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# A LYAPUNOV–TYPE INEQUALITY FOR A CLASS OF HIGHER–ORDER FRACTIONAL BOUNDARY VALUE PROBLEMS

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*Abstract.* This work presents a new Lyapunov-type inequality for a class of higher-order fractional boundary value problem of the fractional Caputo Fabrizio differential equation subject to fractional integral boundary conditions. The derived result is applied to the fractional Sturm-Liouville problem in establishing a lower bound for the eigenvalues. We also provide the necessary condition for nonexistence of the non-trivial solution of the fractional boundary value problem.

## 1. Introduction

In this paper, we derive a new Lyapunov-type inequality for the following higher-order fractional Caputo-Fabrizio differential equation subject to fractional integral boundary conditions

$$ {}^{CF}_{a}D^{\nu}_{x}u(x)+p(x)u(x)=0, \quad 2<\nu\leqslant 3, \quad a\leqslant x\leqslant b, \tag{1}$$

$$ u(a)=u'(a)=0,\ u(b)={}^{CF}_{a}\mathscr{I}^{\alpha}_{t}(hu)(b), \tag{2}$$

where $a$ and $b$ are real constants and ${}^{CF}_{a}D^{\nu}_{x}u(x)$ is the fractional Caputo-Fabrizio derivative of $u$ on $[a,b]$, ${}^{CF}_{a}\mathscr{I}^{\alpha}_{t}(hu)(x)$ denotes the fractional Caputo-Fabrizio integral of $hu$ on $[a,b]$ for a given $p,h\in C([a,b],\mathbb{R})$. Since the trivial solution $u\equiv 0$ satisfies the problem (1)–(2), we only consider the non-trivial solution of the problem.

We will prove that a necessary condition that the problem (1)–(2) will have a nonzero solution for any $\nu\in(2,3]$ under some condition on the real and continuous function $p\in C[a,b]$.

Lyapunov investigated that a necessary condition for the existence of non-trivial solution of the boundary value problem

$$\begin{aligned} &y''(x)+p(x)y(x)=0, \quad x\in(a,b),\\ &y(a)=y(b)=0, \end{aligned} \tag{3}$$

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is that

$$\int_a^b |p(x)|\,dx > \frac{4}{b-a}, \tag{4}$$

where $p \in C[a,b]$ is a real-valued function and $a$ and $b$ are consecutive zeros of $u$. The inequality (4) is the famous Lyapunov inequality [12]. Here, the constant 4 cannot be replaced by a larger number. The Lyapunov inequality is an important tool in many problems including disconjugacy and Sturm-Lioville eigenvalue problems for proving the existence of nontrivial solutions [3], [15]. Many extensions and generalization of Lyapunov inequality for the differential equations with integer orders have been studied and presented in the literature (e.g., see [18], [4], [17], [24], [25], [22], [7], [16] and references therein). For example, Lyapunov-type inequalities for even order differential equations is proved in [7] and for odd-order differential equations in [25]. Some new and improved inequalities for higher order differential equations with anti-periodic boundary conditions was proved in [23]. In [16], the authors provided new Lyapunov-type inequalities for the third order linear differential equation.

Recently, some authors pay attention to the study of Lyapunov-type inequalities for the fractional boundary value problems. Ferreira [5] initiated in deriving a Lyapunov-type inequality for the following Riemann-Liouville fractional boundary value problem

$$ {}^{RL}_{a}D^{\alpha}_{t}u(t) + p(t)y(t) = 0, \quad t \in [a,b], \quad \alpha \in (1,2), \tag{5}$$

$$u(a) = u(b) = 0 \tag{6}$$

where ${}^{RL}_{a}D^{\alpha}_{t}u(t)$ is the Riemann-Liouville derivative of $u(t)$ of order $\alpha \in (1,2]$, $a$ and $b$ are consecutive zeros of $u$ and $p$ is a real and continuous function. The author proved that if $p \in C[a,b]$ and $u$ is a nontrivial solution of the problem (5)–(6), then

$$\int_a^b |p(s)|\,ds > \Gamma(\alpha)\left(\frac{2^{2(\alpha-1)}}{(b-a)^{\alpha-1}}\right). \tag{7}$$

Note that when $\alpha = 2$, one can recover the original Lypunov inequality (4).

