1$. T is any Calderón-Zygmund singular integral operator. Then, given any function $b\in BMO(\mathbb{R}^n)$, $[b,T]$ satisfies the following inequality

$$\Big(\int_{\mathbb{R}^n}|[b,T]f(x)|^p w(x)dx\Big)^{\frac{1}{p}}\leqslant C\|b\|_{BMO}\Big(\int_{\mathbb{R}^n}|f(x)|^p w(x)dx\Big)^{\frac{1}{p}}. \tag{2.4}$$

We also need the following conclusions which are given in [3].

LEMMA 2.3. *Given $0<\alpha<n$ and $1<p<\frac{n}{\alpha}$, define q by $\frac{1}{p}-\frac{1}{q}=\frac{\alpha}{n}$. Given $w\in A_{p,q}$ and a function b, if the commutator $[b,I_\alpha]:L^p(w^p)\to L^q(w^q)$, then $b\in BMO(\mathbb{R}^n)$.*

LEMMA 2.4. *For $1<p<\infty$ and $w\in A_p$, given a regular singular integral operator T and a function b, if $[b,T]$ is bounded on $L^p(w)$, then $b\in BMO(\mathbb{R}^n)$.*

Now, we are ready to give the sufficient and necessary condition for the boundedness of commutators of fractional integral operators on differential forms.

THEOREM 2.5. *Given $0<\alpha<n$ and $1<p<\frac{n}{\alpha}$, define q by $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let $w\in A_{p,q}$, $u(x)\in L^p(\Lambda^l,w^p)$ be a differential l-form, $l=0,1,\cdots,n$, I_α be the fractional integral operator on differential forms. Then, given any function $b\in BMO(\mathbb{R}^n)$, there exists a constant C, independent of u, such that*

$$\|[b,I_\alpha]u\|_{q,w^q}\leqslant C\|b\|_{BMO}\|u\|_{p,w^p}, \tag{2.5}$$

that is $[b,I_\alpha]:L^p(\Lambda^l,w^p)\to L^q(\Lambda^l,w^q)$. Conversely, given a function b, if $[b,I_\alpha]:L^p(\Lambda^l,w^p)\to L^q(\Lambda^l,w^q)$, then $b\in BMO(\mathbb{R}^n)$.

Proof. First, we prove the sufficiency. Let $u=\sum\limits_I u_I(x)dx_I$ be a differential l-form, then

$$\begin{aligned}\|[b,I_\alpha]u\|_{q,w^q}^q&=\int_{\mathbb{R}^n}|[b,I_\alpha]u(x)|^q w(x)^q dx\\&=\int_{\mathbb{R}^n}\left|\sum_I\left(\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}u_I(y)dy\right)^2\right|^{\frac{q}{2}}w(x)^q dx.\end{aligned} \tag{2.6}$$

Using the elementary inequality $|\sum_{i=1}^N t_i|^s\leqslant N^{s-1}\sum_{i=1}^N|t_i|^s$, for constants $N,s>0$, it follows that

$$\begin{aligned}\|[b,I_\alpha]u\|_{q,w^q}^q&\leqslant\int_{\mathbb{R}^n}C_1\sum_I\left|\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}u_I(y)dy\right|^q w(x)^q dx\\&=C_1\sum_I\int_{\mathbb{R}^n}\left|\int_{\mathbb{R}^n}\frac{b(x)-b(y)}{|x-y|^{n-\alpha}}u_I(y)dy\right|^q w(x)^q dx\\&=C_1\sum_I\int_{\mathbb{R}^n}|[b,I_\alpha]u_I(x)|^q w(x)^q dx.\end{aligned} \tag{2.7}$$

From Lemma 2.1, we have

$$\left(\int_{\mathbb{R}^n}|[b,I_\alpha]u_I(x)|^q w(x)^q dx\right)^{\frac{1}{q}}\leqslant C_2\|b\|_{BMO}\left(\int_{\mathbb{R}^n}|u_I(x)|^p w(x)^p dx\right)^{\frac{1}{p}}. \tag{2.8}$$

The combination of (2.7) and (2.8) yields that

$$\|[b,I_\alpha]u\|_{q,w^q}^q\leqslant C_3\|b\|_{BMO}^q\sum_I\left(\int_{\mathbb{R}^n}|u_I(x)|^p w(x)^p dx\right)^{\frac{q}{p}}. \tag{2.9}$$

