

ON APPROXIMATELY CONVEX AND AFFINE FUNCTIONS

ANGSHUMAN R. GOSWAMI AND ZSOLT PÁLES

(Communicated by M. Krnić)

Abstract. A real valued function f defined on a real open interval I is called Φ -convex if, for all $x, y \in I$, $t \in [0, 1]$ it satisfies

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|),$$

where $\Phi : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ is a nonnegative error function. If f and $-f$ are simultaneously Φ -convex, then f is said to be a Φ -affine function. In the main results of the paper, we describe the structural and inclusion properties of these two classes. We characterize these two classes of functions and investigate their relationship with approximately monotone and approximately-Hölder functions. We also introduce a subclass of error functions that enjoys the so-called Γ property and we show that the error function which is the most optimal for a Φ -convex function has to belong to this subclass. The properties of this subclass of error function are investigated as well. Then we offer two formulas for the lower Φ -convex envelop. Besides, a sandwich type theorem is also added.

1. Introduction

The motivation for our paper comes from the theory of approximate convexity which has become an active field of research and many important contributions have been made, see for instance [3, 4, 5, 6, 7, 8, 9, 10, 11, 14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40, 41, 42, 43]. In these papers several aspects of approximate convexity were investigated: stability problems, Bernstein–Doetsch-type theorems, Hermite–Hadamard type inequalities, etc. The notions and results have various applications in nonsmooth and convex analysis and optimization theory, and also in the theory of functional equations and inequalities. The main concepts and results of this paper are distilled from the following elementary observations; primarily studied in the paper [30].

Assume that I is a nonempty interval and a function $f : I \rightarrow \mathbb{R}$ satisfies the following inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + \varepsilon(t((1-t)|x-y|)^p + (1-t)(t|x-y|)^p),$$

$$(x, y \in I, t \in [0, 1]) \tag{1}$$

Mathematics subject classification (2020): Primary 26A48; Secondary 26A12, 26A16, 26A45, 39B72, 39C05.

Keywords and phrases: Φ -monotone function, Φ -Hölder function, Φ -monotone envelope, Φ -Hölder envelope, Ostrowski-type inequality, Hermite–Hadamard-type inequality.

The research of the second author was supported by the K-134191 NKFIH Grant.

for some nonnegative constant ε and real constant $p \in \mathbb{R}$. In other words, f is approximately convex with an error term described in terms of the p th power function. Clearly, if $\varepsilon = 0$, then the above condition is equivalent to the convexity of f . One can easily notice that every convex function f satisfies (1). On the other hand, if $p = 0$, then f satisfies (1) if and only if it can be decomposed as $f = g + h$, where g is convex and $\|h\|_\infty \leq \varepsilon/2$. In the case $p = 1$ the results of the paper [39] show that f fulfills (1) if and only if $f = g + \ell$, where g is convex and ℓ is Lipschitzian with a Lipschitz modulus $c\varepsilon$. Surprisingly, for $p > 2$, the situation is completely different. Then (1) holds if and only if f is convex.

The above described observations and results motivate the investigation of classes of functions that obey a more general approximate convexity and also the related approximately affine property. In fact, the class of Φ -convex functions was introduced in the paper [30], but this property was only characterized therein. In this work, we describe the structural properties of this function class and determine the error function which is the most optimal one. We show that optimal error functions for approximate convexity must possess the so-called Γ property. Then we offer a precise formula for the Φ -convex envelope and also obtain sandwich-type theorems.

2. Basic results

Let I be a nonempty open real interval throughout this paper and let $\ell(I) \in]0, \infty]$ denote its length. The symbols \mathbb{R} and \mathbb{R}_+ denote the sets of real and nonnegative real numbers, respectively.

The class of all functions $\Phi: [0, \ell(I)[\rightarrow \mathbb{R}_+$, called error functions, will be denoted by $\mathcal{E}(I)$. Obviously, $\mathcal{E}(I)$ is a convex cone, i.e., it is closed with respect to addition and multiplication by nonnegative scalars. In what follows, we are going to define four properties related to an error function $\Phi \in \mathcal{E}(I)$. First we recall the notions of Φ -monotone and Φ -Hölder functions that have been introduced in our former papers [12, 13].

A function $f: I \rightarrow \mathbb{R}$ will be called Φ -monotone if, for all $x, y \in I$ with $x \leq y$,

$$f(x) \leq f(y) + \Phi(y - x). \quad (2)$$

If this inequality is satisfied with the identically zero error function Φ , then we say that f is monotone (increasing). The class of Φ -monotone functions on I will be denoted by $\mathcal{M}_\Phi(I)$.

A function $f: I \rightarrow \mathbb{R}$ will be called Φ -Hölder if, for all $x, y \in I$,

$$|f(x) - f(y)| \leq \Phi(|x - y|). \quad (3)$$

The class of Φ -Hölder functions on I will be denoted by $\mathcal{H}_\Phi(I)$.

The following concept was introduced in the paper [30]. We say that a function $f: I \rightarrow \mathbb{R}$ is Φ -convex if, for all $x, y \in I$ and $t \in [0, 1]$,

$$f(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + t\Phi((1 - t)|x - y|) + (1 - t)\Phi(t|x - y|). \quad (4)$$

If the above inequality is satisfied with the identically zero error function Φ , then we say that f is *convex*. The class of Φ -convex functions on I will be denoted by $\mathcal{C}_\Phi(I)$. We mention that a similar type of approximate convexity, termed F -convexity, was defined in the paper [1] by Adamek. In the notion, the structure of the error term is different from ours.

Finally, a function $f : I \rightarrow \mathbb{R}$ is said to be Φ -*affine* if, for all $x, y \in I$ and $t \in [0, 1]$,

$$|f(tx + (1 - t)y) - tf(x) - (1 - t)f(y)| \leq t\Phi((1 - t)|x - y|) + (1 - t)\Phi(t|x - y|). \tag{5}$$

If, in particular, the above inequality is satisfied with the identically zero error function Φ , then we say that f is *affine*. The class of Φ -affine functions on I will be denoted by $\mathcal{A}_\Phi(I)$.

PROPOSITION 2.1. *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}_+$. Then*

$$\alpha_1\mathcal{C}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{C}_{\Phi_n}(I) \subseteq \mathcal{C}_{\alpha_1\Phi_1 + \dots + \alpha_n\Phi_n}(I).$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the class $\mathcal{C}_\Phi(I)$ is convex.

Proof. To prove the inclusion, let $f \in \alpha_1\mathcal{C}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{C}_{\Phi_n}(I)$. Then, f is of the form

$$f = \alpha_1f_1 + \dots + \alpha_nf_n \tag{6}$$

for some elements f_1, \dots, f_n belonging to $\mathcal{C}_{\Phi_1}(I), \dots, \mathcal{C}_{\Phi_n}(I)$, respectively. Then, for all $x, y \in I$, $t \in [0, 1]$, and $i \in \{1, \dots, n\}$, we have

$$f_i(tx + (1 - t)y) \leq tf_i(x) + (1 - t)f_i(y) + t\Phi_i((1 - t)|x - y|) + (1 - t)\Phi_i(t|x - y|).$$

Multiplying this inequality by α_i and summing up side by side, we will arrive at the inequality (4), where $\Phi := \sum_{i=1}^n \alpha_i\Phi_i$. This proves that $f \in \mathcal{C}_\Phi(I)$ and completes the proof of the second inclusion. \square

The following result is the counterpart of the previous statement.

PROPOSITION 2.2. *Let $\Phi_1, \dots, \Phi_n \in \mathcal{E}(I)$ and $\alpha_1, \dots, \alpha_n \in \mathbb{R}$. Then*

$$\alpha_1\mathcal{A}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{A}_{\Phi_n}(I) \subseteq \mathcal{A}_{|\alpha_1\Phi_1 + \dots + \alpha_n\Phi_n}(I).$$

In particular, for all functions $\Phi \in \mathcal{E}(I)$, the class $\mathcal{A}_\Phi(I)$ is convex and central symmetric, i.e., $\mathcal{A}_\Phi(I)$ is closed with respect to multiplication by (-1) .