In [6], Ferreira also considered a similar problem to the problem (5)–(6) in which the Riemann-Liouville fractional derivative is replaced by the fractional Caputo derivative and he derived a Lyapunov-type inequality. Using this inequality, he investigated real zeros of some Mittag-Leffler functions on some intervals. Since then, some extensions of Lyapunov-type inequalities for the fractional boundary value problems using different boundary conditions have been studied. For example, [21] provided a Lyapunov-type inequality for fractional boundary value problem subject to the fractional boundary conditions, a new Lyapunov-type inequality for fractional boundary value problem subject to a Robin boundary condition was derived in [8], a Lyapunov-type inequality for fractional boundary value problem with a mixed boundary condition was provided in [9] and Lyapunov-type inequalities for a class of fractional boundary value problem have been studied in [14]. The above-mentioned Lyapunov-type inequalities have been derived for fractional boundary value problems involving fractional derivatives with singular kernel. In 2015, a new fractional derivative with non singular kernel has been introduced by Caputo and Fabrizio [2]. The solutions of fractional differential equations in the sense of this new fractional derivative contain no singular functions, and it describes better for modelling material heterogeneity and structures with different scale. For more discussion on this new fractional derivative, we refer the reader to [13].

Lyapunov-type inequality for higher-order fractional boundary value problems is very rare. Motivated by the above studies, we will obtain a new Lyapunov-type inequality for higher-order fractional boundary value problems in the sense of the Caputo-Fabrizio derivative. Lyapunov-type inequalities for fractional boundary value problems of the Caputo-Fabrizio derivative are studied in [11], [19], [20].

The rest of the paper is organized as follows. Preliminaries and definitions have been recalled in Section 2. The main results are given in Section 3. Some applications of the derived inequality are presented in Section 4.

## 2. Preliminaries

We introduce the definitions of the Riemann-Liouville fractional integral and fractional derivative and the Caputo fractional derivative and the fractional Caputo-Fabrizio integral and derivative. We also recall some previous results which will be used in our analysis.

DEFINITION 1. [10] Let $\alpha \geqslant 0$. Assume that $f$ is a real-valued function on $[a,b]$. The Riemann-Lioville fractional integral of order $\alpha$ is defined by ${}_aI^0 f = f$ and

$$\left({}_aI^{\alpha} f\right)(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s)\,ds, \quad \alpha > 0,\ t \in [a,b].$$

DEFINITION 2. [10] Let $\alpha \geqslant 0$ and $n \in \mathbb{N}$. Assume that $f$ is real-valued and absolutely continuous function on $[a,b]$. The Riemann-Lioville fractional derivative of order $\alpha \in (n,n+1]$ is defined by

$$ {}^{RL}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{\partial^n}{\partial t^n} \int_a^t (t-s)^{n-1-\alpha} f(s)\,ds, \quad t > a. \tag{8}$$

DEFINITION 3. [10] Let $\alpha \geqslant 0$ and $f \in C^{n+1}[a,b]$. The Caputo fractional derivative of order $\alpha \in (n,n+1]$ of the function $f \in C^n([a,b]$ is defined by

$$ {}^{C}_{a}D_t^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-s)^{n-1-\alpha} f^{(n)}(s)\,ds, \quad t > a. \tag{9}$$

DEFINITION 4. [2] Let $f \in H^1(a,b)$ and $\alpha \in (0,1]$. The fractional Caputo-Fabrizio derivative is defined as

$$ {}^{CF}_{a}D_t^{\alpha} f(t) = \frac{(2-\nu)M(\nu)}{2(1-\nu)} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'(s)\,ds, \quad t \geqslant a, \tag{10}$$

where $M(\nu)$ is a normalization function with $M(0) = M(1) = 1$.

