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A NEW PARTIAL ORDER BASED ON CORE PARTIAL ORDER AND STAR PARTIAL ORDER

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Abstract. In this paper, we introduce a new partial order called CS partial order, which is based on the core partial order and star partial order. We give some characteristics of CS partial order by using polar decomposition. Then we apply the polar-like decomposition to introduce another new partial order called WC (weak CS) partial order, which is an extension of the CS partial order. We illustrate the relationships of the two partial orders with some well-known partial orders, such as minus, Löwner, GL, CL partial order. At the end of the paper, the application of CS and WC partial orders is briefly discussed from the perspective of quantum physics.

1. Introduction

A binary relation on a nonempty set is called partial order if it satisfies reflexivity, transitivity, and antisymmetry. In recent years, more and more mathematicians have turned their attention to matrix partial ordering: Jan Hauke and Augustyn Markiewicz [8] introduced the GL partial order on the set of rectangular matrices; Baksalary and Trenkler [1] studied the core partial order of complex matrices; Hongxing Wang and Xiaoji Liu [16] introduced the CL partial order on the class of core matrices; Hongxing Wang and Xiaoji Liu [18] defined the WL partial order on the set of rectangular matrices, which is the extension of GL partial order, by applying polar-like decomposition. In this paper, a new partial order called CS partial order is introduced on the complex matrix set by polar decomposition and core and star partial orders. Then, we introduce another new partial order called WC partial order which is the extension of the CS order by polar-like decomposition.

First of all, we use the following notation. The symbols $\mathbb{C}_{m,n}$ and $\mathbb{C}_{n,n}$ denote the set of $m\times n$ and $n\times n$ matrices with complex entries, respectively. The subset of $\mathbb{C}_{n,n}$ consisting of Hermitian nonnegative definite matrices will be denoted by $\mathbb{C}_n^{\geqslant}$, and its subset consisting of positive definite matrices by $\mathbb{C}_n^{>}$. A^*, $\mathfrak{R}(A)$ and $rk(A)$ denote the conjugate transpose, range space (or column space) and rank of $A\in\mathbb{C}_{m,n}$, respectively. The smallest positive integer k for which $rk\left(A^{k+1}\right)=rk\left(A^k\right)$ is called the index of $A\in\mathbb{C}_{n,n}$ and is denoted by $\mathrm{Ind}(A)$. The symbol $\mathbb{C}_n^{CM}$ stands for a set of $n\times n$ matrices

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of index less than or equal to 1. Moreover, I_n is the identity matrix of order n. The Moore-Penrose inverse of $A \in \mathbb{C}_{m,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,m}$ satisfying the equations

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (3)\ (AX)^* = AX,\ (4)\ (XA)^* = XA,$$

and is usually denoted by $X = A^{\dagger}$. A matrix X is called a generalized inverse of A, denoted as $X = A^{-}$, if it satisfies $AXA = A$. Furthermore, we denote $P_A = AA^{\dagger}$. The group inverse of $A \in \mathbb{C}_{n,n}$ is defined as the unique matrix $X \in \mathbb{C}_{n,n}$ satisfying the equations

$$(1)\ AXA = A,\ (2)\ XAX = X,\ (5)\ AX = XA,$$

and is usually denoted as $X = A^{\#}$.

We give the definitions of some well-known partial orders such as minus, Löwner, star, core, GL and CL partial orders, which are defined in the following:

(a) $A \overset{-}{\leqslant} B:\ A, B \in \mathbb{C}_{m,n},\ rk(B) - rk(A) = rk(B - A)$;

(b) $A \overset{L}{\leqslant} B:\ A, B \in \mathbb{C}_{m,n},\ B - A = KK^*$;

(c) $A \overset{*}{\leqslant} B:\ A, B \in \mathbb{C}_{m,n},\ AA^* = BA^*$ and $A^*A = A^*B$;

(d) $A \overset{\#\!\!\!\bigcirc}{\leqslant} B:\ A, B \in \mathbb{C}_n^{CM},\ A^{\#\!\!\!\bigcirc}A = A^{\#\!\!\!\bigcirc}B$ and $AA^{\#\!\!\!\bigcirc} = BA^{\#\!\!\!\bigcirc}$;

