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SHARP EXPONENTIAL TYPE INEQUALITIES FOR THE ARC LEMNISCATE SINE FUNCTION WITH APPLICATIONS

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Abstract. In this paper, by proving some monotonicity theorems of certain combinations of the arc lemniscate sine function and elementary functions, we obtain two classes of exponential type inequalities for the arc lemniscate sine function. As applications, sharp bounds for the lemniscatic mean in terms of the arithmetic, harmonic and geometric means are given, which extend some previously known results.

1. Introduction

For $p,q \in (1,+\infty)$, the generalized inverse trigonometric sine function with two parameters is defined on $[-1,1]$ by

$$\operatorname{arcsin}_{p,q}(x) = \begin{cases} \displaystyle\int_0^x \frac{dt}{(1-t^q)^{1/p}}, & 0 \leqslant x \leqslant 1, \\ -\operatorname{arcsin}_{p,q}(-x), & -1 \leqslant x < 0 \end{cases}$$

(cf. [11, 15]). Let

$$\pi_{p,q} = 2\operatorname{arcsin}_{p,q}(1) = 2\int_0^1 \frac{dt}{(1-t^q)^{1/p}},$$

then it is easy to see that the function $x \mapsto \operatorname{arcsin}_{p,q}(x)$ is strictly increasing from $[-1,1]$ onto $[-\pi_{p,q}/2, \pi_{p,q}/2]$, and it has the inverse function $\sin_{p,q}(x)$ defined on $[-\pi_{p,q}/2, \pi_{p,q}/2]$. Particularly, when $p=q=2$, $\sin_{p,q}(x) = \sin(x)$, $\operatorname{arcsin}_{p,q}(x) = \arcsin(x)$ and $\pi_{p,q} = \pi$, so we also call $\sin_{p,q}$ and $\pi_{p,q}$ the generalized trigonometric sine function with two parameters and generalized circumference ratio respectively. For more basic knowledge of the above functions and their applications in the theories of differential equations and function spaces (cf. [4, 5, 6, 10, 13, 14, 22, 23]).

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As one of the most important special cases of $\arcsin_{p,q}(x)$, for $x \in [-1,1]$, the arc lemniscate sine function $\mathrm{arcsl}\,(x)$ is defined as follows

$$\mathrm{arcsl}\,(x) = \arcsin_{2,4}(x) = \int_0^x \frac{dt}{\sqrt{1-t^4}} \tag{1.1}$$

(cf. [7, p. 259], [8, (2.5)]). It is well known that the arc lemniscate sine function has a simple geometric interpretation that the arc length measured from the origin to a point with polar coordinates (r,θ) on the Bernoulli lemniscate $r^2 = \cos\theta$ is $\mathrm{arcsl}\,(r)$. As usual, we denote

$$\omega = \mathrm{arcsl}\,(1) = \frac{1}{\sqrt{2}}\mathscr{K}\,(1/\sqrt{2}) = \frac{\Gamma^2(1/4)}{4\sqrt{2\pi}} = 1.31103\cdots$$

the first lemniscate constant, where $\mathscr{K}\,(r) = \int_0^{\pi/2}(1-r^2\sin^2\theta)^{-1/2}d\theta\,(0<r<1)$ and $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t}dt\,(\mathrm{Re}(x)>0)$ are the complete elliptic integral of the first kind and the classical Euler gamma function respectively (cf. [1, 2, 3, 12, 20]).

Recently, the arc lemniscate sine function has attracted the attention of several researchers. In particular, many remarkable inequalities for $\mathrm{arcsl}\,(r)$ and other related special functions can be found in the literature [9, 16, 17, 18, 19, 25, 27, 28, 29]. For example, in 2012, Neuman[17, Theorem 5.1, (5.1)] proved that inequality

$$\left(\frac{3+2\sqrt{1-x^4}}{5}\right)^{-1/2} < \frac{\mathrm{arcsl}\,(x)}{x} < \frac{1}{(1-x^4)^{1/10}}$$

holds for all $0<|x|<1$.

In 2020, Zhao, Qian and Chu[28] refined Neuman's result and obtained that the double inequality

$$\left(\frac{3}{5}+\frac{2}{5}\sqrt{1-x^4}\right)^{-1/2} < \frac{\mathrm{arcsl}\,(x)}{x} < \left[\frac{1}{\omega^2}+\left(1-\frac{1}{\omega^2}\right)\sqrt{1-x^4}\right]^{-1/2}, \tag{1.2}$$

are valid for all $0<|x|<1$ with the best possible constants $3/5$ and $1/\omega^2$. Wei, He and Wang[25] proved that the inequalities

$$\left(\frac{4}{5}+\frac{1}{5}\sqrt{1-x^4}\right)^{-1} < \frac{\mathrm{arcsl}\,(x)}{x} < \left[\frac{1}{\omega}+\left(1-\frac{1}{\omega}\right)\sqrt{1-x^4}\right]^{-1} \tag{1.3}$$

and

$$\left[\frac{1}{\omega}+\left(1-\frac{1}{\omega}\right)\sqrt[4]{1-x^4}\right]^{-1} < \frac{\mathrm{arcsl}\,(x)}{x} < \left[\frac{3}{5}+\frac{2}{5}\sqrt[4]{1-x^4}\right]^{-1} \tag{1.4}$$

hold for all $0<|x|<1$.