Proof. To prove the inclusion, let $f \in \alpha_1\mathcal{A}_{\Phi_1}(I) + \dots + \alpha_n\mathcal{A}_{\Phi_n}(I)$. Then, f is of the form (6) for some elements f_1, \dots, f_n belonging to $\mathcal{A}_{\Phi_1}(I), \dots, \mathcal{A}_{\Phi_n}(I)$, respectively. Then, for all $x, y \in I$, $t \in [0, 1]$, and $i \in \{1, \dots, n\}$, we have

$$|f_i(tx + (1 - t)y) - tf_i(x) - (1 - t)f_i(y)| \leq t\Phi_i((1 - t)|x - y|) + (1 - t)\Phi_i(t|x - y|).$$

Multiplying this inequality by $|\alpha_i|$ and summing up side by side, we will arrive at the inequality

$$\begin{aligned} &|f(tx + (1-t)y) - tf(x) - (1-t)f(y)| \\ &= \left| \sum_{i=1}^n \alpha_i (f_i(tx + (1-t)y) - tf_i(x) - (1-t)f_i(y)) \right| \\ &\leq \sum_{i=1}^n |\alpha_i| |f_i(tx + (1-t)y) - tf_i(x) - (1-t)f_i(y)| \\ &\leq \sum_{i=1}^n |\alpha_i| (t\Phi_i((1-t)|x-y|) + (1-t)\Phi_i(t|x-y|)) \\ &= t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|), \end{aligned}$$

where $\Phi := \sum_{i=1}^n |\alpha_i| \Phi_i$. This proves that (5) holds, i.e., $f \in \mathcal{A}_\Phi(I)$ and completes the proof of the second inclusion. \square

PROPOSITION 2.3. *Let $\Phi \in \mathcal{E}(I)$. Then*

$$\mathcal{H}_\Phi(I) \subseteq \mathcal{A}_\Phi(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I)). \tag{7}$$

Proof. To prove the inclusion, let $f \in \mathcal{H}_\Phi(I)$. Let $x, y \in I$ and $t \in [0, 1]$. By the Φ -Hölder property of f , we have

$$\begin{aligned} f(tx + (1-t)y) &\leq f(x) + \Phi((1-t)|x-y|), \\ f(tx + (1-t)y) &\leq f(y) + \Phi(t|x-y|). \end{aligned}$$

Multiplying the above inequalities by t and $1-t$, then summing up the inequalities so obtained side by side, we get

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|),$$

which shows that $f \in \mathcal{C}_\Phi(I)$. Repeating the same argument with $(-f)$ instead of f , it follows that $-f \in \mathcal{C}_\Phi(I)$. Therefore, $f \in \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I)) = \mathcal{A}_\Phi(I)$.

To show the equality assertion, assume that f is a Φ -affine function. Then, for any $x, y \in I$ and $t \in [0, 1]$, f will satisfy the inequality (5) and hence the inequalities

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y) + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|).$$

and

$$tf(x) + (1-t)f(y) \leq f(tx + (1-t)y) + t\Phi((1-t)|x-y|) + (1-t)\Phi(t|x-y|). \tag{8}$$

holds. Rearranging these two inequalities, we have that both f and $-f$ are Φ -affine. That is $f \in \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I))$.

To show the inverse inclusion, let $f \in \mathcal{C}_\Phi(I) \cap (-\mathcal{C}_\Phi(I))$. Due to the property of Φ -convexity of the two classes of function, f will satisfy the two inequalities in (8). Hence, inequality (5) holds for any $x, y \in I$ and $t \in [0, 1]$. This inequality being symmetric in x and y , we get that (5) is satisfied for all $x, y \in I$. \square

COROLLARY 2.4. *Let $\Phi, \Psi \in \mathcal{E}(I)$. Then*

$$\mathcal{H}_\Phi(I) + \mathcal{C}_\Psi(I) \subseteq \mathcal{C}_{\Phi+\Psi}(I) \quad \text{and} \quad \mathcal{H}_\Phi(I) + \mathcal{A}_\Psi(I) \subseteq \mathcal{A}_{\Phi+\Psi}(I)$$

Proof. By Proposition 2.5, we have $\mathcal{H}_\Phi(I) \subseteq \mathcal{A}_\Phi(I) \subseteq \mathcal{C}_\Phi(I)$. Therefore, the statement follows from Proposition 2.1. \square

We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise supremum* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is a subfamily of \mathcal{F} with a pointwise supremum $f : I \rightarrow \mathbb{R}$, i.e.,

$$f(x) = \sup_{\gamma \in \Gamma} f_\gamma(x) \quad (x \in I), \tag{9}$$

then $f \in \mathcal{F}$. Similarly, we can define that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise infimum*. A family $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\}$ is called a *chain* if, for all $\alpha, \beta \in \Gamma$, either $f_\alpha \leq f_\beta$ or $f_\beta \leq f_\alpha$ holds on I . We say that a family \mathcal{F} of real valued functions is *closed with respect to the pointwise chain supremum (chain infimum)* if $\{f_\gamma : I \rightarrow \mathbb{R} \mid \gamma \in \Gamma\} \subseteq \mathcal{F}$ is a chain with a pointwise supremum (infimum) $f : I \rightarrow \mathbb{R}$, then $f \in \mathcal{F}$.

The following result was established in the paper [30], therefore its proof is omitted.

PROPOSITION 2.5. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{C}_\Phi(I)$ is closed under pointwise supremum and pointwise chain infimum.*

COROLLARY 2.6. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{C}_\Phi(I)$ is closed with respect to the limsup operation.*

Proof. Assume that $f : I \rightarrow \mathbb{R}$ is the upper limit of a sequence $f_n : I \rightarrow \mathbb{R}$. Then

$$f = \inf_{n \in \mathbb{N}} g_n, \quad \text{where} \quad g_n := \sup_{k \geq n} f_k.$$

If all the functions f_n are Φ -convex, then for all $n \in \mathbb{N}$, the function g_n is Φ -convex. (Because these classes are closed with respect to the pointwise supremum.) On the other hand, the sequence (g_n) is decreasing, therefore f is the pointwise chain infimum of $\{g_n \mid n \in \mathbb{N}\}$, thus f is also Φ -convex. \square

PROPOSITION 2.7. *Let $\Phi \in \mathcal{E}(I)$. Then the class $\mathcal{A}_\Phi(I)$ is closed under pointwise chain infimum and pointwise chain supremum. Consequently $\mathcal{A}_\Phi(I)$ is closed with respect to the pointwise limit operation.*

Proof. The statement easily follows from Proposition 2.5. The statement concerning the pointwise limit operation is obvious. \square

THEOREM 2.8. *Let $\Phi \in \mathcal{E}_0(I)$, let $f : I \rightarrow \mathbb{R}$ be a Φ -convex function and $g : \mathbb{R} \rightarrow \mathbb{R}_+$ be an increasing and sublinear function. Then $(g \circ f)$ is a $(g \circ \Phi)$ -convex function.*

Proof. Using the Φ -convexity of f , the nondecreasingness and sublinearity and properties of g , we arrive at

$$\begin{aligned} & (g \circ f)(tx + (1-t)y) \\ & \leq g\left(t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|))\right) \\ & \leq t(g \circ f(x) + g \circ \Phi((1-t)|y-x|)) + (1-t)(g \circ f(y) + g \circ \Phi(t|y-x|)). \end{aligned}$$

Therefore, we arrive at our desired conclusion. \square

3. Characterizations of Φ -convex and Φ -affine functions

The equivalence of assertions (i), (ii) and (iii) of the following result was established in the paper [30]. For the convenience of the reader, we provide the complete proof.

THEOREM 3.1. *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:*

(i) f is Φ -convex;

(ii) For all $x, u, y \in I$ with $x < u < y$,

$$\frac{f(u) - f(x) - \Phi(u-x)}{u-x} \leq \frac{f(y) - f(u) + \Phi(y-u)}{y-u};$$

(iii) There exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$f(u) + (x-u)\varphi(u) \leq f(x) + \Phi(|u-x|); \quad (10)$$

(iv) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$f(t_1x_1 + \dots + t_nx_n) \leq \sum_{i=1}^n t_i (f(x_i) + \Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|)).$$

Proof. (i) \Rightarrow (ii): Assume that f is Φ convex and let $x < u < y$ be arbitrary elements of I . Choose $t \in [0, 1]$ such that $u = tx + (1-t)y$. That is let $t = \frac{y-u}{y-x}$. Then (4) can be rewritten as

$$f(u) \leq \frac{y-u}{y-x}f(x) + \frac{u-x}{y-x}f(y) + \frac{y-u}{y-x}\Phi(u-x) + \frac{u-x}{y-x}\Phi(y-u).$$

Therefore,

$$\begin{aligned} (y-u+u-x)f(u) & \leq (y-u)f(x) + (u-x)f(y) \\ & \quad + (y-u)\Phi(u-x) + (u-x)\Phi(y-u). \end{aligned}$$

Rearranging this inequality, it follows that

$$\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \frac{f(y) - f(u) + \Phi(y - u)}{y - u}.$$

(ii) \Rightarrow (iii): Assume that (ii) holds and define the function φ on I by

$$\varphi(u) := \sup_{x < u} \left(\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \right) \quad (u \in I).$$

In view of condition (ii), for all $x < u < y$ in I , we have

$$\frac{f(u) - f(x) - \Phi(u - x)}{u - x} \leq \varphi(u) \leq \frac{f(y) - f(u) + \Phi(y - u)}{y - u}. \tag{11}$$

From the left hand side inequality in (11), we get

$$f(u) + (x - u)\varphi(u) \leq f(x) + \Phi(u - x) \quad (x \in I, x < u). \tag{12}$$

Similarly, from the right hand side inequality in (11) (replacing y by x , it follows that

$$f(u) + (x - u)\varphi(u) \leq f(x) + \Phi(x - u) \quad (x \in I, u < x). \tag{13}$$

Now, combining inequalities (12) and (13), the condition (iii) follows (also in the case $x = u$).