(e) $A \overset{GL}{\leqslant} B:\ A, B \in \mathbb{C}_{m,n},\ \mathfrak{R}(A) \subseteq \mathfrak{R}(B),\ \mathfrak{R}(A^*) \subseteq \mathfrak{R}(B^*),\ \lambda_{\max}\left(B^{\dagger}A\right) \leqslant 1,\ AB^* = (AA^*)^{\frac{1}{2}}(BB^*)^{\frac{1}{2}}$;

(f) $A \overset{CL}{\leqslant} B:\ A, B \in \mathbb{C}_n^{CM},\ A^{\#\!\!\!\bigcirc}A \overset{\#\!\!\!\bigcirc}{\leqslant} B^{\#\!\!\!\bigcirc}B$ and $A^2A^{\#\!\!\!\bigcirc} \overset{L}{\leqslant} B^2B^{\#\!\!\!\bigcirc}$.

In recent years, matrix partial order has attracted a lot of attentions and researches in various aspects. For example, the simultaneous polar decomposability of a pair of rectangular matrices is derived in [13]. The unique weighted polar decomposition theorem and the WGL partial order is given in [21]. The core inverse and core partial order can refer to [17, 15, 3, 12]. And the researches about other matrix partial orders can be referred to [2, 4, 19, 6, 14, 10, 11, 5, 20].

In this paper, we consider the matrices on complex field. We introduce the first new partial order on the set of $m \times n$ matrices based on the core partial order and the star partial order by applying the polar decomposition, and then introduce another new partial order which is the extension of the first new partial order by the polar-like decomposition.

2. Preliminary results

In this section, we give some preliminary results which can refer to [1, Theorem 1], [7, Corollary 6], [1, Lemma 3], [8, Lemma 3], [8, Lemma 2], [1, Theorem 7], [18, Theorem 2.1], [18, Theorem 2.6], [1, Lemma 5] and these results will be used in the next section to induce the new partial orders and characterize them.

LEMMA 1. *Let $A \in \mathbb{C}_n^{CM}$. Then*

$$A^{\#\!\!\!\bigcirc} = A^{\#}AA^{\dagger}. \tag{1}$$

LEMMA 2. *Let $A \in \mathbb{C}_{n,n}$, and $rk(A) = r$. Then A can be represented in the form*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \tag{2}$$

where $U \in \mathbb{C}_{n,n}$ is unitary, $\Sigma = diag(\sigma_1 I_{r_1}, \ldots, \sigma_t I_{r_t})$ is the diagonal matrix of singular values of A, $\sigma_1 > \sigma_2 > \cdots > \sigma_t$, $r_1 + r_2 + \cdots + r_t = r$, and $K \in \mathbb{C}_{r,r}$, $L \in \mathbb{C}_{r,n-r}$ satisfy

$$KK^* + LL^* = I_r.$$

LEMMA 3. *Let $A, B \in \mathbb{C}_n^{CM}$, and let A be of the form (2). Then $A \overset{\#}{\leqslant} B$ if and only if*

$$B = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix} U^*, \tag{3}$$

where ΣK is nonsingular and $Z \in \mathbb{C}_{n-r,n-r}$ is some matrix of index one.

LEMMA 4. *Let $A \in \mathbb{C}_{m,n}$. Then A can be written as*

$$A = G_A E_A = E_A H_A, \tag{4}$$

where $E_A \in \mathbb{C}_{m,n}$ is a partial isometry, i.e., $E_A^* = E_A^\dagger$, and $G_A, H_A \in \mathbb{C}_m^{\geqslant}$. The matrices E_A, G_A, H_A are uniquely determined by $\Re(E_A) = \Re(G_A)$, $\Re(E_A^*) = \Re(H_A)$ in which case $G_A = |A|$, $H_A = |A^*|$ and E_A is given by $E_A = G_A^\dagger A = A H_A^\dagger$.

LEMMA 5. *Let $A, B \in \mathbb{C}_{m,n}$ with $rk(A) = a$ and $rk(B) = b$. Then $A \overset{*}{\leqslant} B$ if and only if there exist unitary matrices U and V such that*

$$U^*AV = \begin{bmatrix} D_a & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad U^*BV = \begin{bmatrix} D_b & 0 & 0 \\ 0 & D & 0 \\ 0 & 0 & 0 \end{bmatrix}, \tag{5}$$

where both matrices $D_a \in \mathbb{C}_a^{>}$ and $D_b \in \mathbb{C}_b^{>}$ are diagonal.