Note that $\frac{q}{p}>1$, since $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Thus, applying the fundamental inequality $\|a\|^s+\|b\|^s\leqslant(\|a\|+\|b\|)^s$, $s\geqslant 1$, it follows that

$$\begin{aligned}\|[b,I_\alpha]u\|_{q,w^q}^q&\leqslant C_3\|b\|_{BMO}^q\sum_I\left(\int_{\mathbb{R}^n}|u_I(x)|^p w(x)^p dx\right)^{\frac{q}{p}}\\&\leqslant C_4\|b\|_{BMO}^q\left(\sum_I\int_{\mathbb{R}^n}|u_I(x)|^p w(x)^p dx\right)^{\frac{q}{p}}\end{aligned}$$

$$= C_4\|b\|_{BMO}^q\left(\int_{\mathbb{R}^n}\sum_I |u_I(x)|^p w(x)^p dx\right)^{\frac{q}{p}}$$

$$\leqslant C_5\|b\|_{BMO}^q\left(\int_{\mathbb{R}^n}\left(\sum_I |u_I(x)|\right)^p w(x)^p dx\right)^{\frac{q}{p}}. \tag{2.10}$$

Using the inequality $|\sum_{i=1}^N t_i|^s \leqslant N^{s-1}\sum_{i=1}^N |t_i|^s$ again, we can easily have

$$\left(\sum_I |u_I(x)|\right)^2 \leqslant 4\sum_I |u_I(x)|^2. \tag{2.11}$$

Substituting (2.11) into (2.10) gives

$$\begin{aligned}\|[b,I_\alpha]u\|_{q,w^q}^q &\leqslant C_6\|b\|_{BMO}^q\left(\int_{\mathbb{R}^n}\left(\sum_I |u_I(x)|^2\right)^{\frac{p}{2}} w(x)^p dx\right)^{\frac{q}{p}}\\ &= C_6\|b\|_{BMO}^q\left(\int_{\mathbb{R}^n}|u(x)|^p w(x)^p dx\right)^{\frac{q}{p}}\\ &= C_6\|b\|_{BMO}^q\|u\|_{p,w^p}^q.\end{aligned} \tag{2.12}$$

That is, $\|[b,I_\alpha]u\|_{q,w^q} \leqslant C\|b\|_{BMO}\|u\|_{p,w^p}$. This completes the proof of the sufficiency.

Next, we prove the necessity. According to the assumption, we have $[b,I_\alpha]: L^p(\Lambda^l,w^p)\to L^q(\Lambda^l,w^q)$ for any differential l-form, $l=0,1,\cdots,n$. Therefore, we could select $u(x)$ as the special case 0-form, then we obtain

$$[b,I_\alpha]: L^p(w^p)\to L^q(w^q). \tag{2.13}$$

From Lemma 2.3, we have $b\in BMO(\mathbb{R}^n)$, which completes the proof of Theorem 2.5. □

Selecting $w(x)=1$ in Lemma 2.1 and restricting f in $L^p(\Omega)$, we obviously have the following inequality for $[b,I_\alpha]$

$$\|[b,I_\alpha]f\|_{q,\Omega} \leqslant C\|b\|_{BMO}\|f\|_{p,\Omega}, \tag{2.14}$$

which shows that the local strong type (p,q) estimate holds for functions. Starting with (2.14) and using the same method as we did in the proof of Theorem 2.5, we can prove the following theorem.

THEOREM 2.6. *Given $0<\alpha<n$ and $1<p<\frac{n}{\alpha}$, define q by $\frac{1}{q}=\frac{1}{p}-\frac{\alpha}{n}$. Let $u(x)\in L^p(\Omega,\Lambda^l)$, $l=0,1,\cdots,n$, I_α be the fractional integral operator on differential forms. Then, given any function $b\in BMO(\mathbb{R}^n)$, $[b,I_\alpha]$ satisfies the strong (p,q) inequality*

$$\|[b,I_\alpha]u\|_{q,\Omega} \leqslant C\|b\|_{BMO}\|u\|_{p,\Omega} \tag{2.15}$$

for any Ω with $\Omega\subset\mathbb{R}^n$.

Analogously, we can obtain the following boundedness results for commutators of Calderón-Zygmund singular integral operators on differential forms which are based on Lemma 2.2 and Lemma 2.4 and the approach developed in Theorem 2.5. The precise proofs could be got by a simple adaptation of the proof of Theorem 2.5, thus we omit it here.