(iii) \Rightarrow (iv): To deduce (iv) from (iii), let $x_1, \dots, x_n \in I, t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$ and $u := t_1x_1 + \dots + t_nx_n$. Substituting x by x_i in the inequality of condition (iii), then multiplying this inequality by t_i , finally adding up the inequalities so obtained side by side, we get

$$\sum_{i=1}^n t_i (f(u) + (x_i - u)\varphi(u)) \leq \sum_{i=1}^n t_i (f(x_i) + \Phi(|u - x_i|)).$$

Using that $\sum_{i=1}^n t_i(x_i - u) = 0$, the above inequality simplifies to the inequality of condition (iv).

(iv) \Rightarrow (i): To deduce the Φ -convexity of f from condition (iv), let $x, y \in I$ and $t \in [0, 1]$. Taking $n = 2, x_1 := x, x_2 := y, t_1 := t$ and $t_2 := 1 - t$ in condition (iv), it is immediate to see that the inequality reduces to the defining inequality of Φ -convexity. \square

Motivated by the condition (iii) of Theorem 3.1, we say that $\varphi : I \rightarrow \mathbb{R}$ is a Φ -slope function for f if it satisfies inequality (10) for all $x, u \in I$. Then Theorem 3.1 states that the Φ -convexity of a function f is equivalent to the existence of a Φ -slope function for f .

THEOREM 3.2. *Let $\Phi \in \mathcal{E}(I)$ and $f : I \rightarrow \mathbb{R}$. Then the following conditions are equivalent to each other:*

- (i) f is Φ -affine;

(ii) For all $x, u, y \in I$ with $x < u < y$,

$$\left| \frac{f(u) - f(x)}{u - x} - \frac{f(y) - f(u)}{y - u} \right| \leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u};$$

(iii) Provided that Φ is increasing, there exists a function $\varphi : I \rightarrow \mathbb{R}$ such that, for all $x, u \in I$,

$$|f(u) - f(x) - (u - x)\varphi(u)| \leq \Phi(|u - x|); \tag{14}$$

(iv) For all $n \in \mathbb{N}$, $x_1, \dots, x_n \in I$, $t_1, \dots, t_n \geq 0$ with $t_1 + \dots + t_n = 1$,

$$\left| f(t_1x_1 + \dots + t_nx_n) - \sum_{i=1}^n t_i(f(x_i)) \right| \leq \sum_{i=1}^n t_i\Phi(|(t_1x_1 + \dots + t_nx_n) - x_i|).$$

Proof. Assertions (i), (ii) and (iv) are equivalent by Theorem 3.1.

Assume that (ii) holds and Φ is increasing. First we are going to show the following inequality:

$$\begin{aligned} \max \left(\sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x}, \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} \right) \\ \leq \min \left(\inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}, \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u} \right). \end{aligned} \tag{15}$$

By condition (ii), for all $x, u, y \in I$ with $x < u < y$, we have the following two inequalities

$$\begin{aligned} \frac{f(u) - f(x)}{u - x} - \frac{f(y) - f(u)}{y - u} &\leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u}, \\ -\frac{f(u) - f(x)}{u - x} + \frac{f(y) - f(u)}{y - u} &\leq \frac{\Phi(u - x)}{u - x} + \frac{\Phi(y - u)}{y - u}. \end{aligned} \tag{16}$$

Then, one can easily see that the first and the second of the above inequalities imply

$$\begin{aligned} \sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x} &\leq \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u}, \\ \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} &\leq \inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x} \end{aligned}$$

for all $u \in I$, respectively. In order to show that (15) holds, it remains to prove that

$$\begin{aligned} \sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x} &\leq \inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}, \\ \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} &\leq \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u}. \end{aligned} \tag{17}$$

For the first inequality, we need to prove that, for all $x_1, x_2 < u$,

$$\frac{f(u) - f(x_1) - \Phi(u - x_1)}{u - x_1} \leq \frac{f(u) - f(x_2) + \Phi(u - x_2)}{u - x_2}. \tag{18}$$

If $x_1 = x_2$, this follows from the nonnegativity of Φ . We have to consider two cases according to the possibilities $x_1 < x_2$ or $x_2 < x_1$. In the first case, using (16) for the triplet $x_1 < x_2 < u$, then the monotonicity of Φ , we get

$$\begin{aligned} \frac{f(u) - f(x_1)}{u - x_1} - \frac{f(u) - f(x_2)}{u - x_2} &= \frac{x_2 - x_1}{u - x_1} \left(\frac{f(x_2) - f(x_1)}{x_2 - x_1} - \frac{f(u) - f(x_2)}{u - x_2} \right) \\ &\leq \frac{x_2 - x_1}{u - x_1} \left(\frac{\Phi(x_2 - x_1)}{x_2 - x_1} + \frac{\Phi(u - x_2)}{u - x_2} \right) \\ &\leq \frac{\Phi(x_2 - x_1)}{u - x_1} + \frac{x_2 - x_1}{u - x_1} \frac{\Phi(u - x_2)}{u - x_2} \\ &\leq \frac{\Phi(u - x_1)}{u - x_1} + \frac{\Phi(u - x_2)}{u - x_2}, \end{aligned}$$

which shows (18) in the case $x_1 < x_2$. In the case $x_2 < x_1$, the proof is analogous, and hence the first inequality in (17) has been verified. The second inequality in (17) can be shown in a similar manner. Thus, inequality (15) has been proved, too. Define now the function $\varphi : I \rightarrow \mathbb{R}$ by

$$\varphi(u) := \max \left(\sup_{x < u} \frac{f(u) - f(x) - \Phi(u - x)}{u - x}, \sup_{u < y} \frac{f(y) - f(u) - \Phi(y - u)}{y - u} \right).$$

Then, for all $x < u < y$, we get

$$\begin{aligned} f(u) - f(x) - (u - x)\varphi(u) &\leq \Phi(u - x), \\ f(y) - f(u) + (u - y)\varphi(u) &\leq \Phi(y - u). \end{aligned} \tag{19}$$

Based on inequality (15), we also have

$$\varphi(u) \leq \min \left(\inf_{x < u} \frac{f(u) - f(x) + \Phi(u - x)}{u - x}, \inf_{u < y} \frac{f(y) - f(u) + \Phi(y - u)}{y - u} \right),$$

which implies, for all $x < u < y$, that

$$\begin{aligned} f(x) - f(u) + (u - x)\varphi(u) &\leq \Phi(u - x), \\ f(u) - f(y) - (u - y)\varphi(u) &\leq \Phi(y - u). \end{aligned} \tag{20}$$

The first inequalities in (19) and (20) yield, for all $x < u$, that

$$|f(u) - f(x) - (u - x)\varphi(u)| \leq \Phi(u - x).$$

On the other hand, the second inequalities in (19) and (20) show, for all $u < y$, that

$$|f(u) - f(y) - (u - y)\varphi(u)| \leq \Phi(y - u).$$

Combining these two inequalities, we can see that assertion (iii) must be valid.

Conversely, if assertion (iii) holds, then φ is a Φ -slope function for f , whence it follows that f is Φ -convex. Similarly, (iii) also implies that $(-\varphi)$ is a Φ -slope

function for $(-f)$, hence $(-f)$ is also Φ -convex. These two Φ -convexity properties yield that f is Φ -affine. \square

Motivated by the condition (iii) of Theorem 3.2, we say that $\varphi : I \rightarrow \mathbb{R}$ is an *absolute Φ -slope function* for f if it satisfies inequality (14) for all $x, u \in I$. Then Theorem 3.2 states that a function f is Φ -affine (provided that Φ is increasing) if and only if there exists an absolute Φ -slope function for f .

The next two propositions state that the Φ -slope function of a Φ -convex (resp. absolute Φ -slope function of a Φ -affine) function is approximately monotone (resp. Hölder) with respect to a transformation Φ^* of the error function Φ .