DEFINITION 5. [13] The Caputo-Fabrizio fractional integral of order $\alpha \in (0,1)$ of a continuous function $f$ on $[a,b]$ is defined by

$$ {}^{CF}_{a}\mathscr{I}^{\alpha}_{t} f(t) = \frac{2(1-\alpha)}{(2-\alpha)M(\alpha)} f(t) + \frac{2\alpha}{(2-\alpha)M(\alpha)} \int_0^t f(s)\,ds. \tag{11} $$

Imposing $\dfrac{2(1-\alpha)}{(2-\alpha)M(\alpha)} + \dfrac{2\alpha}{(2-\alpha)M(\alpha)} = 1$, we can have an explicit expressing for $M(\alpha), \quad \alpha \in (0,1]$ given as [13]

$$ M(\alpha) = \frac{2}{2-\alpha}. $$

The high order Caputo-Fabrizio fractional of order $\sigma = \alpha + n$ for $\alpha \in (0,1)$ and $n \in \mathbb{N}$ is defined as

$$ {}^{CF}_{a}D^{\alpha+n}_{t} f(t) := {}^{CF}_{a}D^{\alpha}_{t}\left({}^{CF}_{a}D^{n}_{t} f(t)\right). $$

THEOREM 1. [2] *Let the function $f(t)$ satisfy $f^{(k)}(a) = 0$, $k = 1,2,\ldots,n$, then the following equality holds:*

$$ {}^{CF}_{a}D^{\alpha}_{t}\left({}^{CF}_{a}D^{n}_{t} f(t)\right) = {}^{CF}_{a}D^{n}_{t}\left({}^{CF}_{a}D^{\alpha}_{t} f(t)\right). \tag{12} $$

DEFINITION 6. [2] For $\sigma = \alpha + 2$ with $\alpha \in (0,1)$, the Caputo-Fabrizio fractional derivative of order $\sigma$ defined as

$$ {}^{CF}_{a}D^{\sigma}_{t} f(t) = \frac{1}{1-\alpha} \int_a^t \exp\left(-\frac{\alpha}{1-\alpha}(t-s)\right) f'''(s)\,ds. \tag{13} $$

Note that the equality ${}^{CF}_{a}D^{\alpha}_{t}({}^{CF}_{a}D^{(2)}_{t} f(x)) = {}^{CF}_{a}D^{(2)}_{t}({}^{CF}_{a}D^{\alpha}_{t} f(x))$ is defined unambiguously when $f''(0) = 0$ (see [2]).

DEFINITION 7. Let $\nu \in (n, n+1]$, $n \in \mathbb{N}$ and $f \in C^{n+1}[a,b]$, $a < b$. The fractional Caputo-Fabrizio integral operator of order $\nu$ defined as

$$ {}^{CF}_{a}\mathscr{I}^{\nu}_{t} f(t) = (1+n-\nu) I^{n}_{a} f(t) + (\nu - n) I^{n+1}_{a} f(x), \quad x \geqslant a, \tag{14} $$

where $I^{n}_{a} f(x)$ is the iterated Cauchy integral given by $I^{n}_{a} f(x) = \frac{1}{(n-1)!}\int_a^x (x-t)^{n-1} f(t)\,dt$, with the adaption $I^{0}_{a} f(x) = f(x)$.

LEMMA 1. [11] *Let $\nu \in (n, n+1]$. Then we have the following relation*

$$ {}^{CF}_{a}\mathscr{I}^{\nu}_{t}\, {}^{CF}_{a}D^{\nu}_{t} f(t) = f(t) - \sum_{k=0}^{n} \frac{(t-a)^k}{k!} f^{(k)}(a). $$

LEMMA 2. [1] *Let $\nu \in (n, n+1]$. Then we have the following relation*

$$ {}^{CF}_{a}D^{\nu}_{t}\, {}^{CF}_{a}\mathscr{I}^{\nu}_{t} f(t) = f(t). $$

## 3. Main results

We present a Lyapunov type inequality for high-order fractional differential equations of the Caputo-Fabrizio fractional derivative in this section. We follow Green's function technique consisting in converting the fractional boundary value problem (1)–(2) into an equivalent integral form and obtain the maximum value of the Green's function. We will need the following results in our analysis.

LEMMA 3. *Let $y \in C([a,b],\mathbb{R})$. If $w \in C([a,b],\mathbb{R}) \cap AC([a,b],\mathbb{R})$ is a solution of the following Caputo-Fabrizio fractional boundary value problem*

$$ {}^{CF}_{a}D^{\nu}_{t} w(x) + y(x) = 0, \quad \nu \in (2,3], \quad a \leqslant x \leqslant b, \tag{15}$$

$$ w(a) = w'(a) = 0, \quad w(b) = 0. \tag{16}$$

*then*

$$ w(x) = \int_a^b H(x,t)y(t)dt, $$

*where the Green's function $H(x,t)$ is given by*

$$ \mathrm{H(x,t)} = \begin{cases} h_1(x,t), & a \leqslant t \leqslant x \leqslant b, \\ h_2(x,t), & a \leqslant t \leqslant x \leqslant b, \end{cases} \tag{17}$$