THEOREM 2.7. *Let $u(x) \in L^p(\Lambda^l, w)$, $l = 0, 1, \cdots, n$, $1 < p < \infty$, $w \in A_p$, T be a Calderón-Zygmund singular integral operator on differential forms. Then, given any function $b \in BMO(\mathbb{R}^n)$, there exists a constant C, independent of u, such that*

$$\|[b,T]u\|_{p,w} \leqslant C\|b\|_{BMO}\|u\|_{p,w}, \tag{2.16}$$

that is $[b,T] : L^p(\Lambda^l, w) \to L^p(\Lambda^l, w)$. Conversely, given a function b, if $[b,T] : L^p(\Lambda^l, w) \to L^p(\Lambda^l, w)$, then $b \in BMO(\mathbb{R}^n)$.

THEOREM 2.8. *Let $u \in L^p(\Omega, \Lambda^l)$, $l = 0, 1, \cdots, n$, $1 < p < \infty$, T be a Calderón-Zygmund singular integral operator on differential forms. Then, given any function $b \in BMO(\mathbb{R}^n)$, $[b,T]$ satisfies the strong (p,p) inequality*

$$\|[b,T]u\|_{p,\Omega} \leqslant C\|b\|_{BMO}\|u\|_{p,\Omega} \tag{2.17}$$

for any Ω with $\Omega \subset \mathbb{R}^n$.

REMARK 2.1. We should note that the integral operators defined on differential forms in this paper could be extended to the bilinear cases as follows,

$$I_\alpha(u,v)(x) = \sum_I \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{u_I(y_1)v_I(y_2)}{(|x-y_1|+|x-y_2|)^{2n-\alpha}} dy_1 dy_2 \right) dx_I, \tag{2.18}$$

where $0 < \alpha < 2n$ and

$$T(u,v)(x) = \sum_I \left(\int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \frac{\Omega(x-y_1, x-y_2)}{|(x-y_1, x-y_2)|^n} u_I(y_1)v_I(y_2) dy_1 dy_2 \right) dx_I, \tag{2.19}$$

where $\Omega(x)$ is defined on S^{2n-1}, has mean 0, and is sufficiently smooth. For any bilinear operator L on differential forms and $b \in L^1_{loc}(\mathbb{R}^n)$, the bilinear commutators on differential forms are defined as

$$[b,L]_1(u,v)(x) = b(x)L(u,v)(x) - L(bu,v)(x)$$

and

$$[b,L]_2(u,v)(x) = b(x)L(u,v)(x) - L(u,bv)(x).$$

It is worth pointing out the techniques developed in this section provide an effective mean to study the bilinear commutators of singular and fractional integrals on differential forms, which are more complicated than the linear cases. In a similar way, the sufficient and necessary conditions for the boundedness of the bilinear commutators of fractional and singular integrals on differential forms can be deduced by using the weighted norm inequalities of the bilinear commutators on the function spaces and the related corollaries in [3]. We leave the statements and proofs to the interested readers.

3. Caccioppoli-type inequalities for commutators of Calderón-Zygmund singular integrals

In this section, we mainly establish the Caccioppoli-type inequality with $L^{\varphi}(\Omega,\mu)$-norm for commutators of Calderón-Zygmund singular integral operators applied to the solution of the non-homogeneous A-harmonic equation. Before stating our results, we first recall a weight class introduced in [20] and the Orlicz norm.

We say a measurable function $w(x)$ defined on a subset $E \subset \mathbb{R}^n$ satisfies the $A(\alpha,\beta,\gamma;E)$ condition for some positive constants α,β,γ; write $w(x) \in A(\alpha,\beta,\gamma;E)$ if $w(x) > 0$ a.e., and

$$\sup_B \left(\frac{1}{|B|}\int_B w^{\alpha}dx\right)\left(\frac{1}{|B|}\int_B w^{-\beta}dx\right)^{\gamma/\beta} < \infty,$$

where the supremum is over all balls $B \subset E$. An Orlicz function is a continuously increasing function $\varphi : [0,\infty) \to [0,\infty)$ with $\varphi(0) = 0$. The Orlicz space $L^{\varphi}(\Omega,\mu)$ consists of all measurable functions f on Ω such that $\int_{\Omega}\varphi\left(\frac{|f|}{\lambda}\right)d\mu < \infty$ for some $\lambda = \lambda(f) > 0$. $L^{\varphi}(\Omega,\mu)$ is equipped with the nonlinear Luxemburg functional

$$\|f\|_{L^{\varphi}(\Omega,\mu)} = \inf\{\lambda > 0 : \int_{\Omega}\varphi\left(\frac{|f|}{\lambda}\right)d\mu \leqslant 1\},$$

where the Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\gamma;\Omega)$. A convex Orlicz function φ is often called a Young function. If φ is a Young function, then $\|\cdot\|_{L^{\varphi}(\Omega,\mu)}$ defines a norm in $L^{\varphi}(\Omega,\mu)$, which is called the Luxemburg norm or Orlicz norm.