Inspired by the lower and upper bounds for $[\mathrm{arcsl}\,(x)]/x$ in (1.2)–(1.4), in this paper, for $p,q \in \mathbf{R}$ and $\alpha,\beta \in (0,1)$, we introduce two classes of exponential type functions $A(p,\alpha;x)$ and $B(q,\beta;x)$ defined on $(0,1)$ as follows

$$A(p,\alpha;x) = \begin{cases} \left[(1-\alpha)+\alpha(1-x^4)^p\right]^{-1/(2p)}, & p\neq 0,\\ (1-x^4)^{-\alpha/2}, & p=0, \end{cases} \tag{1.5}$$

$$B(q,\beta;x)=\begin{cases}\left[(1-\beta)+\beta(1-x^4)^q\right]^{-1/(4q)}, & q\neq 0,\\ (1-x^4)^{-\beta/4}, & q=0.\end{cases} \tag{1.6}$$

Using (1.5) and (1.6), it is not difficult to verify that both $\alpha\mapsto A(p,\alpha;x)$ and $\beta\mapsto B(q,\beta;x)$ are increasing on $(0,1)$.

The main purpose of this paper is to find the best parameters $\alpha,\beta\in(0,1)$ depending on any fixed $p\in\mathbf{R}$, and $\lambda,\mu\in(0,1)$ depending on any fixed $q\in\mathbf{R}$ such that the double inequalities

$$A(p,\alpha;x)<\frac{\operatorname{arcsl}(x)}{x}<A(p,\beta;x),\quad B(q,\lambda;x)<\frac{\operatorname{arcsl}(x)}{x}<B(q,\mu;x)$$

hold for all $0<|x|<1$.

It is worth pointing out that arc lemniscate sine function $\operatorname{arcsl}(x)$, as well as the generalized inverse trigonometric sine function with two parameters $\arcsin_{p,q}(x)$, can be expressed by

$$\operatorname{arcsl}(x)=xF\left(\frac{1}{2},\frac{1}{4};\frac{5}{4};x^4\right)\quad\text{and}\quad \arcsin_{p,q}(x)=xF\left(\frac{1}{p},\frac{1}{q};1+\frac{1}{q};x^q\right) \tag{1.7}$$

(cf. [6]), where

$$F(a,b;c;x)=\sum_{n=0}^{\infty}\frac{(a,n)(b,n)}{(c,n)}\frac{x^n}{n!},\ (|x|<1)$$

is the classical Gaussian hypergeometric function (cf. [1, 3, 20, 24]) with the real parameters a,b, and $c\neq 0,-1,-2,\cdots$, and $(a,n)=\Gamma(a+n)/\Gamma(a)=a(a+1)(a+2)\cdots(a+n-1)$ is the shifted factorial function or the Pochhammer symbol for $n\in\mathbf{N}$.

2. Lemmas

In order to prove the main theorems we need serval lemmas, which we present in this section.

LEMMA 2.1. [3, Theorem 1.25] Let $a,b\in\mathbf{R}$ with $a<b$, $f,g:[a,b]\to\mathbf{R}$ be continuous on $[a,b]$ and differentiable on (a,b). Let $g'(x)\neq 0$ on (a,b). Then, if f'/g' is increasing (decreasing) on (a,b), then so are the functions

$$\frac{f(x)-f(a)}{g(x)-g(a)}\quad\text{and}\quad\frac{f(x)-f(b)}{g(x)-g(b)}.$$

If f'/g' is strict monotone, then the monotonicity in the conclusion is also strict.

LEMMA 2.2. [21, Lemma 2.1] Let $f(x)=\sum\limits_{k=0}^{\infty}a_kx^k$ and $g(x)=\sum\limits_{k=0}^{\infty}b_kx^k$ be two real power series converging on $(-r,r)$ and $b_k>0$ for all k. If the non-constant sequence $\{a_k/b_k\}$ is increasing (decreasing) for all k, then the function $x\mapsto f(x)/g(x)$ is strictly increasing (decreasing) on $(0,r)$.

LEMMA 2.3. [26, p.298, Example 11] *If* $a+b+1/2=c$, *then*

$$[F(a,b;c;x)]^2=\frac{\Gamma(c)\Gamma(2c-1)}{\Gamma(2a)\Gamma(2b)\Gamma(a+b)}\sum_{n=0}^{\infty}\frac{\Gamma(2a+n)\Gamma(a+b+n)\Gamma(2b+n)}{n!\Gamma(c+n)\Gamma(2c-1+n)}x^n$$

for all $x\in(-1,1)$.

LEMMA 2.4. *Let* $x\in(-1,1)$. *Then one has the following Maclaurin formulas*

$$[\operatorname{arcsl}(x)]^2=\sum_{n=0}^{\infty}\frac{(3/4,n)}{(2n+1)(5/4,n)}x^{4n+2} \tag{2.1}$$

and

$$\frac{\operatorname{arcsl}(x)}{\sqrt{1-x^4}}=\sum_{n=0}^{\infty}\frac{(3/4,n)}{(5/4,n)}x^{4n+1}. \tag{2.2}$$

Proof. Putting $a=1/2$, $b=1/4$, $c=5/4$ in Lemma 2.3 together with (1.7) yields

$$\begin{aligned}[\operatorname{arcsl}(x)]^2&=\left[xF(1/2,1/4;5/4;x^4)\right]^2\\&=x^2\frac{\Gamma(5/4)\Gamma(3/2)}{\Gamma(1)\Gamma(1/2)\Gamma(3/4)}\sum_{n=0}^{\infty}\frac{\Gamma(n+1)\Gamma(n+3/4)\Gamma(n+1/2)}{n!\Gamma(n+5/4)\Gamma(n+3/2)}x^{4n}\\&=\sum_{n=0}^{\infty}\frac{\Gamma(5/4)\Gamma(n+3/4)}{(2n+1)\Gamma(3/4)\Gamma(n+5/4)}x^{4n+2}=\sum_{n=0}^{\infty}\frac{(3/4,n)}{(2n+1)(5/4,n)}x^{4n+2}.\end{aligned}$$