LEMMA 6. *Let $A, B \in \mathbb{C}_n^{CM}$, and let A be EP. Then $A \overset{\#}{\leqslant} B$ if and only if $A \overset{*}{\leqslant} B$.*

LEMMA 7. *Let $A \in \mathbb{C}_{m,n}$. Then the polar-like decomposition of A can be written as*

$$A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}, \tag{6}$$

where E_A, G_A, H_A are given in Lemma 4.

LEMMA 8. *Let $A \in \mathbb{C}_{m,n}$. Then the polar-like decomposition of $A^\dagger$ can be written as*

$$A^\dagger = \left(H_A^\dagger\right)^{\frac{1}{2}} E_A^* \left(G_A^\dagger\right)^{\frac{1}{2}}, \tag{7}$$

where E_A, G_A, H_A are given in Lemma 4.

LEMMA 9. *Let $A, B \in \mathbb{C}_n^{CM}$,*

$$A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*, \quad B = U \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} U^*, \tag{8}$$

where U, Σ, K, L are given in Lemma 2, $W \in \mathbb{C}_{r,r}$, $Z \in \mathbb{C}_{n-r,n-r}$. Then $A^2 \overset{\#}{\leqslant} B^2$ if and only if

(i) $YW + ZY = 0$,
(ii) $W^2 + XY = (\Sigma K)^2$,
(iii) $WX + XZ = \Sigma K \Sigma L$.

3. Main results

First, we introduce the definition of the CS order.

DEFINITION 1. Let $A, B \in \mathbb{C}_{m,n}$ and $AA^*, BA^* \in \mathbb{C}_n^{CM}$, $\mathfrak{R}(A^*) \subseteq \mathfrak{R}(B^*)$. We say A is below B under the CS order if

$$AA^* \overset{\#}{\leqslant} BA^*, \tag{9}$$

if so, we write $A \overset{CS}{\leqslant} B$.

THEOREM 1. *The CS order is a partial order on $\mathbb{C}_{m,n}$.*

Proof. The reflexivity is obvious.

Antisymmetry: If $A \overset{CS}{\leqslant} B$ and $B \overset{CS}{\leqslant} A$, $rk(A) = a$, $rk(B) = b$, $a \leqslant b$, we can apply the singular value decompositions for $A = V_A \Lambda_A W_A^*$ and $B = V_B \Lambda_B W_B^*$, where $\Lambda_A \in \mathbb{C}_a^>$ and $\Lambda_B \in \mathbb{C}_b^>$ are diagonal matrices, while $V_A \in \mathbb{C}_{m,a}, W_A \in \mathbb{C}_{n,a}, V_B \in \mathbb{C}_{m,b}$ and $W_B \in \mathbb{C}_{n,b}$ are isometries, i.e., $V_A^* V_A = W_A^* W_A = I_a$ and $V_B^* V_B = W_B^* W_B = I_b$. By calculating we have that $V_A^* V_B = W_A^* W_B$ and $W_A = W_B W_B^* W_A$. Then with the Definition 1, $A \overset{CS}{\leqslant} B \Leftrightarrow V_A \Lambda_A W_A^* W_A \Lambda_A V_A^* \overset{\#}{\leqslant} V_B \Lambda_B W_B^* W_A \Lambda_A V_A^*$. According to $V_A^* V_A = W_A^* W_A = I_a$ and $V_A^* V_B = W_A^* W_B$, we can convert the inequality to $V_A \Lambda_A V_A^* V_A \Lambda_A V_A^* \overset{\#}{\leqslant} V_B \Lambda_B V_B^* V_A \Lambda_A V_A^*$. Then postmultiplying the inequality by $(V_A \Lambda_A V_A^*)^{-1}$ and we can get $V_A \Lambda_A V_A^* \overset{\#}{\leqslant} V_B \Lambda_B V_B^*$. Similarly, we can convert $B \overset{CS}{\leqslant} A$ to $V_B \Lambda_B V_B^* \overset{\#}{\leqslant} V_A \Lambda_A V_A^*$. With the definition of core partial order, we can have that $V_A \Lambda_A V_A^* = V_B \Lambda_B V_B^*$. Moreover, postmultiplying the equality by $V_B W_B^*$ and we derive that $V_A \Lambda_A W_A^* = V_B \Lambda_B W_B^*$, that is $A = B$, the antisymmetry condition holds.