The following subclass of Young functions and the related property in Lemma 3.1 are given in [13].

DEFINITION 3.1. A Young function $\varphi : [0,\infty) \longrightarrow [0,\infty)$ is said to be in the class $NG(p,q)$ if φ satisfies the nonstandard growth condition

$$p\varphi(t) \leqslant t\varphi'(t) \leqslant q\varphi(t),\ 1 < p \leqslant q < \infty. \tag{3.1}$$

The first inequality in (3.1) is equivalent to that $\frac{\varphi(t)}{t^p}$ is increasing, and the second inequality in (3.1) is equivalent to $\triangle_2$-condition, i.e., for each $t > 0$, $\varphi(2t) \leqslant K\varphi(t)$, where $K > 1$, and $\frac{\varphi(t)}{t^q}$ is decreasing with t.

LEMMA 3.1. *Suppose φ is a continuous function in the class $NG(p,q)$, $1 < p \leqslant q < \infty$. For any $t > 0$, setting*

$$A(t) = \int_0^t \left(\frac{\varphi(s^{1/q})}{s}\right)^{\frac{n+q}{q}}ds,\ \ K(t) = \frac{\left(\varphi(t^{1/q})\right)^{\frac{n+q}{q}}}{t^{n/q}}. \tag{3.2}$$

Then, $A(t)$ is a concave function, and there exists a constant C, such that

$$K(t) \leqslant A(t) \leqslant CK(t),\ \ \forall t > 0. \tag{3.3}$$

We need the following two lemmas appearing in [8] and [9], respectively.

LEMMA 3.2. *Let $u \in D'(\Omega,\Lambda^l)$, $l = 0,1,\cdots,n-1$, be a solution of the non-homogeneous A-harmonic equation (1.7) in a domain Ω. Then, there exists a constant C, independent of u, such that*

$$\|du\|_{p,B} \leqslant C|B|^{-1/n}\|u-c\|_{p,B} \tag{3.4}$$

for all balls B with $B \subset \Omega$ and all closed forms. Here $1 < p < \infty$.

LEMMA 3.3. *Let u be a solution of the non-homogeneous A-harmonic equation (1.7) in a domain Ω and $0 < s,t < \infty$. Then, there exists a constant C, independent of u, such that*

$$\|du\|_{s,B} \leqslant C|B|^{(t-s)/st}\|du\|_{t,\sigma B} \tag{3.5}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$.

In [21], the results in Lemma 3.2 and Lemma 3.3 were extended to the versions of $A(\alpha,\beta,\alpha;\Omega)$-weight as follows.

LEMMA 3.4. *Let $u \in L^s_{loc}(\Omega,\Lambda^l)$ be a solution of the non-homogeneous A-harmonic equation (1.7) in a bounded domain Ω, $l = 0,1,\cdots,n-1$ and $\frac{1+\beta}{\beta} < p < \infty$, Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^s d\mu\right)^{1/s} \leqslant C|B|^{-1/n}\left(\int_{\sigma B} |u-c|^s d\mu\right)^{1/s} \tag{3.6}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$ and any closed form c, where the Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

LEMMA 3.5. *Let u be a solution of the non-homogeneous A-harmonic equation (1.7) in a domain Ω, $0 < p,q < \infty$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |du|^q d\mu\right)^{1/q} \leqslant C(\mu(B))^{(p-q)/pq}\left(\int_{\sigma B} |du|^p d\mu\right)^{1/p} \tag{3.7}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$, where the Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

Combining Theorem 2.8 and Lemma 3.2, we can easily give the following Caccioppoli-type inequality for $[b,T]$.

THEOREM 3.6. *Let u be a solution of the non-homogeneous A-harmonic equation (1.7) in Ω and $du \in L^p(\Omega,\Lambda^{l+1})$, $1 < p < \infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b \in BMO(\mathbb{R}^n)$. Then, there exists a constant C, independent of u, such that*

$$\|[b,T](du)\|_{p,B} \leqslant C|B|^{-1/n}\|u-c\|_{p,B} \tag{3.8}$$

for all balls B with $B \subset \Omega$ and all closed forms c.

We now extend the Theorem 3.6 to the case of $A(\alpha,\beta,\gamma;\Omega)$-weight, we begin with the following lemma.