Thus

$$\frac{\operatorname{arcsl}(x)}{\sqrt{1-x^4}}=\frac{1}{2}\frac{d\,[\operatorname{arcsl}(x)]^2}{dx}=\sum_{n=0}^{\infty}\frac{(3/4,n)}{(5/4,n)}x^{4n+1}. \quad \square$$

LEMMA 2.5. *For* $x\in(0,1)$, *let*

$$\begin{aligned}\varphi_1(x)&=x(3-x^4)\operatorname{arcsl}(x)/\sqrt{1-x^4}-4\,[\operatorname{arcsl}(x)]^2+x^2,\\\varphi_2(x)&=2x\operatorname{arcsl}(x)/\sqrt{1-x^4}-(1+x^4)\,[\operatorname{arcsl}(x)]^2-x^2,\\\varphi_3(x)&=x(2-x^4)\operatorname{arcsl}(x)/\sqrt{1-x^4}-[\operatorname{arcsl}(x)]^2-x^2.\end{aligned}$$

Then we have the following conclusions:

(1) *The function* $\xi(x)=\varphi_1(x)/\varphi_2(x)$ *is strictly increasing from* $(0,1)$ *onto* $(2/3,1)$;
(2) *The function* $\zeta(x)=\varphi_1(x)/\varphi_3(x)$ *is strictly increasing from* $(0,1)$ *onto* $(16/9,2)$.

Proof. (1) Employing (2.1) and (2.2) gives

$$\begin{aligned}\varphi_1(x)&=x(3-x^4)\sum_{n=0}^{\infty}\frac{(3/4,n)}{(5/4,n)}x^{4n+1}-4\sum_{n=0}^{\infty}\frac{(3/4,n)}{(2n+1)(5/4,n)}x^{4n+2}+x^2\\&=\sum_{n=1}^{\infty}\frac{3(3/4,n)}{(5/4,n)}x^{4n+2}-\sum_{n=1}^{\infty}\frac{4(3/4,n)}{(2n+1)(5/4,n)}x^{4n+2}-\sum_{n=0}^{\infty}\frac{(3/4,n)}{(5/4,n)}x^{4n+6}\end{aligned}$$

$$= \sum_{n=0}^{\infty} \frac{(6n+5)(3/4,n+1)}{(2n+3)(5/4,n+1)} x^{4n+6} - \sum_{n=0}^{\infty} \frac{(3/4,n)}{(5/4,n)} x^{4n+6}$$
$$= \sum_{n=0}^{\infty} \frac{16(n+1)(n+2)(4n+3)(3/4,n)}{(2n+5)(4n+5)(4n+9)(5/4,n)} x^{4n+10},$$
$$\varphi_2(x) = 2x \sum_{n=0}^{\infty} \frac{(3/4,n)}{(5/4,n)} x^{4n+1} - (1+x^4) \sum_{n=0}^{\infty} \frac{(3/4,n)}{(2n+1)(5/4,n)} x^{4n+2} - x^2$$
$$= \sum_{n=1}^{\infty} \frac{2(3/4,n)}{(5/4,n)} x^{4n+2} - \sum_{n=1}^{\infty} \frac{(3/4,n)}{(2n+1)(5/4,n)} x^{4n+2} - \sum_{n=0}^{\infty} \frac{(3/4,n)}{(2n+1)(5/4,n)} x^{4n+6}$$
$$= \sum_{n=0}^{\infty} \frac{(4n+5)(3/4,n+1)}{(2n+3)(5/4,n+1)} x^{4n+6} - \sum_{n=0}^{\infty} \frac{(3/4,n)}{(2n+1)(5/4,n)} x^{4n+6}$$
$$= \sum_{n=0}^{\infty} \frac{8(n+1)(n+2)(4n+3)(3/4,n)}{(2n+3)(2n+5)(4n+5)(5/4,n)} x^{4n+10}$$

and therefore

$$\xi(x) = \frac{\sum_{n=0}^{\infty} a_n x^{4n}}{\sum_{n=0}^{\infty} b_n x^{4n}},$$

where

$$a_n = \frac{16(n+1)(n+2)(4n+3)(3/4,n)}{(2n+5)(4n+5)(4n+9)(5/4,n)}, \quad b_n = \frac{8(n+1)(n+2)(4n+3)(3/4,n)}{(2n+3)(2n+5)(4n+5)(5/4,n)}. \tag{2.3}$$

Since $a_n/b_n = 1 - 3/(4n+9)$ is strictly increasing with respect to n, then $\xi(x)$ is strictly increasing on $(0,1)$ by application of Lemma 2.2. Moreover, $\xi(0^+) = a_0/b_0 = 2/3$ and $\xi(1^-) = 1$.