PROPOSITION 3.3. *Let $\Phi \in \mathcal{E}(I)$, $f : I \rightarrow \mathbb{R}$ be a Φ -convex function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f . Then φ is Φ^* -monotone, where*

$$\Phi^*(t) := \begin{cases} 2\frac{\Phi(t)}{t} & \text{if } 0 < t < \ell(I), \\ 0 & \text{if } t = 0. \end{cases} \quad (21)$$

Proof. Let $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f and let $x, y \in I$ with $x < y$ be arbitrary. By obvious substitution into the inequality (10), we get

$$\begin{aligned} f(x) + (y-x)\varphi(x) &\leq f(y) + \Phi(y-x), \\ f(y) + (x-y)\varphi(y) &\leq f(x) + \Phi(y-x). \end{aligned}$$

Adding these inequalities side by side, the values $f(x)$ and $f(y)$ cancel out, hence we arrive at

$$(y-x)(\varphi(x) - \varphi(y)) \leq 2\Phi(y-x).$$

Dividing by $y-x$, this inequality implies that φ is Φ^* -monotone, which was to be proved. \square

PROPOSITION 3.4. *Let $\Phi \in \mathcal{E}(I)$ be increasing, $f : I \rightarrow \mathbb{R}$ be a Φ -affine function and $\varphi : I \rightarrow \mathbb{R}$ be a Φ -slope function for f . Then φ is Φ^* -Hölder, where Φ^* is defined by (21).*

Proof. Let $\varphi : I \rightarrow \mathbb{R}$ be an absolute Φ -slope function for f and let $x, y \in I$ with $x \neq y$ be arbitrary. By obvious substitution into the inequality (14), we get

$$\begin{aligned} |f(x) - f(y) - (x-y)\varphi(x)| &\leq \Phi(|x-y|), \\ |f(y) - f(x) - (y-x)\varphi(y)| &\leq \Phi(|y-x|). \end{aligned}$$

Therefore, by the triangle inequality,

$$\begin{aligned} |(y-x)(\varphi(x) - \varphi(y))| &\leq |f(x) - f(y) - (x-y)\varphi(x)| + |f(y) - f(x) - (y-x)\varphi(y)| \\ &\leq 2\Phi(|y-x|) \end{aligned}$$

Dividing by $|y-x|$, this inequality shows that φ is Φ^* -Hölder, which was to be proved. \square

4. Optimal error functions

We say that an error function $\Phi \in \mathcal{E}(I)$ possesses the property Γ if it satisfies the inequality

$$\Phi(x+y) \leq \Phi(x) + \frac{2x+y}{y}\Phi(y) \quad (x \geq 0, y > 0, x+y < \ell(I)). \quad (22)$$

The subclass of error functions in $\mathcal{E}(I)$ with the property Γ will be denoted by $\mathcal{E}^\Gamma(I)$. The subset of $\mathcal{E}^\Gamma(I)$ whose elements also satisfy $\Phi(0) = 0$ will be denoted by $\mathcal{E}_0^\Gamma(I)$.

One can easily see that any subadditive error function satisfies property Γ , however, as we will see later, the reversed implication is not true.

The next result establishes some necessary and some sufficient conditions for the Γ property.

THEOREM 4.1. *Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then $\sqrt{\Phi}$ and the map $t \mapsto t^{-1}\Phi(t)$ is subadditive on $]0, \ell(I)[$. If, in addition, $\varphi : [0, \ell(I)[\rightarrow \mathbb{R}_+$ is decreasing on $]0, \ell(I)[$, then $\varphi \cdot \Phi \in \mathcal{E}^\Gamma(I)$. In particular, if $\Psi \in \mathcal{E}(I)$ and $t \mapsto t^{-2}\Psi(t)$ is decreasing on $]0, \ell(I)[$, then $\Psi \in \mathcal{E}^\Gamma(I)$.*

Proof. To prove the subadditivity of $\sqrt{\Phi}$ on $]0, \ell(I)[$, let $x, y > 0$ with $x+y < \ell(I)$. We have

$$\Phi(x+y) \leq \Phi(x) + \frac{2x+y}{y}\Phi(y) = \Phi(x) + \Phi(y) + 2\frac{x}{y}\Phi(y).$$

and, interchanging the roles of x and y , we also have

$$\Phi(x+y) \leq \Phi(x) + \Phi(y) + 2\frac{y}{x}\Phi(x).$$

These two inequalities imply that

$$\Phi(x+y) \leq \Phi(x) + \Phi(y) + 2 \min\left(\frac{x}{y}\Phi(y), \frac{y}{x}\Phi(x)\right). \quad (23)$$

By using that the geometric mean of two numbers exceeds their minimum, we get

$$\Phi(x+y) \leq \Phi(x) + \Phi(y) + 2\sqrt{\Phi(x)\Phi(y)} = (\sqrt{\Phi(x)} + \sqrt{\Phi(y)})^2.$$

Taking square root of this inequality side by side, it follows that

$$\sqrt{\Phi(x+y)} \leq \sqrt{\Phi(x)} + \sqrt{\Phi(y)},$$

which completes the proof of the subadditivity of $\sqrt{\Phi}$.

To show the subadditivity of $t \mapsto t^{-1}\Phi(t)$, we use that the minimum of two numbers is smaller than their arithmetic mean. Thus, from (23) we arrive at

$$\Phi(x+y) \leq \Phi(x) + \Phi(y) + \left(\frac{x}{y}\Phi(y) + \frac{y}{x}\Phi(x)\right) = (x+y)\frac{\Phi(x)}{x} + (x+y)\frac{\Phi(y)}{y}.$$

Dividing both sides by $x + y$, we can reach at our desired conclusion.

To prove the next assertion, denote $\Psi := \varphi \cdot \Phi$ and let $x > 0, y > 0$ with $x + y < \ell(I)$. Using the Γ property of Φ and then the decreasingness of φ , we get

$$\begin{aligned} \Psi(x+y) &= \varphi(x+y)\Phi(x+y) \leq \varphi(x+y)\Phi(x) + \frac{2x+y}{y}\varphi(x+y)\Phi(y) \\ &\leq \varphi(x)\Phi(x) + \frac{2x+y}{y}\varphi(y)\Phi(y) = \Psi(x) + \frac{2x+y}{y}\Psi(y), \end{aligned}$$

which establishes the Γ property of Ψ .

Finally, let $\Psi \in \mathcal{E}(I)$ be arbitrary such that $t \mapsto t^{-2}\Psi(t) =: \varphi(t)$ is decreasing on $]0, \ell(I)[$. Since $\Phi(t) := t^2$ possesses the Γ property, it follows that $\Psi = \varphi \cdot \Phi$ belongs to $\mathcal{E}^\Gamma(I)$. \square

COROLLARY 4.2. *Let $p \in \mathbb{R}$ and define $\Phi_p : [0, \infty[\rightarrow \mathbb{R}_+$ by $\Phi_p(t) := t^p$ for $t > 0$ and $\Phi_p(0) := 0$. Then $\Phi_p \in \mathcal{E}^\Gamma(\mathbb{R}_+)$ if and only if $p \leq 2$.*

Proof. Assume that Φ_p possesses the Γ property. Then, according to the first part of Theorem 4.1, $t \mapsto t^{-1}\Phi_p(t) = t^{p-1}$ is subadditive on \mathbb{R}_+ . Therefore, $p - 1 \leq 1$, which yields $p \leq 2$.

On the other hand, if $p \leq 2$, then $t \mapsto t^{-2}\Phi_p(t) = t^{p-2}$ is decreasing, therefore the last part of Theorem 4.1 yields that $\Phi_p \in \mathcal{E}^\Gamma(\mathbb{R}_+)$. \square

PROPOSITION 4.3. *Let $\Phi \in \mathcal{E}^\Gamma(I)$. Then, for all $n \in \mathbb{N}$ and for all $u_1 \geq 0, u_2, \dots, u_n > 0$ with $u_1 + \dots + u_n < \ell(I)$, the following inequality holds*

$$\Phi(u_1 + \dots + u_n) \leq \Phi(u_1) + \frac{2u_1 + u_2}{u_2}\Phi(u_2) + \dots + \frac{2(u_1 + \dots + u_{n-1}) + u_n}{u_n}\Phi(u_n). \tag{24}$$

Proof. Assume that Φ possesses the Γ property, i.e., (22) is satisfied.