*with*

$$ h_1(x,t) := h_2(x,t) - \frac{2(3-\nu)(x-t)(b-a)^2 + (\nu-2)(x-t)^2(b-a)^2}{2(b-a)^2}, $$

*and*

$$ h_2(x,t) := \frac{2(3-\nu)(x-a)^2(b-t) + (\nu-2)(x-a)^2(b-t)^2}{2(b-a)^2}. $$

*Proof.* By Lemma 1, we get, for $\nu \in (2,3]$

$$ {}^{CF}_{a}\mathscr{I}^{\nu}_{x}\, {}^{CF}_{a}D^{\nu}_{x} w(x) = w(x) - w(a) - (x-a)w'(a) - \frac{(x-a)^2}{2} w''(a). \tag{18}$$

We apply the high order fractional Caputo-Fabrizio integral operator (14) to the equation (15) and using (18) , we obtain at once

$$ w(x) - w(a) - (x-a)w'(a) - \frac{(x-a)^2}{2} w''(a) = -I^{\nu}_{a} y(x). $$

The boundary conditions $w(a) = w'(a) = 0$ imply that

$$ w(x) = \frac{(x-a)^2}{2} w''(a) - I^{\nu}_{a} y(x). $$

Using the boundary condition $w(b) = 0$, we obtain

$$w(x) = \frac{(3-\nu)}{(b-a)^2}\int_a^b (x-a)^2(b-t)y(t)dt + \frac{\nu-2}{2\,(b-a)^2}\int_a^b (x-a)^2(b-t)^2y(t)dt \\ -(3-\nu)\int_a^x (x-t)y(t)dt - \frac{a-2}{2}\int_a^x (x-t)^2y(t)dt$$

or, equivalently

$$\begin{aligned} \mathrm{u}(\mathrm{x}) &= \int_{\mathrm{a}}^{\mathrm{x}} h_1(x,t)y(t)dt + \int_x^b h_2(x,t)y(t)dt \\ &= \int_{\mathrm{a}}^{\mathrm{b}} H(x,t)y(t)dt. \end{aligned}$$

The prof is now completed. □

LEMMA 4. *If $u \in C([a,b],\mathbb{R}) \cap AC([a,b],\mathbb{R})$ is a solution of the problem (1)–(2), then we have*

$$u(x) = \int_a^b H(x,t)(p(t)+h(t))\,dt \\ +(3-\nu)\int_a^x (x-t)h(t)u(t)\,dt + \frac{\nu-2}{2}\int_a^x (x-t)^2h(t)u(t)\,dt,$$

*where $H(x,t)$ is the Green's function given by* (17).

*Proof.* Let $u \in C([a,b],\mathbb{R}) \cap AC([a,b],\mathbb{R})$ be a solution of the problem (1)–(2). Define

$$v(x) := u(x) - {}^{CF}_{a}\mathscr{I}^{\nu}_{x}(hu)(x), \quad a \leqslant x \leqslant b. \tag{19}$$

More precisely,

$$v(x) = u(x) - (3-\nu)\int_a^x (x-t)h(t)u(t)\,dt - \frac{\nu-2}{2}\int_a^x (x-t)^2h(t)u(t)\,dt, \quad x \in [a,b].$$

Applying the Caputo-Fabrizio fractional derivative on both sides of the above equation and using Lemma 2, we have, for $x \in (a,b)$

$$\begin{aligned} {}^{CF}_{a}D^{\nu}_{x}v(x) &= {}^{CF}_{a}D^{\nu}_{x}u(x) - {}^{CF}_{a}D^{\nu}_{x}\,{}^{CF}_{a}\mathscr{I}^{\nu}_{x}(hu)(x) \\ &= {}^{CF}_{a}D^{\nu}_{x}u(x) - h(x)u(x). \end{aligned}$$

Using (1), from the above equation we get

$$\begin{aligned} {}^{CF}_{a}D^{\nu}_{x}v(x) &= {}^{CF}_{a}D^{\nu}_{x}u(x) - h(x)u(x) \\ &= -p(x)u(x) - h(x)u(x) \\ &= -(p(x)+h(x))u(x). \end{aligned}$$