(2) Lemma 2.4 also shows

$$\varphi_3(x) = x(2-x^4) \sum_{n=0}^{\infty} \frac{(3/4,n)}{(5/4,n)} x^{4n+1} - \sum_{n=0}^{\infty} \frac{(3/4,n)}{(2n+1)(5/4,n)} x^{4n+2} - x^2$$
$$= \sum_{n=1}^{\infty} \frac{2(3/4,n)}{(5/4,n)} x^{4n+2} - \sum_{n=1}^{\infty} \frac{(3/4,n)}{(2n+1)(5/4,n)} x^{4n+2} - \sum_{n=0}^{\infty} \frac{(3/4,n)}{(5/4,n)} x^{4n+6}$$
$$= \sum_{n=0}^{\infty} \frac{(4n+3)(3/4,n)}{(2n+3)(5/4,n)} x^{4n+6} - \sum_{n=0}^{\infty} \frac{(3/4,n)}{(5/4,n)} x^{4n+6}$$
$$= \sum_{n=0}^{\infty} \frac{2(n+1)(4n+3)(3/4,n)}{(2n+5)(4n+5)(5/4,n)} x^{4n+10},$$

and therefore

$$\zeta(x) = \frac{\sum_{n=0}^{\infty} a_n x^{4n}}{\sum_{n=0}^{\infty} c_n x^{4n}},$$

where a_n is defined in (2.3), and

$$c_n = \frac{2(n+1)(4n+3)(3/4,n)}{(2n+5)(4n+5)(5/4,n)}.$$

Note that $a_n/c_n = 2-2/(4n+9)$, making use of Lemma 2.2, we conclude that $\zeta(x)$ increases on $(0,1)$. For the limiting values, $\zeta(0^+) = a_0/c_0 = 16/9$ and $\zeta(1^-) = 2$. □

LEMMA 2.6. *Let $p \in \mathbf{R}$ and $p_0 = (\log 5 - 2\log 2)/(2\log\omega) = 0.41198\cdots$. Define the function F_p on $(0,1)$ by*

$$F_p(x) = \begin{cases} \dfrac{1-[x/\mathrm{arcsl}\,(x)]^{2p}}{1-(1-x^4)^p}, & p \neq 0, \\ \dfrac{\log[x/\mathrm{arcsl}\,(x)]}{\log\left(\sqrt{1-x^4}\right)}, & p = 0. \end{cases} \tag{2.4}$$

Then the following statements are true:

(1) *$F_p(x)$ is strictly increasing on $(0,1)$ if and only if $p \in [1/2, +\infty)$, in which case the range of $F_p(x)$ on $(0,1)$ is $(1/5, 1-\omega^{-2p})$;*

(2) *$F_p(x)$ is strictly decreasing on $(0,1)$ if and only if $p \in (-\infty, 1/3]$, and $F_p(x)$ ranges from $(0,1)$ onto $(1-\omega^{-2p}, 1/5)$ if $p \in (0, 1/3]$, and while onto $(0, 1/5)$ if $p \in (-\infty, 0]$;*

(3) *If $p \in (1/3, 1/2)$, then there exists $x_0 = x_0(p) \in (0,1)$ such that $F_p(x)$ is strictly increasing on $(0, x_0)$ and strictly decreasing on $(x_0, 1)$. Consequently, if $p \in (1/3, p_0]$, then the inequality*

$$1-\omega^{-2p} < F_p(x) \leqslant \sigma_0 \tag{2.5}$$

holds for all $x \in (0,1)$, and while $p \in (p_0, 1/2)$,

$$1/5 < F_p(x) \leqslant \sigma_0 \tag{2.6}$$

holds for all $x \in (0,1)$, where $\sigma_0 = \sigma_0(p) = F_p(x_0)$.

Proof. First, we investigate the case of $p = 0$. Let $g_1(x) = \log[x/\mathrm{arcsl}\,(x)]$, $g_2(x) = \log(\sqrt{1-x^4})$. Then

$$g_1(0^+) = g_2(0) = 0, \; F_p(x) = g_1(x)/g_2(x),$$

$$\frac{g_1'(x)}{g_2'(x)} = \frac{\sqrt{1-x^4}\left[x - \sqrt{1-x^4}\,\mathrm{arcsl}\,(x)\right]}{2x^4\,\mathrm{arcsl}\,(x)},$$

$$\left[\frac{g_1'(x)}{g_2'(x)}\right]' = \frac{x(3-x^4)\,\mathrm{arcsl}\,(x)/\sqrt{1-x^4} - 4[\mathrm{arcsl}\,(x)]^2 + x^2}{2x^5[\mathrm{arcsl}\,(x)]^2} = -\frac{\varphi_1(x)}{2x^5[\mathrm{arcsl}\,(x)]^2}. \tag{2.7}$$

Here $\varphi_1(x)$ is defined in Lemma 2.5.

It follows from Lemma 2.5 and (2.7) that the function $g_1'(x)/g_2'(x)$ is strictly decreasing on $(0,1)$, so is $F_p(x)$ by application of Lemma 2.1. By l'Höptial's rule, we obtain $F_p(0^+) = 1/5$. Clearly, $F_p(1^-) = 0$.

Following we assume that $p \neq 0$. Let $f_1(x) = 1 - [x/\operatorname{arcsl}(x)]^{2p}$, $f_2(x) = 1 - (1 - x^4)^p$. Then simple computations lead to

$$f_1(0^+) = f_2(0) = 0,\ F_p(x) = f_1(x)/f_2(x), \tag{2.8}$$

$$f_1'(x) = 2p\left[\frac{x}{\operatorname{arcsl}(x)}\right]^{2p-1}\frac{x - \sqrt{1-x^4}\operatorname{arcsl}(x)}{\sqrt{1-x^4}[\operatorname{arcsl}(x)]^2},$$

$$f_2'(x) = 4px^3(1-x^4)^{p-1},$$

$$\frac{f_1'(x)}{f_2'(x)} = \left[\frac{x}{\operatorname{arcsl}(x)}\right]^{2p-1}\frac{x - \sqrt{1-x^4}\operatorname{arcsl}(x)}{2x^3(1-x^4)^{p-1/2}[\operatorname{arcsl}(x)]^2},$$

$$\left[\frac{f_1'(x)}{f_2'(x)}\right]' = \frac{x^{2p-5}\varphi_2(x)}{2(1-x^4)^p[\operatorname{arcsl}(x)]^{2p+2}}[2p - \xi(x)], \tag{2.9}$$

where $\varphi_2(x)$ and $\xi(x)$ are defined in Lemma 2.5.