LEMMA 3.7. *Let $u \in L^p_{loc}(\Omega,\Lambda^l)$ be a solution of the non-homogeneous A-harmonic equation (1.7) in Ω, $l = 0,1,\cdots,n-1$, $1 < p < \infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b \in BMO(\mathbb{R}^n)$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |[b,T](du)|^p d\mu\right)^{1/p} \leqslant C\left(\int_{\sigma B} |du|^p d\mu\right)^{1/p} \tag{3.9}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$, where Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

Proof. Taking $s = \alpha p/(\alpha-1)$ and $t = \beta p/(1+\beta)$, then we have $s > p > t > 0$. Applying Lemma 3.3 to $s,t > 0$ yields

$$\|du\|_{s,B} \leqslant C_1 |B|^{\frac{t-s}{ts}} \|du\|_{t,\sigma_1 B}, \tag{3.10}$$

where $\sigma_1 > 1$ is a constant. Using the Hölder inequality with $\frac{1}{p} = \frac{1}{s} + \frac{s-p}{ps}$, Theorem 2.8 and (3.10), we have

$$\begin{aligned}
\left(\int_B |[b,T](du)|^p d\mu\right)^{1/p} &= \left(\int_B |[b,T](du)|^p w(x)dx\right)^{1/p}\\
&= \left(\int_B \left(|[b,T](du)|w^{1/p}(x)\right)^p dx\right)^{1/p}\\
&\leqslant \left(\int_B |[b,T](du)|^s dx\right)^{1/s}\left(\int_B w^{s/(s-p)}(x)dx\right)^{(s-p)/ps}\\
&\leqslant C_1\left(\int_B |du|^s dx\right)^{1/s}\left(\int_B w^{s/(s-p)}(x)dx\right)^{(s-p)/ps}\\
&\leqslant C_2 |B|^{\frac{t-s}{ts}} \|du\|_{t,\sigma_1 B}\left(\int_B w^{\alpha}(x)dx\right)^{1/\alpha p}.
\end{aligned} \tag{3.11}$$

By the Hölder inequality with $\frac{1}{t} = \frac{1}{p} + \frac{p-t}{tp}$, we find that

$$\begin{aligned}
\|du\|_{t,\sigma_1 B} &= \left(\int_{\sigma_1 B} \left(|du|(w(x))^{1/p}(w(x))^{-1/p}\right)^t dx\right)^{1/t}\\
&\leqslant \left(\int_{\sigma_1 B} |du|^p w(x)dx\right)^{1/p}\left(\int_{\sigma_1 B} \left(w^{-1/p}(x)\right)^{\frac{tp}{p-t}} dx\right)^{\frac{p-t}{tp}}\\
&\leqslant \left(\int_{\sigma_1 B} |du|^p w(x)dx\right)^{1/p}\left(\int_{\sigma_1 B} \left(w(x)\right)^{\frac{-t}{p-t}} dx\right)^{\frac{p-t}{tp}}\\
&\leqslant \left(\int_{\sigma_1 B} |du|^p d\mu\right)^{1/p}\left(\int_{\sigma_1 B} w^{-\beta}(x)dx\right)^{1/\beta p}.
\end{aligned} \tag{3.12}$$

Substituting (3.12) into (3.11) yields that

$$\begin{aligned}&\Big(\int_B |[b,T](du)|^p d\mu\Big)^{1/p}\\&\leqslant C_2|B|^{\frac{t-s}{ts}}\Big(\int_{\sigma_1 B}|du|^p d\mu\Big)^{1/p}\Big(\int_{\sigma_1 B} w^{-\beta}(x)dx\Big)^{1/\beta p}\Big(\int_B w^{\alpha}(x)dx\Big)^{1/\alpha p}.\end{aligned}\tag{3.13}$$

Since $w \in A(\alpha,\beta,\alpha;\Omega)$, we have

$$\begin{aligned}&\Big(\int_B w^{\alpha}(x)dx\Big)^{1/\alpha p}\Big(\int_{\sigma_1 B} w^{-\beta}(x)dx\Big)^{1/\beta p}\\&\leqslant\left(\Big(\int_{\sigma_1 B} w^{\alpha}(x)dx\Big)\Big(\int_{\sigma_1 B} w^{-\beta}(x)dx\Big)^{\alpha/\beta}\right)^{1/\alpha p}\\&=\left(|\sigma_1 B|^{1+\alpha/\beta}\Big(\frac{1}{|\sigma_1 B|}\int_{\sigma_1 B} w^{\alpha}(x)dx\Big)\Big(\frac{1}{|\sigma_1 B|}\int_{\sigma_1 B} w^{-\beta}(x)dx\Big)^{\alpha/\beta}\right)^{1/\alpha p}\\&\leqslant C_3|B|^{1/\alpha p+1/\beta p}.\end{aligned}\tag{3.14}$$