If $n = 1$, then the statement trivially holds with equality. If $n = 2$, then (22) with $x := u_1$ and $y := u_2$ is equivalent to (24). Assume that (24) has been proved for some $n \geq 2$ and let $u_1 \geq 0, u_2, \dots, u_{n+1} > 0$ with $u_1 + \dots + u_{n+1} < \ell(I)$. Then, by the Γ property with $x := u_1 + \dots + u_n, y := u_{n+1}$ and then by inductive assumption, we get

$$\begin{aligned} &\Phi(u_1 + \dots + u_{n+1}) \\ &\leq \Phi(u_1 + \dots + u_n) + \frac{2(u_1 + \dots + u_n) + u_{n+1}}{u_{n+1}}\Phi(u_{n+1}) \\ &\leq \Phi(u_1) + \dots + \frac{2(u_1 + \dots + u_{n-1}) + u_n}{u_n}\Phi(u_n) + \frac{2(u_1 + \dots + u_n) + u_{n+1}}{u_{n+1}}\Phi(u_{n+1}), \end{aligned}$$

which is exactly the inequality to be proved for $n + 1$ variables. \square

PROPOSITION 4.4. *The classes $\mathcal{E}^\Gamma(I)$ and $\mathcal{E}_0^\Gamma(I)$ are closed with respect to addition, multiplication by nonnegative scalars, pointwise maximum, pointwise chain infimum and the limsup operation.*

Proof. The proof is similar to that of Proposition 2.1 and Proposition 2.5, therefore it is left to the reader. \square

In view of the above proposition, we can see that the supremum of all error functions with the Γ property which are smaller than a given error function $\Phi \in \mathcal{E}(I)$ is the largest error function below Φ with the Γ property, which will be denoted as Φ^Γ and called the Γ -envelope of Φ . In what follows, we describe a construction for Φ^Γ in the class $\mathcal{E}_0^\Gamma(I)$. For this aim, for any error function $\Phi \in \mathcal{E}_0(I)$, we define its γ -transform $\Phi^\gamma : [0, \ell(I)[\rightarrow \mathbb{R}_+$ of Φ by

$$\Phi^\gamma(0) := 0, \quad \Phi^\gamma(u) := \inf \left\{ \Phi(x) + \frac{2x+y}{y} \Phi(y) \mid x \geq 0, y > 0 : x+y = u \right\} \quad (u > 0).$$

Obviously, taking $x = 0$ in the above definition, it follows from $\Phi(0) = 0$ that $\Phi^\gamma \leq \Phi$ on $[0, \ell(I)[$.

THEOREM 4.5. *Let $\Phi \in \mathcal{E}_0(I)$ and define the sequence $(\Phi_n)_{n=0}^\infty$ by*

$$\Phi_1 := \Phi, \quad \Phi_{n+1} := \Phi_n^\gamma \quad (n \in \mathbb{N}). \tag{25}$$

Then (Φ_n) is a pointwise decreasing sequence on $[0, \ell(I)[$ whose pointwise limit function equals Φ^Γ on $[0, \ell(I)[$.

Proof. The inequality $\Phi_2 \leq \Phi_1$ is the consequence of $\Phi^\gamma \leq \Phi$. Now, using the obvious monotonicity of the mapping $\Phi \mapsto \Phi^\gamma$, by induction on n , it follows that $\Phi_{n+1} \leq \Phi_n$ holds on $[0, \ell(I)[$, which proves the decreasingness of the sequence (Φ_n) . This implies that the pointwise limit function of this sequence, denoted as Φ_0 , exists and is not greater than Φ .

By the construction, for all $n \in \mathbb{N}$, we have that $\Phi_n(0) = 0$ and, for all $0 < u < \ell(I)$

$$\Phi_{n+1}(u) = \Phi_n^\gamma(u) = \inf \left\{ \Phi_n(x) + \frac{2x+y}{y} \Phi_n(y) \mid x \geq 0, y > 0 : x+y = u \right\}.$$

This implies that

$$\Phi_{n+1}(x+y) \leq \Phi_n(x) + \frac{2x+y}{y} \Phi_n(y) \quad (x \geq 0, y > 0, x+y < \ell(I)).$$

Upon taking the limit as $n \rightarrow \infty$, it follows that Φ_0 possesses the Γ property.

It remains to show that Φ_0 is nonsmaller than any error function $\Psi \in \mathcal{E}(I)$ with $\Psi \leq \Phi$ which possesses the Γ property. We prove, by induction on n , that $\Psi \leq \Phi_n$ on $[0, \ell(I)[$. The inequality $\Psi \leq \Phi_1 = \Phi$ holds by assumption. Assume that $\Psi \leq \Phi_n$ is valid for some $n \in \mathbb{N}$. Let $u \in]0, \ell(I)[$ and let $x \geq 0, y > 0$ such that $u = x+y$. Then

$$\Psi(u) = \Psi(x+y) \leq \Psi(x) + \frac{2x+y}{y} \Psi(y) \leq \Phi_n(x) + \frac{2x+y}{y} \Phi_n(y).$$

Upon taking the infimum for all $x \geq 0, y > 0$ with $u = x+y$, it follows that

$$\Psi(u) \leq \Phi_n^\gamma(u) = \Phi_{n+1}(u).$$

This shows that $\Psi \leq \Phi_{n+1}$ holds on $[0, \ell(I)[$, which was to be verified.

Finally, taking the limit $n \rightarrow \infty$, we can conclude that $\Psi \leq \Phi_0$ is valid on $[0, \ell(I)[$, which demonstrates that Φ_0 is the largest error function below Φ possessing the Γ property. Therefore, $\Phi_0 = \Phi^\Gamma$ holds. \square

The next result establishes a sufficient condition that ensures the Γ envelope to be zero.

THEOREM 4.6. *Let $\Phi \in \mathcal{E}(I)$ and assume that*

$$\lim_{t \rightarrow 0^+} t^{-2}\Phi(t) = 0. \tag{26}$$

Then $\Phi^\Gamma(u) = 0$ holds for $u \in]0, \ell(I)[$.

Proof. Let $u \in]0, \ell(I)[$ and let $n \in \mathbb{N}$. Then, with $u_1 := \dots := u_n := \frac{u}{n}$, the inequality (24) applied for Φ^Γ (instead of Φ) and the inequality $\Phi^\Gamma \leq \Phi$ yield that

$$\Phi^\Gamma(u) \leq (1 + 3 + \dots + (2n - 1))\Phi^\Gamma\left(\frac{u}{n}\right) = n^2\Phi^\Gamma\left(\frac{u}{n}\right) \leq n^2\Phi\left(\frac{u}{n}\right) = u^2\left(\frac{n}{u}\right)^2\Phi\left(\frac{u}{n}\right).$$

Upon taking the limit $n \rightarrow \infty$ and using the relation (26), we get that $\Phi^\Gamma(u) = 0$, which proves the statement. \square

COROLLARY 4.7. *Let $p \in \mathbb{R}$ and define Φ_p as in Corollary 4.2. Then*

$$\Phi_p^\Gamma = \begin{cases} \Phi_p & \text{if } p \leq 2, \\ 0 & \text{if } p > 2. \end{cases}$$

Proof. If $p \leq 2$, then, by Corollary 4.2, Φ_p possesses the Γ property, and hence, $\Phi_p^\Gamma = \Phi_p$. In the case $p > 2$, we have that

$$\lim_{t \rightarrow 0^+} t^{-2}\Phi_p(t) = \lim_{t \rightarrow 0^+} t^{p-2} = 0.$$

Therefore, by Theorem 4.6, it follows that $\Phi_p^\Gamma = 0$. \square

THEOREM 4.8. *Let $\Phi \in \mathcal{E}_0(I)$. Then*

$$\mathcal{C}_\Phi(I) = \mathcal{C}_{\Phi^\Gamma}(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) = \mathcal{A}_{\Phi^\Gamma}(I).$$

Proof. Due to the inequality $\Phi^\Gamma \leq \Phi$, it follows that

$$\mathcal{C}_\Phi(I) \supseteq \mathcal{C}_{\Phi^\Gamma}(I) \quad \text{and} \quad \mathcal{A}_\Phi(I) \supseteq \mathcal{A}_{\Phi^\Gamma}(I).$$

To prove the reversed inclusions, we will first verify that $\mathcal{C}_\Phi(I) \subseteq \mathcal{C}_{\Phi^\Gamma}(I)$.