We divide the proof into three cases.

Case 1 $p \in [1/2, +\infty)$. Then Lemma 2.5(1) and (2.9) lead to the conclusion that the function $f_1'(x)/f_2'(x)$ is strictly increasing on $(0,1)$. Hence the asserted monotonicity of $F_p(x)$ in part (1) follows from Lemma 2.1 and (2.8). Moreover, $F_p(0^+) = 1/5$ and $F_p(1^-) = 1 - \omega^{-2p}$.

Case 2 $p \in (-\infty, 0) \cup (0, 1/3]$. Then it follows from Lemma 2.5(1) and (2.9) that the function $f_1'(x)/f_2'(x)$ is strictly decreasing on $(0,1)$, so is $F_p(x)$ by Lemma 2.1. For the limiting values, $F_p(0^+) = 1/5$, and $F_p(1^-) = 1 - \omega^{-2p}$ if $p \in (0, 1/3)$, and while $F_p(1^-) = 0$ if $p \in (-\infty, 0)$.

Case 3 $p \in (1/3, 1/2)$. Noting that

$$F_p'(x) = \left[\frac{f_1(x)}{f_2(x)}\right]' = \frac{f_1'(x)f_2(x) - f_1(x)f_2'(x)}{[f_2(x)]^2} = \frac{f_2'(x)}{[f_2(x)]^2}H_{f_1,f_2}(x), \tag{2.10}$$

where

$$H_{f_1,f_2}(x) = \frac{f_1'(x)}{f_2'(x)}f_2(x) - f_1(x), \tag{2.11}$$

$$H_{f_1,f_2}'(x) = \left[\frac{f_1'(x)}{f_2'(x)}\right]' f_2(x) = \frac{x^{2p-5}[1-(1-x^4)^p]\varphi_2(x)}{2(1-x^4)^p[\operatorname{arcsl}(x)]^{2p+2}}[2p - \xi(x)], \tag{2.12}$$

$$H_{f_1,f_2}(0^+) = \lim_{x\to 0^+}\frac{f_1'(x)}{f_2'(x)}\lim_{x\to 0^+} f_2(x) - \lim_{x\to 0^+} f_1(x) = 0, \quad H_{f_1,f_2}(1^-) = \frac{1}{\omega^{2p}} - 1 < 0. \tag{2.13}$$

Since $p \in (1/3, 1/2)$, then equation (2.12) and Lemma 2.5(1) show that $H_{f_1,f_2}(x)$ is piecewise monotone on $(0,1)$, first increasing then decreasing. This, together with (2.13), gives that there exists $x_0 = x_0(p) \in (0,1)$ such that $H_{f_1,f_2}(x) > 0$ for $x \in (0, x_0)$, and $H_{f_1,f_2}(x) < 0$ for $x \in (x_0, 1)$. Hence $F_p(x)$ is strictly increasing on $(0, x_0)$ and strictly decreasing on $(x_0, 1)$ by (2.10). Consequently,

$$\min\{F_p(0^+), F_p(1^-)\} = \min\{1/5, 1 - \omega^{-2p}\} < F_p(x) \leqslant F_p(x_0) = \sigma_0 \tag{2.14}$$

for all $x \in (0,1)$. Furthermore, it can easily be checked that $p \mapsto 1-\omega^{-2p}$ is strictly increasing on $(1/3,1/2)$, and $1-\omega^{-2p} \leqslant 1/5$ for $p \in (1/3,p_0)$ and $1-\omega^{-2p} > 1/5$ for $p \in (p_0,1/2)$. Therefore, inequalities (2.5) and (2.6) are clear. $\square$

LEMMA 2.7. *Let $q \in \mathbf{R}$ and $q_0 = (\log 5 - \log 3)/(4\log\omega) = 0.4715\cdots$. Define the function G_q on $(0,1)$ by*

$$G_q(x) = \begin{cases} \dfrac{1-[x/\operatorname{arcsl}(x)]^{4q}}{1-(1-x^4)^q}, & q \neq 0, \\[2ex] \dfrac{\log[x/\operatorname{arcsl}(x)]}{\log\left(\sqrt[4]{1-x^4}\right)}, & q = 0. \end{cases} \tag{2.15}$$

Then the following statements are true:

(1) $G_q(x)$ is strictly increasing on $(0,1)$ if and only if $q \in [1/2,+\infty)$, in which case the range of $G_q(x)$ on $(0,1)$ is $(2/5, 1-\omega^{-4q})$;

(2) $G_q(x)$ is strictly decreasing on $(0,1)$ if and only if $q \in (-\infty,4/9]$, and $G_q(x)$ ranges from $(0,1)$ onto $(1-\omega^{-4q},2/5)$ if $q \in (0,4/9]$, and while onto $(0,2/5)$ if $q \in (-\infty,0]$;

(3) If $p \in (4/9,1/2)$, then there exists $x_1 = x_1(q) \in (0,1)$ such that $G_q(x)$ is strictly increasing on $(0,x_1)$ and strictly decreasing on $(x_1,1)$. Consequently, if $q \in (4/9,q_0]$, then the inequality

$$1-\omega^{-4q} < G_q(x) \leqslant \tau_0$$

holds for all $x \in (0,1)$, and while $q \in (q_0,1/2)$,

$$2/5 < G_q(x) \leqslant \tau_0$$

holds for all $x \in (0,1)$, where $\tau_0 = \tau_0(q) = G_q(x_1)$.