Combining (3.13) and (3.14), we obtain

$$\begin{aligned}&\Big(\int_B |[b,T](du)|^p d\mu\Big)^{1/p}\\&\leqslant C_2|B|^{\frac{t-s}{ts}}\Big(\int_{\sigma_1 B}|du|^p d\mu\Big)^{1/p}\Big(\int_{\sigma_1 B} w^{-\beta}(x)dx\Big)^{1/\beta p}\Big(\int_B w^{\alpha}(x)dx\Big)^{1/\alpha p}\\&\leqslant C_4|B|^{\frac{t-s}{ts}}|B|^{1/\alpha p+1/\beta p}\Big(\int_{\sigma_1 B}|du|^p d\mu\Big)^{1/p}\\&=C_4\Big(\int_{\sigma_1 B}|du|^p d\mu\Big)^{1/p},\end{aligned}\tag{3.15}$$

which completes the proof of Lemma 3.7. □

Using Lemma 3.4 and Lemma 3.7, we have the following Caccioppoli-type inequality with the weight $A(\alpha,\beta,\gamma;\Omega)$.

THEOREM 3.8. *Let $u \in L^p_{loc}(\Omega,\Lambda^l)$ be a solution of the non-homogeneous A-harmonic equation (1.7) in Ω, $l = 0,1,\cdots,n-1$, $\frac{1+\beta}{\beta} < p < \infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b \in BMO(\mathbb{R}^n)$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B |[b,T](du)|^p d\mu\right)^{1/p} \leqslant C|B|^{-1/n}\left(\int_{\sigma B}|u-c|^p d\mu\right)^{1/p}\tag{3.16}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$ and any closed form c, where the Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

Next, we further extend the Theorem 3.8 into the version with Orlicz norm as follows.

THEOREM 3.9. *Let φ be a Young function in the class $NG(p,q)$ with $\frac{1+\beta}{\beta} < p \leqslant q < \infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b \in BMO(\mathbb{R}^n)$. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega,\mu)$ and u is a solution of the non-homogeneous A-harmonic equation (1.7) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|[b,T](du)\|_{L^{\varphi}(B,\mu)} \leqslant C\|u-c\|_{L^{\varphi}(\sigma B,\mu)} \tag{3.17}$$

for all balls B with $\sigma B \subset \Omega$ and $|B| \geqslant \varepsilon_0 > 0$, where $\sigma > 1$ is a constant, c is any closed form and Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

Proof. We may assume that $w(x) \geqslant 1$ a.e. in Ω reasonably. Thus, for any ball $B \subset \Omega$, we have

$$\mu(B) = \int_B d\mu = \int_B w(x)dx \geqslant \int_B dx = |B|.$$

Using the Hölder inequality with $1 = \frac{q}{n+q} + \frac{n}{n+q}$, we obtain

$$\begin{aligned}
&\int_B \varphi\Big(|[b,T](du)|\Big)d\mu \\
&= \int_B \frac{\varphi\Big(|[b,T](du)|\Big)}{|[b,T](du)|^{\frac{nq}{n+q}}}|[b,T](du)|^{\frac{nq}{n+q}}d\mu \\
&\leqslant \Big(\int_B \frac{\varphi(|[b,T](du)|)^{\frac{n+q}{q}}}{|[b,T](du)|^n}d\mu\Big)^{\frac{q}{n+q}}\Big(\int_B |[b,T](du)|^q d\mu\Big)^{\frac{n}{n+q}}.
\end{aligned} \tag{3.18}$$