Transitivity: If $A \overset{CS}{\leqslant} B$ and $B \overset{CS}{\leqslant} C$, using the singular value decompositions and the Definition 1 once again and we can get $A \overset{CS}{\leqslant} B \Leftrightarrow V_A \Lambda_A W_A^* W_A \Lambda_A V_A^* \overset{\#}{\leqslant} V_B \Lambda_B W_B^* W_A \Lambda_A V_A^*$. Then, converting the inequality to $V_A \Lambda_A V_A^* V_A \Lambda_A V_A^* \overset{\#}{\leqslant} V_B \Lambda_B V_B^* V_A \Lambda_A V_A^*$ and postmultiplying inequality by $(V_A \Lambda_A V_A^*)^{-1}$, and we can get $V_A \Lambda_A V_A^* \overset{\#}{\leqslant} V_B \Lambda_B V_B^*$. Similarly, we can convert $B \overset{CS}{\leqslant} C$ to $V_B\Lambda_B V_B^* \overset{\textcircled{\#}}{\leqslant} V_C\Lambda_C V_C^*$. Then we can derive that $V_A\Lambda_A V_A^* \overset{\textcircled{\#}}{\leqslant} V_C\Lambda_C V_C^*$ with the definition of core partial order. Postmultiplying the inequality $V_A\Lambda_A V_A^* \overset{\textcircled{\#}}{\leqslant} V_C\Lambda_C V_C^*$ by $V_C W_C^* W_A \Lambda_A V_A^*$ and we derive that $AA^* \overset{\textcircled{\#}}{\leqslant} CA^*$, which is $A \overset{CS}{\leqslant} C$ with the Definition 1. The transitivity condition holds.

The proof is complete. $\square$

THEOREM 2. *For* $A,B \in \mathbb{C}_{m,n}$, $A \overset{CS}{\leqslant} B$ *if and only if* $A^* \overset{CS}{\leqslant} B^*$.

Proof. Let $A,B \in \mathbb{C}_{m,n}$, and $rk(A) = a$, $rk(B) = b$, $a \leqslant b$. We consider the singular value decompositions for $A = V_A\Lambda_A W_A^*$ and $B = V_B\Lambda_B W_B^*$, where $\Lambda_A \in \mathbb{C}_a^>$ and $\Lambda_B \in \mathbb{C}_b^>$ are diagonal matrices, while $V_A \in \mathbb{C}_{m,a}, W_A \in \mathbb{C}_{n,a}, V_B \in \mathbb{C}_{m,b}$ and $W_B \in \mathbb{C}_{n,b}$ are isometries, i.e., $V_A^*V_A = W_A^*W_A = I_a$ and $V_B^*V_B = W_B^*W_B = I_b$. With Definition 1, we have

$$V_A\Lambda_A W_A^* W_A\Lambda_A V_A^* \overset{\textcircled{\#}}{\leqslant} V_B\Lambda_B W_B^* W_A\Lambda_A V_A^*. \tag{10}$$

Then, postmultiplying the inequality (10) by $(W_A\Lambda_A V_A^*)^{-1}$, and we get the equivalent form

$$V_A\Lambda_A W_A^* \overset{\textcircled{\#}}{\leqslant} V_B\Lambda_B W_B^*. \tag{11}$$

We notice that

$$V_A^*V_B = W_A^*W_B \quad \text{and} \quad W_A = W_B W_B^* W_A. \tag{12}$$

Premultiplying the inequality (11) by $W_B V_B^*$ and postmultiplying by $W_B V_B^*$, with (12), we can get

$$W_A\Lambda_A V_A^* \overset{\textcircled{\#}}{\leqslant} W_B\Lambda_B V_B^*. \tag{13}$$

Then postmultiplying inequality (13) by $V_A\Lambda_A W_A^*$, we obtain $A^*A \overset{\textcircled{\#}}{\leqslant} B^*A$, and applying Definition 1 once more, i.e., $A^* \overset{CS}{\leqslant} B^*$. The proof is complete. $\square$

It should be noted that in the proof of Theorem 2, we consider the property of inequality not the core partial order. The proof can be done by a single transformation, which is premultiplying the inequality (10) by $W_B V_B^*$ and postmultiplying by $(W_A\Lambda_A V_A^*)^{-1} W_B V_B^* V_A \Lambda_A W_A^*$.