Proof. The monotonicity property and range of $G_0(x)$ on $(0,1)$ directly follow from those of $F_0(x)$ in Lemma 2.6. Following it suffices to investigate the case of $q \neq 0$. Suppose that $q \neq 0$. Let $h_1(x) = 1-[x/\operatorname{arcsl}(x)]^{4q}$, $h_2(x) = 1-(1-x^4)^q$. Then $h_1(0^+) = h_2(0) = 0$, $G_q(x) = h_1(x)/h_2(x)$ and

$$h_1'(x) = 4q\left[\frac{x}{\operatorname{arcsl}(x)}\right]^{4q-1}\frac{x-\sqrt{1-x^4}\operatorname{arcsl}(x)}{\sqrt{1-x^4}[\operatorname{arcsl}(x)]^2}, \quad h_2'(x) = 4qx^3(1-x^4)^{q-1}, \tag{2.16}$$

$$\frac{h_1'(x)}{h_2'(x)} = \left[\frac{x}{\operatorname{arcsl}(x)}\right]^{4q-1}\frac{x-\sqrt{1-x^4}\operatorname{arcsl}(x)}{x^3(1-x^4)^{q-1/2}[\operatorname{arcsl}(x)]^2}, \tag{2.17}$$

$$\left[\frac{h_1'(x)}{h_2'(x)}\right]' = \frac{x^{4q-5}\varphi_3(x)}{(1-x^4)^q[\operatorname{arcsl}(x)]^{2q+2}}\left[4q-\zeta(x)\right], \tag{2.18}$$

where $\varphi_3(x)$ and $\zeta(x)$ are defined in Lemma 2.5. Therefore, parts (1) and (2) follow from Lemma 2.5(2), Lemma 2.1 and (2.16)–(2.18) and together with the limiting values

$$G_q(0^+) = \lim_{x\to 0^+} \frac{h_1'(x)}{h_2'(x)} = \frac{2}{5}, \quad G_q(1^-) = \begin{cases} 1-\omega^{-4q}, & q>0, \\ 0, & q\leqslant 0. \end{cases}$$

For part (3), employing the auxiliary function $H_{f,g} = (f'/g')g - f$ defined in (2.11), one has

$$G_q'(x) = \left[\frac{h_1(x)}{h_2(x)}\right]' = \frac{h_2'(x)}{[h_2(x)]^2} H_{h_1,h_2}(x),$$

$$H_{h_1,h_2}'(x) = \left[\frac{h_1'(x)}{h_2'(x)}\right]' h_2(x) = \frac{x^{4q-5}[1-(1-x^4)^q]\varphi_3(x)}{(1-x^4)^q[\operatorname{arcsl}(x)]^{2q+2}}\left[4q-\zeta(x)\right],$$

$$H_{h_1,h_2}(0^+) = 0, \quad H_{h_1,h_2}(1^-) = \frac{1}{\omega^{4q}} - 1 < 0.$$

Thus there exists $x_1 = x_1(q) \in (0,1)$ such that $H_{h_1,h_2}(x) > 0$ for $x\in(0,x_1)$ and $H_{h_1,h_2}(x) < 0$ for $x\in(x_1,1)$, which shows that $G_q(x)$ is strictly increasing on $(0,x_1)$ and strictly decreasing on $(x_1,1)$. The proof of the remaining conclusions are completely similar to those given earlier for $F_p(x)$ with $p\in(1/3,1/2)$ in Lemma 2.6(3) and so is omitted. □

3. Main results

THEOREM 3.1. *Let $\alpha,\beta\in[0,1]$, and let $p_0 = (\log 5 - 2\log 2)/(2\log\omega) = 0.41198\cdots$ and σ_0 be defined in Lemma 2.6. Then for any fixed $p\in\mathbf{R}$, the double inequality*

$$A(p,\alpha;x) \leqslant \frac{\operatorname{arcsl}(x)}{x} \leqslant A(p,\beta;x) \tag{3.1}$$

holds for all $0<|x|<1$ if and only if $\alpha\leqslant\alpha_0$ and $\beta\geqslant\beta_0$, where

$$\alpha_0 = \begin{cases} 0, & p\in(-\infty,0], \\ 1-\omega^{-2p}, & p\in(0,p_0], \\ 1/5, & p\in(p_0,+\infty), \end{cases} \qquad \beta_0 = \begin{cases} 1/5, & p\in(-\infty,1/3], \\ \sigma_0, & p\in(1/3,1/2), \\ 1-\omega^{-2p}, & p\in[1/2,+\infty). \end{cases}$$

Inequality (3.1) becomes equality only for the case of $\beta=\beta_0=\sigma_0$.

Proof. Without loss of generality, we assume that $x\in(0,1)$. A routine computation shows that the inequality (3.1) is equivalent to

$$\alpha \leqslant F_p(x) \leqslant \beta, \tag{3.2}$$

where $F_p(x)$ is defined in (2.4).

Therefore, Theorem 3.1 follows directly from Lemma 2.6 and (3.2). □

THEOREM 3.2. *Let* $\lambda,\mu\in[0,1]$, *and let* $q_0=(\log 5-\log 3)/(4\log\omega)=0.4715\cdots$ *and* τ_0 *be defined in Lemma 2.7. Then for any fixed* $q\in\mathbf{R}$, *the double inequality*

$$B(q,\lambda;x)\leqslant\frac{\operatorname{arcsl}(x)}{x}\leqslant B(q,\mu;x) \tag{3.3}$$

holds for all $0<|x|<1$ *if and only if* $\lambda\leqslant\lambda_0$ *and* $\mu\geqslant\mu_0$, *where*

$$\lambda_0=\begin{cases}0, & q\in(-\infty,0],\\ 1-\omega^{-4q}, & q\in(0,q_0],\\ 2/5, & q\in(q_0,+\infty),\end{cases}\qquad \mu_0=\begin{cases}2/5, & q\in(-\infty,4/9],\\ \tau_0, & q\in(4/9,1/2),\\ 1-\omega^{-4q}, & q\in[1/2,+\infty).\end{cases}$$

Inequality (3.3) *becomes equality only for the case of* $\mu=\mu_0=\tau_0$.