Using Lemma 3.1 and noticing $A(t)$ is a concave function, it follows that

$$\begin{aligned}
&\int_B \varphi\Big(|[b,T](du)|\Big)d\mu \\
&\leqslant \left(\int_B K(|[b,T](du)|^q)d\mu\right)^{\frac{q}{n+q}}\left(\int_B |[b,T](du)|^q d\mu\right)^{\frac{n}{n+q}} \\
&\leqslant \left(\int_B A(|[b,T](du)|^q)d\mu\right)^{\frac{q}{n+q}}\left(\int_B |[b,T](du)|^q d\mu\right)^{\frac{n}{n+q}} \\
&\leqslant A^{\frac{q}{n+q}}\left(\int_B |[b,T](du)|^q d\mu\right)\left(\int_B |[b,T](du)|^q d\mu\right)^{\frac{n}{n+q}} \\
&\leqslant C_1(n,q)K^{\frac{q}{n+q}}\left(\int_B |[b,T](du)|^q d\mu\right)\left(\int_B |[b,T](du)|^q d\mu\right)^{\frac{n}{n+q}}
\end{aligned}$$

$$
\begin{aligned}
&= C_1(n,q)\frac{\varphi\left(\left(\int_B |[b,T](du)|^q d\mu\right)^{1/q}\right)}{\left(\int_B |[b,T](du)|^q d\mu\right)^{\frac{n}{n+q}}}\left(\int_B |[b,T](du)|^q d\mu\right)^{\frac{n}{n+q}} \\
&= C_1(n,q)\varphi\left(\left(\int_B |[b,T](du)|^q d\mu\right)^{1/q}\right).
\end{aligned} \tag{3.19}
$$

Combining Lemma 3.7, Lemma 3.4 and Lemma 3.5 gives

$$
\begin{aligned}
&\left(\int_B |[b,T](du)|^q d\mu\right)^{1/q} \\
&\leqslant C_1\left(\int_B |du|^q d\mu\right)^{1/q} \\
&\leqslant C_2\,(\mu(B))^{(p-q)/pq}\left(\int_{\sigma_1 B} |du|^p d\mu\right)^{1/p} \\
&\leqslant C_3\,(\mu(B))^{(p-q)/pq}|B|^{-1/n}\left(\int_{\sigma_2 B} |u-c|^p d\mu\right)^{1/p},
\end{aligned} \tag{3.20}
$$

where $\sigma_2 > \sigma_1$ is a constant. Note that

$$
(\mu(B))^{(p-q)/pq}|B|^{-1/n} < |B|^{(p-q)/pq-1/n} < |\varepsilon_0|^{(p-q)/pq-1/n}, \tag{3.21}
$$

since $\mu(B) \geqslant |B| \geqslant \varepsilon_0$ and $p < q$. Combining (3.19), (3.20) and (3.21) and noticing φ is increasing and satisfies Δ_2-condition, we have

$$
\int_B \varphi\left(|[b,T](du)|\right)d\mu \leqslant C_4\varphi\left(\left(\int_{\sigma_2 B} |u-c|^p d\mu\right)^{1/p}\right). \tag{3.22}
$$

Taking $h(t) = \int_0^t \frac{\varphi(s)}{s}ds$ and using the fact that $\varphi(t)/t^q$ is decreasing with t, we obtain

$$
h(t) = \int_0^t \frac{\varphi(s)}{s}ds = \int_0^t \frac{\varphi(s)}{s^q}s^{q-1}ds \geqslant \frac{\varphi(t)}{t^q}\frac{1}{q}s^q|_0^t = \frac{1}{q}\varphi(t).
$$

Similarly, we have $h(t) \leqslant \frac{1}{p}\varphi(t)$ since $\varphi(t)/t^p$ is increasing with t. Hence,

$$
\frac{1}{q}\varphi(t) \leqslant h(t) \leqslant \frac{1}{p}\varphi(t). \tag{3.23}
$$

Let $g(t) = h(t^{1/p})$, then $\left(h(t^{1/p})\right)' = \frac{1}{p}\frac{\varphi(t^{1/p})}{t}$ is increasing. Thus, g is a convex function. According to definitions of g and h and using Jensen's inequality to g, we have

$$
h\left(\left(\int_B |u|^p d\mu\right)^{1/p}\right) = g\left(\int_B |u|^p d\mu\right) \leqslant \int_B g(|u|^p)d\mu = \int_B h(|u|)d\mu. \tag{3.24}
$$

Combining (3.22), (3.23) and (3.24), we have

$$\begin{aligned}\int_B \varphi\Big(|[b,T](du)|\Big)d\mu &\leqslant C_4\varphi\left(\left(\int_{\sigma_2 B}|u-c|^p d\mu\right)^{1/p}\right)\\ &\leqslant C_5 h\left(\left(\int_{\sigma_2 B}|u-c|^p d\mu\right)^{1/p}\right)\\ &\leqslant C_5\int_{\sigma_2 B} h(|u-c|)d\mu\\ &\leqslant C_6\int_{\sigma_2 B}\varphi(|u-c|)d\mu, \end{aligned} \tag{3.25}$$

which implies (3.17) holds. This completes the proof of Theorem 3.9. □

Choosing $\varphi(t)=t^p\log_+^{\alpha} t$ in Theorem 3.9, we obtain the following Caccioppoli-type inequality with the $L^p(\log_+^{\alpha} L)$-norms.