Let f be a Φ -convex function and let $x, y \in I$ with $x < y$ and $t \in]0, 1[$. Define $u := tx + (1 - t)y$ and then choose $v \in [x, u[$ and $w \in]u, y]$ arbitrarily. Applying the Φ -convexity of f three times, we obtain

$$\begin{aligned} f(u) &\leq \frac{w-u}{w-v} (f(v) + \Phi(u-v)) + \frac{u-v}{w-v} (f(w) + \Phi(w-u)) \\ &\leq \frac{w-u}{w-v} \left(\frac{u-v}{u-x} (f(x) + \Phi(v-x)) + \frac{v-x}{u-x} (f(u) + \Phi(u-v)) + \Phi(u-v) \right) \\ &\quad + \frac{u-v}{w-v} \left(\frac{y-w}{y-u} (f(u) + \Phi(w-u)) + \frac{w-u}{y-u} (f(y) + \Phi(y-w)) + \Phi(w-u) \right). \end{aligned}$$

After a simple calculation, this reduces to the inequality

$$\begin{aligned} f(u) &\leq \frac{y-u}{y-x} \left(f(x) + \Phi(v-x) + \frac{u+v-2x}{u-v} \Phi(u-v) \right) \\ &\quad + \frac{u-x}{y-x} \left(f(y) + \Phi(y-w) + \frac{2y-u-w}{w-u} \Phi(w-u) \right). \end{aligned} \tag{27}$$

Observe, that by the definition of Φ^γ , we have that

$$\inf_{v \in [x, u[} \left(\Phi(v-x) + \frac{u+v-2x}{u-v} \Phi(u-v) \right) = \Phi^\gamma(u-x)$$

and

$$\inf_{w \in]u, y]} \left(\Phi(y-w) + \frac{2y-u-w}{w-u} \Phi(w-u) \right) = \Phi^\gamma(y-u).$$

Therefore, upon taking the infimum with respect to $v \in [x, u[$ and $w \in]u, y]$ in (27), it follows that

$$f(u) \leq \frac{y-u}{y-x} (f(x) + \Phi^\gamma(u-x)) + \frac{u-x}{y-x} (f(y) + \Phi^\gamma(y-u)),$$

which proves that f is Φ^γ -convex.

Now define the sequence of error function Φ_n by the iteration (25). By the assumption, f is $\Phi_1 = \Phi$ -convex. On the other hand, using what we have proved above, it follows that if f is Φ_n -convex, then it is also $\Phi_{n+1} = \Phi_n^\gamma$ -convex. Therefore, f is Φ_n -convex for all $n \in \mathbb{N}$. Upon taking the limit $n \rightarrow \infty$ and using Theorem 4.5, we obtain that f is Φ^Γ -convex.

For the second part of the assertion, observe that if f is Φ -affine, then by Proposition 2.3, f and $(-f)$ are Φ -convex. Thus, by the first part, we get that f and $(-f)$ are Φ^Γ -convex, which according to Proposition 2.3 again implies that f is Φ^Γ -affine. Therefore, the reversed inclusion $\mathcal{A}_\Phi(I) \subseteq \mathcal{A}_{\Phi^\Gamma}(I)$ also holds. \square

COROLLARY 4.9. *Let $p > 2$ and define Φ_p as in Corollary 4.2. Then*

$$\mathcal{E}_{\Phi_p}(I) = \mathcal{E}_0(I) \quad \text{and} \quad \mathcal{A}_{\Phi_p}(I) = \mathcal{A}_0(I).$$

Proof. The statement directly follows from Theorem 4.8 and Corollary 4.7. \square

In the following result we obtain a statement which is analogous to [1, Corollary 1.9].

THEOREM 4.10. *Let $\Phi \in \mathcal{E}(I)$ and assume that*

$$\lim_{t \rightarrow 0^+} t^{-2}\Phi(t) = c \quad \text{and} \quad \Phi(t) \geq ct^2 \quad \text{for } t \in [0, \ell(I)[. \tag{28}$$

Then $f : I \rightarrow \mathbb{R}$ is Φ -convex if and only if the function $g : I \rightarrow \mathbb{R}$ defined by $g(x) := f(x) + cx^2$ is convex.

Proof. Define $\Psi(t) := \Phi(t) - ct^2$. Then, by the inequality in (28), Ψ is an error function and

$$\lim_{t \rightarrow 0^+} t^{-2}\Psi(t) = \lim_{t \rightarrow 0^+} t^{-2}(\Phi(t) - ct^2) = 0. \tag{29}$$

Assume that f is Φ -convex and let $g(x) := f(x) + cx^2$. Then

$$\begin{aligned} g(tx + (1-t)y) &= f(tx + (1-t)y) + c(tx + (1-t)y)^2 \\ &\leq tf(x) + (1-t)f(y) + t\Phi((1-t)|y-x|) + (1-t)\Phi(t|y-x|) \\ &\quad + c(tx + (1-t)y)^2 \\ &= tg(x) + (1-t)g(y) + t\Phi((1-t)|y-x|) + (1-t)\Phi(t|y-x|) \\ &\quad + c(tx + (1-t)y)^2 - ctx^2 - c(1-t)y^2 \\ &= tg(x) + (1-t)g(y) + t\Phi((1-t)|y-x|) + (1-t)\Phi(t|y-x|) \\ &\quad - ct((1-t)(x-y))^2 - c(1-t)(t(x-y))^2 \\ &= tg(x) + (1-t)g(y) + t\Psi((1-t)|y-x|) + (1-t)\Psi(t|y-x|). \end{aligned}$$

Therefore, g is Ψ -convex. In view of (29), we can apply Theorem 4.6 and obtain that $\Psi^\Gamma = 0$, and hence, by Theorem 4.8, we obtain that g is convex. \square

5. The Φ -convex envelop

For $\Phi \in \mathcal{E}_0(I)$ and $f : I \rightarrow \mathbb{R}$, define $\underline{C}_\Phi(f) : I \rightarrow [-\infty, \infty)$ by

$$\underline{C}_\Phi(f)(u) := \inf_{\substack{x,y \in I, t \in [0,1] \\ u = tx + (1-t)y}} t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)). \tag{30}$$

THEOREM 5.1. *Let $\Phi \in \mathcal{E}_0(I)$. Then the operator \underline{C}_Φ is monotone and concave in the pointwise sense, i.e., for $f, g : I \rightarrow \mathbb{R}$ and $\lambda \in [0, 1]$, we have*

$$\lambda \underline{C}_\Phi(f) + (1-\lambda)\underline{C}_\Phi(g) \leq \underline{C}_\Phi(\lambda f + (1-\lambda)g).$$

Furthermore, for all $f : I \rightarrow \mathbb{R}$, the inequality $\underline{C}_\Phi(f) \leq f$ holds and here equality is valid if and only if f is Φ -convex. In other words, the fixed points of \underline{C}_Φ are exactly the Φ -convex functions.

Proof. The monotonicity of \underline{C}_Φ is obvious. To see its concavity, observe that, by the definition, for all $u \in I$ and $x, y \in I, t \in [0, 1]$ with $u = tx + (1-t)y$, we have

$$\begin{aligned} \lambda \underline{C}_\Phi(f)(u) + (1-\lambda) \underline{C}_\Phi(g)(u) & \\ & \leq \lambda(t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|))) \\ & \quad + (1-\lambda)(t(g(x) + \Phi((1-t)|y-x|)) + (1-t)(g(y) + \Phi(t|y-x|))) \\ & = t((\lambda f + (1-\lambda)g)(x) + \Phi((1-t)|y-x|)) \\ & \quad + (1-t)((\lambda f + (1-\lambda)g)(y) + \Phi(t|y-x|)). \end{aligned}$$

Upon taking the infimum for $x, y \in I, t \in [0, 1]$ with $u = tx + (1-t)y$, it follows that

$$\lambda \underline{C}_\Phi(f)(u) + (1-\lambda) \underline{C}_\Phi(g)(u) \leq \underline{C}_\Phi(\lambda f + (1-\lambda)g)(u).$$

Taking $x := y := u$ in the definition of $\underline{C}_\Phi(f)(u)$ and using $\Phi(0) = 0$, we can see that $\underline{C}_\Phi(f)(u) \leq f(u)$ holds for all $u \in I$.

On the other hand, if $f = \underline{C}_\Phi(f)$ holds for some $f : I \rightarrow \mathbb{R}$, then, for all $x, y \in I$ and $t \in [0, 1]$ with $u := tx + (1-t)y$, we get

$$\begin{aligned} f(tx + (1-t)y) = f(u) = \underline{C}_\Phi(f)(u) & \\ \leq t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)), & \end{aligned}$$

which shows that f is Φ -convex.

Conversely, if f is Φ -convex, then, for all $x, y \in I$ and $t \in [0, 1]$ with $u = tx + (1-t)y$,

$$f(u) = f(tx + (1-t)y) \leq t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)).$$

Now, taking the infimum for $x, y \in I, t \in [0, 1]$ with $u = tx + (1-t)y$, it follows that $f(u) \leq \underline{C}_\Phi(f)(u)$. This implies that, in fact $f(u) = \underline{C}_\Phi(f)(u)$ and completes the proof of the equality $f = \underline{C}_\Phi(f)$. \square

In what follows, we construct the Φ -convex envelope of any function which admits a Φ -convex minorant in terms of the operator \underline{C}_Φ . The following auxiliary result will be useful.