Proof. Suppose that $x\in(0,1)$. Rewrite the inequality (3.3) as

$$\lambda\leqslant G_q(x)\leqslant\mu, \tag{3.4}$$

where $G_q(x)$ is defined in (2.15).

Therefore, Theorem 3.2 follows easily from Lemma 2.7 and (3.4). □

As an immediate consequence of Theorem 3.1, we obtain

COROLLARY 3.3. *Let* $p_0=(\log 5-2\log 2)/(2\log\omega)=0.41198\cdots$, *and* $p_1,p_2\in\mathbf{R}$. *Then inequality*

$$A\left(p_1,\frac{1}{5};x\right)<\frac{\operatorname{arcsl}(x)}{x}<A\left(p_2,\frac{1}{5};x\right) \tag{3.5}$$

holds for all $0<|x|<1$ *with the best possible constants* $p_1=p_0$ *and* $p_2=1/3$.

Proof. Inequality (3.5) can be directly derived from Theorem 3.1. Following it suffices to prove that p_0 and $1/3$ are the best possible constants. In fact, if $p\in(1/3,p_0)$, then from Theorem 3.1 we obtain that

$$A\left(p,1-\omega^{-2p};x\right)<\frac{\operatorname{arcsl}(x)}{x}\leqslant A\left(p,\sigma_0;x\right)$$

holds for all $0<|x|<1$ with the optimal parameters $1-\omega^{-2p}$ and σ_0. This, in conjunction with the fact that $1-\omega^{-2p}<1/5$ and $\sigma_0>1/5$ for $p\in(1/3,p_0)$ and the monotonicity of $\alpha\mapsto A(p,\alpha;x)$, shows that there exists $x_1,x_2\in(0,1)$ such that $A(p,1/5;x_1)>[\operatorname{arcsl}(x)]/x$ and $[\operatorname{arcsl}(x)]/x>A(p,1/5;x_2)$. □

Similarly, an application of Theorem 3.2 yields

COROLLARY 3.4. *Let* $q_0=(\log 5-\log 3)/(4\log\omega)=0.4715\cdots$, *and* $q_1,q_2\in\mathbf{R}$. *Then inequality*

$$B\left(q_1,\frac{2}{5};x\right)<\frac{\operatorname{arcsl}(x)}{x}<B\left(q_2,\frac{2}{5};x\right)$$

holds for all $0<|x|<1$ *with the best possible constants* $q_1=q_0$ *and* $q_2=4/9$.

REMARK 3.5. Put $p = 1/2$, Theorem 3.1 reduces to inequality (1.3), and substitute $1/2$ and $1/4$ for q in (3.3), we also obtain inequalities (1.2) and (1.4) immediately.

4. Applications

In this section, by applying our main results in Section 3, some new inequalities between the lemniscatic mean and several classical means will be established.

For $a,b > 0$ with $a \neq b$, the lemniscatic mean $LM(a,b)$ was introduced by Neumann in [16] as follows:

$$LM(a,b) = \begin{cases} \dfrac{\sqrt{a^2-b^2}}{\left[\operatorname{arcsl}\left(\sqrt[4]{1-b^2/a^2}\right)\right]^2}, & a > b, \\ \dfrac{\sqrt{b^2-a^2}}{\left[\operatorname{arcslh}\left(\sqrt[4]{b^2/a^2-1}\right)\right]^2}, & a < b, \end{cases} \tag{4.1}$$

where

$$\operatorname{arcslh}(x) = \int_0^x \frac{dt}{\sqrt{1+t^4}}, \quad x \in \mathbf{R}$$

is the hyperbolic arc lemniscate sine function (cf. [8, (2.6)]). It is apparent from (4.1) that the lemniscatic mean is homogenous of degree one in its variables and $LM(a,b) \neq LM(b,a)$ for $a \neq b$.

Let $H(a,b) = 2ab/(a+b)$, $A(a,b) = (a+b)/2$ and $G(a,b) = \sqrt{ab}$ be the harmonic, arithmetic and geometric mean of two positive numbers a and b. Then it is well known that $H(a,b) < G(a,b) < A(a,b)$ for all $a,b > 0$ with $a \neq b$. Using $A(a,b)$ and $G(a,b)$ in place of a and b in $LM(a,b)$ respectively, a new mean can be derived (cf. [16, (6.4)]):

$$LM_{AG}(a,b) = LM\left[A(a,b), G(a,b)\right]. \tag{4.2}$$

Since $H(a,b)$, $G(a,b)$, $A(a,b)$ and $LM_{AG}(a,b)$ are symmetric in a and b, without loss of generality, we assume that $a > b > 0$. Then simple computations yield

$$H(a,b) = A(a,b)(1-x^4), \quad G(a,b) = A(a,b)\sqrt{1-x^4} \tag{4.3}$$

and

$$LM_{AG}(a,b) = A(a,b)\left[\frac{x}{\operatorname{arcsl}(x)}\right]^2, \tag{4.4}$$

where $x = \sqrt{(a-b)/(a+b)} \in (0,1)$.

Combining (4.3) and (4.4) with our main results in Section 3 leads to the following propositions.