Therefore, we have

$$ {}^{CF}_{a}D^{\nu}_{x} v(x) = -(p(x)+h(x))u(x), \quad a \leqslant x \leqslant b. \tag{20} $$

With the help of (19), one has for $x \in [a,b]$

$$ v'(x) = u'(x) - (3-\nu)\int_a^x h(t)u(t)\,dt - (\nu-2)\int_a^x (x-t)h(t)u(t)\,dt. $$

Using (2), we obtain

$$ v(a) = v'(a) = 0, \quad v(b) = 0. \tag{21} $$

Then, $v \in C([a,b],\mathbb{R}) \cap AC([a,b],\mathbb{R})$ be a solution of the problem (20)–(21). Lemma 3 implies that $v$ also satisfies the following integral equation

$$ v(x) = \int_a^b H(x,t)(p(t)+h(t))\,dt, \quad a \leqslant x \leqslant b. $$

Hence, from (19) we get

$$ \begin{aligned} u(x) &= \int_a^b H(x,t)(p(t)+h(t))\,dt \\ &\quad + (3-\nu)\int_a^x (x-t)h(t)u(t)\,dt + \frac{\nu-2}{2}\int_a^x (x-t)^2 h(t)u(t)\,dt, \quad a \leqslant x \leqslant b, \end{aligned} $$

which is the desired conclusion. □

LEMMA 5. *The Green function $H(x,t)$ defined by (17) satisfies the following bound:*

$$ |H(x,t)| \leqslant \frac{2(3-\nu)(b-a)+(\nu-2)(b-a)^2}{2}, \;\; 2<\nu\leqslant 3, \; a\leqslant x, \; t\leqslant b. $$

*Proof.* For $a \leqslant t \leqslant x \leqslant b$, we have $x-a \leqslant b-a$. Using this inequality, one has

$$ \frac{2(3-\nu)(x-a)^2(b-t)+(\nu-2)(x-a)^2(b-t)^2}{2(b-a)^2} \leqslant \frac{2(3-\nu)(b-t)+(\nu-2)(b-t)^2}{2}. $$

Thus, we have

$$ h_1(x,t) \leqslant \frac{2(3-\nu)(b-t)+(\nu-2)(b-t)^2}{2} - (3-\nu)(x-t) - \frac{\nu-2}{2}(x-t)^2. $$

For $a \leqslant t \leqslant x \leqslant b$, we get

$$ (3-\nu)((b-t)-(x-t)) = 2(3-\nu)(b-x) \leqslant (3-\nu)(b-a), $$

and

$$ \frac{\nu-2}{2}\left((b-t)^2-(x-t)^2\right) \leqslant \frac{\nu-2}{2}(b-t)^2 \leqslant \frac{\nu-2}{2}(b-a)^2. $$

Consequently, for $a \leqslant t \leqslant x \leqslant b$ we have

$$|H(x,t)| \leqslant \frac{2(3-\nu)(b-a)+(\nu-2)(b-a)^2}{2}. \tag{22}$$

For $a \leqslant x \leqslant t \leqslant b$, we note that $x-a \leqslant t-a \leqslant b-a$. Then, one obtains

$$|h_2(x,t)| \leqslant \frac{2(3-\nu)(t-a)^2(b-t)+(\nu-2)(t-a)^2(b-t)^2}{2(b-a)^2}.$$

Now, using the inequality $2uv \leqslant (u+v)^2$ for $u=t-a$, $v=b-t$, we have

$$|h_2(x,t)| \leqslant \frac{4(3-\nu)(b-a)+(\nu-2)(b-a)^2}{8}.$$

This implies that

$$|H(x,t)| \leqslant \frac{4(3-\nu)(b-a)+(\nu-2)(b-a)^2}{8} \text{ for } a \leqslant x \leqslant t \leqslant b. \tag{23}$$

From (22) and (23), the desired conclusion follows. This finished the proof. □

We have the following Lyapunov-type inequality for the fractional boundary value problem (1)–(2) when $\nu \in (2,3]$.