COROLLARY 3.10. *Let $\varphi(t)=t^p\log_+^{\alpha} t$, $p>\frac{1+\beta}{\beta}$ and $\alpha\in\mathbb{R}$, T be a Calderón-Zygmund singular integral operator on differential forms and $b\in BMO(\mathbb{R}^n)$. Assume that $\varphi(|u|)\in L^1_{loc}(\Omega,\mu)$ and u is a solution of the non-homogeneous A-harmonic equation (1.7). Then, there exists a constant C, independent of u, such that*

$$\|[b,T](du)\|_{L^p(\log_+^{\alpha} L)(B,\mu)}\leqslant C\|u-c\|_{L^p(\log_+^{\alpha} L)(\sigma B,\mu)} \tag{3.26}$$

for all balls B with $\sigma B\subset\Omega$ and $|B|\geqslant\varepsilon_0>0$, where $\sigma>1$ is a constant, c is any closed form and Radon measure μ is defined by $d\mu=w(x)dx$ and $w(x)\in A(\alpha,\beta,\alpha;\Omega)$, $\alpha>1$, $\beta>0$.

It is to be observed that c is any closed form in all Caccioppoli-type inequalities derived in this section. Therefore, selecting $c=0$ in Theorem 3.6, Theorem 3.8, Theorem 3.9 gives the following corollaries.

COROLLARY 3.11. *Let u be a solution of the non-homogeneous A-harmonic equation (1.7) in Ω and $du\in L^p(\Omega,\Lambda^l+1)$, $1<p<\infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b\in BMO(\mathbb{R}^n)$. Then, there exists a constant C, independent of u, such that*

$$\|[b,T](du)\|_{p,B}\leqslant C|B|^{-1/n}\|u\|_{p,B} \tag{3.27}$$

for all balls B with $B\subset\Omega$.

COROLLARY 3.12. *Let $u\in L^p_{loc}(\Omega,\Lambda^l)$ be a solution of the non-homogeneous A-harmonic equation (1.7) in Ω, $l=0,1,\cdots,n-1$, $\frac{1+\beta}{\beta}<p<\infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b\in BMO(\mathbb{R}^n)$. Then, there exists a constant C, independent of u, such that*

$$\left(\int_B|[b,T](du)|^p d\mu\right)^{1/p}\leqslant C|B|^{-1/n}\left(\int_{\sigma B}|u|^p d\mu\right)^{1/p} \tag{3.28}$$

for all balls B with $\sigma B \subset \Omega$ for some $\sigma > 1$, where the Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

COROLLARY 3.13. *Let φ be a Young function in the class $NG(p,q)$ with $\frac{1+\beta}{\beta} < p \leqslant q < \infty$, T be a Calderón-Zygmund singular integral operator on differential forms and $b \in BMO(\mathbb{R}^n)$. Assume that $\varphi(|u|) \in L^1_{loc}(\Omega,\mu)$ and u is a solution of the nonhomogeneous A-harmonic equation (1.7) in Ω. Then, there exists a constant C, independent of u, such that*

$$\|[b,T](du)\|_{L^{\varphi}(B,\mu)} \leqslant C\|u\|_{L^{\varphi}(\sigma B,\mu)} \tag{3.29}$$

for all balls B with $\sigma B \subset \Omega$ and $|B| \geqslant \varepsilon_0 > 0$, where $\sigma > 1$ is a constant and the Radon measure μ is defined by $d\mu = w(x)dx$ and $w(x) \in A(\alpha,\beta,\alpha;\Omega)$, $\alpha > 1$, $\beta > 0$.

REMARK 3.1. It should be noticed that the Caccioppoli-type inequalities derived in this paper can be extended into the global cases using the well-known Covering Lemma, which also means that these Caccioppoli-type inequalities also hold for bounded John domains, L^p-averaging domains or $L^{\varphi}(\mu)$-averaging domains.

REMARK 3.2. Note that the $A(\alpha,\beta,\gamma;E)$-class is an extension of several existing weight classes which contain $A_r^{\lambda}(E)$-weight, $A_r(\lambda,E)$-weight and $A_r(E)$-weight. Thus, these conclusions obtained in this paper will change into the corresponding versions when we take some weight as a special case.

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