PROPOSITION 4.1. *Let $\alpha, \beta \in [0,1]$, and let $p_0 = (\log 5 - 2\log 2)/(2\log \omega) = 0.41198\cdots$ and σ_0 be defined as in Lemma 2.6. Then for any fixed $p \in \mathbf{R}$, the following mean inequality*

$$\left[(1-\beta)A^p(a,b) + \beta H^p(a,b)\right]^{1/p} \leqslant LM_{AG}(a,b) \leqslant \left[(1-\alpha)A^p(a,b) + \alpha H^p(a,b)\right]^{1/p}$$

holds for all $a,b>0$ *with* $a\neq b$ *if and only if* $\beta\geqslant\beta_0$ *and* $\alpha\leqslant\alpha_0$*, where*

$$\beta_0=\begin{cases}1/5, & p\in(-\infty,1/3],\\ \sigma_0, & p\in(1/3,1/2),\\ 1-\omega^{-2p}, & p\in[1/2,+\infty),\end{cases}\qquad \alpha_0=\begin{cases}0, & p\in(-\infty,0],\\ 1-\omega^{-2p}, & p\in(0,p_0],\\ 1/5, & p\in(p_0,+\infty).\end{cases}$$

Inequality becomes equality only for the case of $\beta=\beta_0=\sigma_0$.

PROPOSITION 4.2. *Let* $\lambda,\mu\in[0,1]$*, and let* $q_0=(\log 5-\log 3)/(4\log\omega)=0.4715\cdots$ *and* τ_0 *be defined as in Lemma 2.7. Then for any fixed* $q\in\mathbf{R}$*, the double inequality*

$$\left[(1-\mu)A^{2q}(a,b)+\mu G^{2q}(a,b)\right]^{1/(2q)}\leqslant LM_{AG}(a,b)\leqslant\left[(1-\lambda)A^{2q}(a,b)+\lambda G^{2q}(a,b)\right]^{1/(2q)}$$

holds for all all $a,b>0$ *with* $a\neq b$ *if and only if* $\mu\geqslant\mu_0$ *and* $\lambda\leqslant\lambda_0$*, where*

$$\mu_0=\begin{cases}2/5, & q\in(-\infty,4/9],\\ \tau_0, & q\in(4/9,1/2),\\ 1-\omega^{-4q}, & q\in[1/2,+\infty),\end{cases}\qquad \lambda_0=\begin{cases}0, & q\in(-\infty,0],\\ 1-\omega^{-4q}, & q\in(0,q_0],\\ 2/5, & q\in(q_0,+\infty).\end{cases}$$

Inequality becomes equality only for the case of $\mu=\mu_0=\tau_0$.

PROPOSITION 4.3. *Let* $p_0=(\log 5-2\log 2)/(2\log\omega)=0.41198\cdots$*, and* $p_1,p_2\in\mathbf{R}$*. Then the inequality*

$$\left[\frac{4}{5}A^{p_2}(a,b)+\frac{1}{5}H^{p_2}(a,b)\right]^{1/p_2}<LM_{AG}(a,b)<\left[\frac{4}{5}A^{p_1}(a,b)+\frac{1}{5}H^{p_1}(a,b)\right]^{1/p_1}$$

holds for all $a,b>0$ *with* $a\neq b$ *with the best possible constants* $p_1=p_0$ *and* $p_2=1/3$.

By (4.1) and (4.2), Proposition 4.2 can be rewritten as

PROPOSITION 4.4. *Let* $\lambda^*,\mu^*\in[0,1]$*, and let* $q_0^*=(\log 5-\log 3)/(2\log\omega)=0.9431\cdots$*. Then for any fixed* $q\in\mathbf{R}$*, the double inequality*

$$[(1-\mu)a^q+\mu b^q]^{1/q}\leqslant LM(a,b)\leqslant[(1-\lambda)a^q+\lambda b^q]^{1/q}$$

holds for all all $a>b>0$ *if and only if* $\mu^*\geqslant\mu_0^*$ *and* $\lambda^*\leqslant\lambda_0^*$*, where*

$$\mu_0^*=\begin{cases}2/5, & q\in(-\infty,8/9],\\ \tau_0^*, & q\in(8/9,1),\\ 1-\omega^{-2q}, & q\in[1,+\infty),\end{cases}\qquad \lambda_0^*=\begin{cases}0, & q\in(-\infty,0],\\ 1-\omega^{-2q}, & q\in(0,q_0^*],\\ 2/5, & q\in(q_0^*,+\infty),\end{cases}$$

and $\tau_0^*=\tau_0^*(q)=\max\limits_{x\in(0,1)}[1-(x/\operatorname{arcsl}(x))^{2q}]/[1-(1-x^4)^{q/2}]$*. Inequality becomes equality only for the case of* $\mu^*=\mu_0^*=\tau_0^*$.

PROPOSITION 4.5. *Let* $q_0^* = (\log 5 - \log 3)/(2\log\omega) = 0.9431\cdots$, *and* $q_1^*, q_2^* \in$ $\mathbf{R}$. *Then the inequality*

$$\left[\frac{3}{5}a^{q_2^*} + \frac{2}{5}b^{q_2^*}\right]^{1/q_2^*} < LM(a,b) < \left[\frac{3}{5}a^{q_1^*} + \frac{2}{5}b^{q_1^*}\right]^{1/q_1^*}$$

holds for all all $a > b > 0$ *with the best possible constants* $q_1^* = q_0^*$ *and* $q_2^* = 8/9$.

REMARK 4.6. If we put $p = 1$, then Proposition 4.1 reduces to [28, Theorem 3.2].

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