LEMMA 5.2. *Let $\Phi \in \mathcal{E}_0(I)$ be increasing and subadditive. Then the function $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. In particular, for $p \in [0, 1]$, the function $\Psi_p(u) := -\Phi_p(|u|)$ is Φ_p -convex on \mathbb{R} .*

Proof. Let $x, y \in J$ and $t \in [0, 1]$. Then

$$\begin{aligned} -\Psi(x) = \Phi(|x|) & \leq \Phi(|tx + (1-t)y| + |(1-t)(x-y)|) \\ & \leq \Phi(|tx + (1-t)y|) + \Phi(|(1-t)(x-y)|) \\ & = -\Psi(tx + (1-t)y) + \Phi(|(1-t)(x-y)|) \end{aligned}$$

and

$$\begin{aligned}
 -\Psi(y) = \Phi(|y|) &\leq \Phi(|tx + (1-t)y| + |t(y-x)|) \\
 &\leq \Phi(|tx + (1-t)y|) + \Phi(|t(x-y)|) \\
 &= -\Psi(tx + (1-t)y) + \Phi(|t(x-y)|).
 \end{aligned}$$

Multiplying the first inequality by t and the second one by $(1-t)$ and then adding up the inequalities so obtained side by side, we obtain that

$$\begin{aligned}
 -t\Psi(x) - (1-t)\Psi(y) \\
 \leq -\Psi(tx + (1-t)y) + t\Phi(|(1-t)(x-y)|) + (1-t)\Phi(|t(x-y)|),
 \end{aligned}$$

which proves that Ψ is Φ -convex.

The last assertion of the lemma follows from the fact the Φ_p is nondecreasing and subadditive provided that $p \in [0, 1]$. \square

THEOREM 5.3. *Let $\Phi \in \mathcal{E}_0(I)$ such that $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .*

Proof. The inequality $\underline{C}_\Phi(f) \leq f$ follows from the previous theorem.

Now suppose g is a Φ -convex function such that $g \leq f$ holds (by the assumption, there is at least one such function g). Then, according to Theorem 5.1, $g = \underline{C}_\Phi(g) \leq \underline{C}_\Phi(f)$, which proves that $g \leq \underline{C}_\Phi(f)$.

To complete the proof, it will be sufficient to show that $g := \underline{C}_\Phi(f)$ is Φ -convex. Let $u \in I$ be fixed arbitrarily. Then, for all $x < u < y$, with $t := \frac{y-u}{y-x}$, we have $u = tx + (1-t)y$. Thus, the definition of $g(u) = \underline{C}_\Phi(f)(u)$ yields

$$g(u) \leq \frac{y-u}{y-x}(f(x) + \Phi(u-x)) + \frac{u-x}{y-x}(f(y) + \Phi(y-u)).$$

This inequality implies

$$\varphi(u) := \sup_{x < u} \frac{g(u) - f(x) - \Phi(u-x)}{u-x} \leq \inf_{u < y} \frac{f(y) + \Phi(y-u) - g(u)}{y-u}.$$

Therefore, for every $z \in I$, we have

$$g(u) + \varphi(u)(z-u) \leq f(z) + \Phi(|z-u|). \tag{31}$$

Let $v \in I$ and $\varepsilon > 0$ be arbitrary. Then, by the definition of $g(v) = \underline{C}_\Phi(f)(v)$, there exist $x, y \in I$, $t \in [0, 1]$ with $v = tx + (1-t)y$ such that

$$t(f(x) + \Phi((1-t)|y-x|)) + (1-t)(f(y) + \Phi(t|y-x|)) < g(v) + \varepsilon.$$

Now, applying the inequality (31) for $z := x$ and $z := y$, and in the last step using the Φ -convexity of Ψ , it follows that

$$\begin{aligned} &g(u) + \varphi(u)(v - u) \\ &= t(g(u) + \varphi(u)(x - u)) + (1 - t)(g(u) + \varphi(u)(y - u)) \\ &\leq t(f(x) + \Phi(|x - u|)) + (1 - t)(f(y) + \Phi(|y - u|)) \\ &< g(v) + \varepsilon + t(\Phi(|x - u|) - \Phi((1 - t)|y - x|)) + (1 - t)(\Phi(|y - u|) - \Phi(t|y - x|)) \\ &= g(v) + \varepsilon + t(-\Psi(x - u) - \Phi((1 - t)|y - x|)) + (1 - t)(-\Psi(y - u) - \Phi(t|y - x|)) \\ &\leq g(v) + \varepsilon - \Psi((tx + (1 - t)y) - u) = g(v) + \varepsilon + \Phi(|v - u|). \end{aligned}$$

Upon taking the limit $\varepsilon \rightarrow 0^+$, we obtain that g satisfies the inequality

$$g(u) + \varphi(u)(v - u) \leq g(v) + \Phi(|v - u|) \quad (u, v \in I).$$

Therefore, according to the characterization theorem of Φ -convexity, this implies that g is Φ -convex. \square

Combining the assertions of Lemma 5.2 and Theorem 5.3, we immediately obtain the following consequence.

COROLLARY 5.4. *Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing and subadditive. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant. Then the function $\underline{C}_\Phi(f)$ is the largest Φ -convex function which is smaller than or equal to f .*

The next result is an extension of a sandwich-type theorem of Baron–Matkowski–Nikodem [2].

THEOREM 5.5. *Let $\Phi \in \mathcal{E}_0(I)$ and let $f, g : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ for some Φ -convex function $h : I \rightarrow \mathbb{R}$. Then, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality*

$$g(tx + (1 - t)y) \leq tf(x) + (1 - t)f(y) + t(\Phi(1 - t)|y - x|) + (1 - t)\Phi(t|y - x|) \quad (32)$$

holds. Conversely, if $\Psi(u) := -\Phi(|u|)$ is Φ -convex on $J := I - I$ and (32) is valid for all $x, y \in I$ and $t \in [0, 1]$, then there exists a Φ -convex function $h : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ holds on I .

Proof. Let $f, g : I \rightarrow \mathbb{R}$ such that $g \leq h \leq f$ holds for some Φ -convex function h . Then, for all $x, y \in I$ and $t \in [0, 1]$, we have

$$\begin{aligned} g(tx + (1 - t)y) &\leq h(tx + (1 - t)y) \\ &\leq th(x) + (1 - t)h(y) + t(\Phi(1 - t)|y - x|) + (1 - t)\Phi(t|y - x|) \\ &\leq tf(x) + (1 - t)f(y) + t(\Phi(1 - t)|y - x|) + (1 - t)\Phi(t|y - x|). \end{aligned}$$

This shows the validity of (32) on the indicated domain.

Assume that $f, g : I \rightarrow \mathbb{R}$ satisfy (32). Then, this inequality yields that $g \leq \underline{C}_\Phi(f)$. In view of Theorem 5.3 and the Φ -convexity of Ψ , it follows that $\underline{C}_\Phi(f)$ is Φ -convex. Therefore, the statement is fulfilled with $h := \underline{C}_\Phi(f)$. \square

Combining the assertions of Lemma 5.2 and Theorem 5.5, we immediately obtain the following consequence.

COROLLARY 5.6. *Let $\Phi \in \mathcal{E}_0(I)$ be nondecreasing, subadditive and let $f, g : I \rightarrow \mathbb{R}$. Then the inequalities $g \leq h \leq f$ hold for some Φ -convex function $h : I \rightarrow \mathbb{R}$ if and only if, for all $x, y \in I$ and $t \in [0, 1]$, the functional inequality (32) is satisfied.*

THEOREM 5.7. *Let $\Phi \in \mathcal{E}_0(I)$. Let $f : I \rightarrow \mathbb{R}$ be a function which admits a Φ -convex minorant and define the sequence $f_n : I \rightarrow \mathbb{R}$ by*

$$f_1 := f, \quad f_{n+1} := \underline{C}_\Phi(f_n) \quad (n \in \mathbb{N}).$$

Then the sequence (f_n) is pointwise decreasing and its limit function is the largest Φ -convex function which is smaller than or equal to f .

Proof. The pointwise monotonicity of the sequence follows from the property $\underline{C}_\Phi(f) \leq f$ established in Theorem 5.1. Let f_0 denote the pointwise limit function of the sequence (f_n) . Now suppose g is an arbitrary Φ -convex function such that $g \leq f$. Then $g \leq \underline{C}_\Phi(g) \leq \underline{C}_\Phi(f) = \underline{C}_\Phi(f_1) = f_2$. Using the same iterative argument, we can show that $g \leq f_n$ for all $n \in \mathbb{N}$. Upon taking the limit, we get that $g \leq f_0$ holds.