THEOREM 2. *Let $\nu \in (2,3]$ and $p \in C([a,b])$. If $u \in C([a,b],\mathbb{R}) \cap AC([a,b],\mathbb{R})$ is a nontrivial solution of the fractional boundary value problem (1)–(2), then the function $p$ satisfies the following condition*

$$\begin{aligned} &\int_a^b \Big|(p(t)+h(t)\Big|dt + \int_a^b \frac{2(3-\nu)(b-t)+(\nu-2)(b-t)^2}{2(3-\nu)(b-a)+(\nu-2)(b-a)^2}|h(t)|\,dt \\ &\geqslant \frac{2}{2(3-\nu)(b-a)+(\nu-2)(b-a)^2}. \end{aligned} \tag{24}$$

*Proof.* Let $C[a,b]$ be the Banach space with maximum norm, that is,

$$\|u\| = \max_{x\in[a,b]} |u(x)|, \quad u \in C[a,b].$$

By Lemma 4, the solution of the fractional boundary value problem (1)–(2) has the integral form

$$\begin{aligned} u(x) &= \int_a^b H(x,t)(p(t)+h(t))\,dt \\ &\quad + (3-\nu)\int_a^x (x-t)h(t)u(t)\,dt + \frac{\nu-2}{2}\int_a^x (x-t)^2 h(t)u(t)\,dt, \quad x,t \in [a,b]. \end{aligned}$$

Taking the maximum norm of the both side of the above equation yields

$$\|u\| \leqslant \Big( \max_{x\in[a,b]} \int_a^b \Big|H(x,t)(p(t)+h(t)\Big|dt \\ + (3-\nu)\int_a^b (b-t)||h(t)|\,dt + \frac{\nu-2}{2}\int_a^b (b-t)^2|h(t)|\,dt\Big)\|u\|.$$

Since $u$ is non-zero, we have $\|u\| \neq 0$. Hence, using Lemma 5 one can show that

$$2 \leqslant \Big(2(3-\nu)(b-a)+(\nu-2)(b-a)^2\Big)\int_a^b \Big|(p(t)+h(t)\Big|dt \\ + 2(3-\nu)\int_a^b (b-t)||h(t)|\,dt + (\nu-2)\int_a^b (b-t)^2|h(t)|\,dt,$$

which yields

$$\int_a^b \Big|(p(t)+h(t)\Big|dt + \int_a^b \frac{2(3-\nu)(b-t)+(\nu-2)(b-t)^2}{2(3-\nu)(b-a)+(\nu-2)(b-a)^2}|h(t)|\,dt \\ \geqslant \frac{2}{2(3-\nu)(b-a)+(\nu-2)(b-a)^2}.$$

This gives the desired result (24). Thus, we complete the proof. □

REMARK 1. In Theorem 2, if we let $h \equiv 0$, then we recover the Lyapunov-type inequality derived in [20].

We now present some applications of the obtained result in Theorem 2. We present a lower bound for the eigenvalues of some nonlocal fractional boundary value problems.

THEOREM 3. *Let $\nu \in (2,3]$ and $\lambda \in \mathbb{R}$. If $u$ is a nontrivial solution of the following fractional boundary value problem,*

$$ {}^{CF}_{a}D_t^{\nu} u(x) + \lambda u(x) = 0, \quad a \leqslant x \leqslant b, \tag{25}$$

$$ u(a) = u'(0) = 0, \quad u(1) = 0, \tag{26}$$

*then the eigenvalues $\lambda \in \mathbb{R}$ must satisfy*

$$|\lambda| > \frac{2}{2(3-\nu)(b-a)^2+(\nu-2)(b-a)^3}.$$

THEOREM 4. *If $\nu \in (2,3]$, $\int_0^1 p(t)\,dt \leqslant \frac{2}{4-\nu}$, then the following fractional boundary value problem*

$$\begin{aligned} &{}^{CF}_{a}D_t^{\nu} u(x) + p(x)u(x) = 0 \quad 0 < x < 1, \\ &u(0) = u'(0) = 0, \quad u(1) = 0, \end{aligned} \tag{27}$$

*has no nontrivial solution.*

*Proof.* If there were nonzero solution to the problem (27), then we would have that $\int_0^1 p(x)\,dt > \dfrac{2}{4-\nu}$ by Theorem 2. However, this contradicts the hypothesis that $\int_0^1 p(x)\,dx \leqslant \dfrac{2}{4-\nu}$. Therefore, the only solution to the boundary value problem is the trivial solution. □

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