THEOREM 3. *For any* $A,B \in \mathbb{C}_{m,n}$,

$$\text{if } A \overset{*}{\leqslant} B \text{ then } A \overset{CS}{\leqslant} B. \tag{14}$$

Proof. It can be straightforward derived from Definition 1, Lemma 5 and the definition of star partial order. $\square$

The next theorem asserts that the CS partial order can imply the star partial order under special conditions.

THEOREM 4. *Let A be EP. Then,*

$$A \overset{CS}{\leqslant} B \Rightarrow A \overset{*}{\leqslant} B. \tag{15}$$

Proof. If $A \overset{CS}{\leqslant} B$, with Definition 1, we know that $AA^* \overset{\#}{\leqslant} BA^*$ and $AA^*, BA^* \in \mathbb{C}_n^{CM}$. As A is EP matrix, we have AA^* is still the EP matrix. Applying Lemma 6, we can get

$$AA^* \overset{*}{\leqslant} BA^*, \tag{16}$$

and postmultiplying the inequality by $(A^*)^{-1}$, we have $A \overset{*}{\leqslant} B$. □

THEOREM 5. *Let $A, B \in \mathbb{C}_{m,n}$, the matrices $A = G_A E_A$ and $B = G_B E_B$ are their polar decompositions, where $\mathfrak{R}(E_A) = \mathfrak{R}(G_A)$, $\mathfrak{R}(E_B) = \mathfrak{R}(G_B)$, $G_A, G_B \in \mathbb{C}_m^{CM} \cap \mathbb{C}_m^{\geqslant}$. Then*

$$A \overset{CS}{\leqslant} B \Leftrightarrow G_A \overset{\#}{\leqslant} G_B \quad and \quad E_A \overset{*}{\leqslant} E_B. \tag{17}$$

Proof. According to $G_A \overset{\#}{\leqslant} G_B$, Lemma 2, Lemma 3 and the conditions of G_A, G_B, we can get

$$G_A = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{bmatrix} U^*; \; G_B = U \begin{bmatrix} \Sigma_1 & 0 \\ 0 & \Sigma_2 \end{bmatrix} U^*, \tag{18}$$

where $U \in \mathbb{C}_{m,m}$ is unitary matrix, Σ_1 is the diagonal matrix of singular values of G_A, $\Sigma_1 + \Sigma_2$ is the diagonal matrix of singular values of G_B. With Definition 1 and Lemma 4, we have

$$G_A E_A (G_A E_A)^* \overset{\#}{\leqslant} G_B E_B (G_A E_A)^*. \tag{19}$$

This inequality (19) can be written as

$$G_A E_A E_A^* G_A^* \overset{\#}{\leqslant} G_B E_B E_A^* G_A^*. \tag{20}$$

We consider $E_A E_A^*$, $E_B E_A^*$ as Q_1, Q_2, respectively. According to the conditions of E_A and E_B, we can denote $Q_1 = U \begin{bmatrix} X_{11} & 0 \\ 0 & 0 \end{bmatrix} U^*$, $Q_2 = U \begin{bmatrix} Y_{11} & 0 \\ 0 & 0 \end{bmatrix} U^*$ where X_{11} is the diagonal matrix of singular values of $E_A E_A^*$, Y_{11} is upper-triangular matrix which the main diagonal is singular values of $E_B E_A^*$. Then,

$$G_A E_A E_A^* G_A^* = U \begin{bmatrix} S_1 & 0 \\ 0 & 0 \end{bmatrix} U^*; \; G_B E_B E_A^* G_A^* = U \begin{bmatrix} S_2 & 0 \\ 0 & 0 \end{bmatrix} U^*, \tag{21}$$

where $S_1 = \Sigma_1 X_{11} (\Sigma_1)^*$, $S_2 = \Sigma_1 Y_{11} (\Sigma_1)^*$.

With Lemma 1, we can calculate that

$$(G_A E_A E_A^* G_A^*)^{\#} = U \begin{bmatrix} S_1^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*. \tag{22}$$

Applying the definition of core partial order and (21), (22) to calculate, we have $X_{11} = Y_{11}$, and thus $Q_1 = Q_2$ which can indicate that $E_A E_A^* = E_B E_A^*$.