Finally, we show the Φ -convexity of f_0 . Let $x, y \in I$ and $t \in [0, 1]$ be arbitrary. From definition of f_{n+1} it follows that

$$f_{n+1}(tx + (1-t)y) \leq t(f_n(x) + \Phi((1-t)|y-x|)) + (1-t)(f_n(y) + \Phi(t|y-x|)).$$

Upon taking the limit as $n \rightarrow \infty$, we get that

$$f_0(tx + (1-t)y) \leq t(f_0(x) + \Phi((1-t)|y-x|)) + (1-t)(f_0(y) + \Phi(t|y-x|)),$$

which shows that f_0 is Φ -convex, indeed. \square

Acknowledgement. The authors are thankful to the anonymous referee who called their attention to some recent papers and corrected some missprints.

REFERENCES

- [1] M. ADAMEK, *On F -Convex Functions*, J. Convex Anal., 28 (1), 2021.
- [2] K. BARON, J. MATKOWSKI, AND K. NIKODEM, *A sandwich with convexity*, Math. Pannon., 5 (1): 139–144, 1994.
- [3] Z. BOROS AND T. NAGY, *Approximately convex functions*, Ann. Univ. Sci. Budapest. Sect. Comput., 40: 143–150, 2013.
- [4] P. BURAI AND A. HÁZY, *On approximately h -convex functions*, J. Convex Anal., 18 (2): 447–454, 2011.
- [5] P. BURAI, A. HÁZY, AND T. JUHÁSZ, *On approximately Breckner s -convex functions*, Control Cybernet., 40 (1): 91–99, 2011.

- [6] S. J. DILWORTH, R. HOWARD, AND J. W. ROBERTS, *Extremal approximately convex functions and estimating the size of convex hulls*, Adv. Math., 148 (1): 1–43, 1999.
- [7] S. J. DILWORTH, R. HOWARD, AND J. W. ROBERTS, *On the size of approximately convex sets in normed spaces*, Studia Math., 140 (3): 213–241, 2000.
- [8] S. J. DILWORTH, R. HOWARD, AND J. W. ROBERTS, *Extremal approximately convex functions and the best constants in a theorem of Hyers and Ulam*, Adv. Math., 172 (1): 1–14, 2002.
- [9] R. GER, *Almost approximately convex functions*, Math. Slovaca, 38 (1): 61–78, 1988.
- [10] A. GILÁNYI, C. GONZÁLEZ, K. NIKODEM, AND ZS. PÁLES, *Bernstein-Doetsch type theorems with Tabor type error terms for set-valued maps*, Set-Valued Var. Anal., 25 (2): 441–462, 2017.
- [11] C. GONZÁLEZ, K. NIKODEM, ZS. PÁLES, AND G. ROA, *Bernstein-Doetsch type theorems for set-valued maps of strongly and approximately convex and concave type*, Publ. Math. Debrecen, 84 (1–2): 229–252, 2014.
- [12] A. GOSWAMI AND ZS. PÁLES, *On approximately monotone and approximately Hölder functions*, Per. Math. Hungar., 81 (1): 65–87, 2020.
- [13] A. GOSWAMI AND ZS. PÁLES, *Characterization of approximately monotone and approximately Hölder functions*, Math. Inequal. Appl., 24 (1): 247–264, 2021.
- [14] J. W. GREEN, *Approximately convex functions*, Duke Math. J., 19: 499–504, 1952.
- [15] D. H. HYERS AND S. M. ULAM, *Approximately convex functions*, Proc. Amer. Math. Soc., 3: 821–828, 1952.
- [16] A. HÁZY, *On approximate t -convexity*, Math. Inequal. Appl., 8 (3): 389–402, 2005.
- [17] A. HÁZY, *Bernstein-Doetsch type results for generalized approximately convex functions*, In Proc. MicroCAD 2010 Int. Sci. Conf., volume H, page 27–32, 2010.
- [18] A. HÁZY AND ZS. PÁLES, *On approximately midconvex functions*, Bull. London Math. Soc., 36 (3): 339–350, 2004.
- [19] A. HÁZY AND ZS. PÁLES, *On approximately t -convex functions*, Publ. Math. Debrecen, 66 (3–4): 489–501, 2005.
- [20] W. JARCZYK AND M. LACZKOVICH, *Convexity on abelian groups*, J. Convex Anal., 16 (1): 33–48, 2009.
- [21] W. JARCZYK AND M. LACZKOVICH, *Almost convex functions on locally compact abelian groups*, Math. Inequal. Appl., 13 (1): 217–225, 2010.
- [22] J. KRZYSZKOWSKI, *Approximately generalized convex functions*, Math. Pannon., 12 (1): 93–104, 2001.
- [23] M. LACZKOVICH, *The local stability of convexity, affinity and of the Jensen equation*, Aequationes Math., 58: 135–142, 1999.
- [24] J. MAKÓ, *A new proof of the approximate convexity of the Takagi function*, Acta Math. Hungar., 151 (2): 456–461, 2017.
- [25] J. MAKÓ AND A. HÁZY, *On approximate Hermite-Hadamard type inequalities*, J. Convex Anal., 24 (2): 349–363, 2017.
- [26] J. MAKÓ AND ZS. PÁLES, *Approximate convexity of Takagi type functions*, J. Math. Anal. Appl., 369 (2): 545–554, 2010.
- [27] J. MAKÓ AND ZS. PÁLES, *Strengthening of strong and approximate convexity*, Acta Math. Hungar., 132 (1–2): 78–91, 2011.
- [28] J. MAKÓ AND ZS. PÁLES, *Implications between approximate convexity properties and approximate Hermite-Hadamard inequalities*, Cent. Eur. J. Math., 10 (3): 1017–1041, 2012.
- [29] J. MAKÓ AND ZS. PÁLES, *Korovkin type theorems and approximate Hermite-Hadamard inequalities*, J. Approx. Theory, 164 (8): 1111–1142, 2012.
- [30] J. MAKÓ AND ZS. PÁLES, *On ϕ -convexity*, Publ. Math. Debrecen, 80 (1–2): 107–126, 2012.
- [31] J. MAKÓ AND ZS. PÁLES, *Approximate Hermite-Hadamard type inequalities for approximately convex functions*, Math. Inequal. Appl., 16 (2): 507–526, 2013.
- [32] J. MAKÓ AND ZS. PÁLES, *On approximately convex Takagi type functions*, Proc. Amer. Math. Soc., 141 (6): 2069–2080, 2013.
- [33] J. MROWIEC, *Remark on approximately Jensen-convex functions*, C. R. Math. Acad. Sci. Soc. R. Can., 23 (1): 16–21, 2001.
- [34] J. MROWIEC, JA. TABOR, AND JÓ. TABOR, *Approximately midconvex functions*, In *Inequalities and applications*, volume 157 of Internat. Ser. Numer. Math., page 261–265, Birkhäuser, Basel, 2009.

- [35] C. T. NG AND K. NIKODEM, *On approximately convex functions*, Proc. Amer. Math. Soc., 118 (1): 103–108, 1993.
- [36] K. NIKODEM AND ZS. PÁLES, *On approximately Jensen-convex and Wright-convex functions*, C. R. Math. Rep. Acad. Sci. Canada, 23 (4): 141–147, 2001.
- [37] K. NIKODEM AND ZS. PÁLES, *Generalized convexity and separation theorems*, J. Convex Anal., 14 (2): 239–248, 2007.
- [38] ZS. PÁLES, *Separation by approximately convex functions*, In D. Gronau and L. Reich, editors, *Contributions to the theory of functional equations, III* (Graz, 2000), volume 344 of Grazer Math. Ber., page 43–50, Karl-Franzens-Univ. Graz, Graz, 2001.
- [39] ZS. PÁLES, *On approximately convex functions*, Proc. Amer. Math. Soc., 131 (1): 243–252, 2003.
- [40] S. ROLEWICZ, *On γ -paraconvex multifunctions*, Math. Japon., 24 (3): 293–300, 1979.
- [41] P. SPUREK AND JA. TABOR, *Numerical verification of condition for approximately midconvex functions*, Aequationes Math., 2012.
- [42] JA. TABOR, JÓ. TABOR, AND M. ŻOŁDAK, *Approximately convex functions on topological vector spaces*, Publ. Math. Debrecen, 77: 115–123, 2010.
- [43] JA. TABOR, JÓ. TABOR, AND M. ŻOŁDAK, *Optimality estimations for approximately midconvex functions*, Aequationes Math., 80: 227–237, 2010.

(Received December 22, 2020)

Angshuman R. Goswami
Doctoral School of Mathematical and Computational Sciences
University of Debrecen
H-4002 Debrecen, Pf. 400, Hungary
e-mail: angshu@science.unideb.hu

Zsolt Páles
Institute of Mathematics
University of Debrecen
H-4002 Debrecen, Pf. 400, Hungary
e-mail: pales@science.unideb.hu