With Lemma 4 and Definition 1, observing that $G_A G_A^\dagger = E_A^* E_A = P_{A^*}$ and $G_B G_B^\dagger = E_B^* E_B = P_{B^*}$; then the relation $\Re(A^*) \subseteq \Re(B^*)$ is equivalent to $E_B^* E_B E_A^* E_A = E_A^* E_A$. In view of $E_A E_A^* = E_B E_A^*$, thus $E_A^* E_A = E_B^* E_A$ which can indicate $E_A^* E_A = E_A^* E_B$. Applying the definition of star partial order, we get $E_A \overset{*}{\leqslant} E_B$. The proof is complete. □

Inspired by the characterizations of the WL partial order, which is a generalization of the GL partial order, in [18]. We apply the polar-like decomposition to the CS partial order, and introduce a new partial order called WC partial order.

THEOREM 6. *Let $A, B \in \mathbb{C}_{m,n}$, the binary operation:*

$$A \overset{WC}{\leqslant} B \Leftrightarrow G_A^{\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\frac{1}{2}},\ E_A \overset{*}{\leqslant} E_B,\ H_A^{\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\frac{1}{2}}, \tag{23}$$

where $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ are the polar-like decompositions of A and B, respectively. Then the binary operation is a partial order.

Proof. Since the polar-like decomposition of a given matrix is unique, with Lemma 7 and Theorem 5, it is easy to check that the binary operation is a partial order. □

THEOREM 7. *Let $A, B \in \mathbb{C}_{m,n}$, then*

$$A \overset{WC}{\leqslant} B \Leftrightarrow A^* \overset{WC}{\leqslant} B^*. \tag{24}$$

Proof. Let $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$, since $G_A^{\frac{1}{2}} = H_{A^*}^{\frac{1}{2}}, E_A = E_{A^*}, H_A^{\frac{1}{2}} = G_{A^*}^{\frac{1}{2}}$, and similarly let $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$, we can derive it. □

THEOREM 8. *Let $A, B \in \mathbb{C}_{m,n}$, $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ be their polar-like decompositions, and $E_A \overset{*}{\leqslant} E_B$. Then,*

$$G_A^{\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\frac{1}{2}} \Leftrightarrow H_A^{\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\frac{1}{2}}. \tag{25}$$

Proof. Considering the singular value decompositions for $A = V_A \Lambda_A W_A^*$ and $B = V_B \Lambda_B W_B^*$, $rk(A) = a$, $rk(B) = b$, and we can know that

$$G_A = V_A \Lambda_A V_A^*,\ E_A = V_A W_A^*,\ H_A = W_A \Lambda_A W_A^*. \tag{26}$$

First let $G_A^{\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\frac{1}{2}}$ and $E_A \overset{*}{\leqslant} E_B$, then

$$V_A \Lambda_A^{\frac{1}{2}} V_A^* \overset{\#}{\leqslant} V_B \Lambda_B^{\frac{1}{2}} V_B^*, \tag{27}$$

$$V_A W_A^* \overset{*}{\leqslant} V_B W_B^*, \tag{28}$$

according to (28) we have $W_A V_A^* V_A W_A^* = W_A V_A^* V_B W_B^*$. With $V_A^* V_A = W_A^* W_A = I_a$ and $W_A^* \left(W_A V_A^* V_A W_A^*\right) W_B = W_A^* \left(W_A V_A^* V_B W_B^*\right) W_B$, we can have

$$W_A^* W_B = V_A^* V_B, \tag{29}$$

with (27) and (29) we get $W_B^* W_A \Lambda_A^{\frac{1}{2}} W_A^* W_B \overset{\#}{\leqslant} \Lambda_B^{\frac{1}{2}}$, then

$$W_B W_B^* W_A \Lambda_A^{\frac{1}{2}} W_A^* W_B W_B^* \overset{\#}{\leqslant} W_B \Lambda_B^{\frac{1}{2}} W_B^*, \tag{30}$$

applying (28) and we have $\left(V_A W_A^*\right)^* V_A W_A^* \overset{*}{\leqslant} \left(V_B W_B^*\right)^* V_B W_B^*$, i.e.,

$$W_A W_A^* \overset{*}{\leqslant} W_B W_B^*, \tag{31}$$

we can imply that $W_A W_A^* W_A W_A^* = W_A W_A^* = W_A W_A^* W_B W_B^*$, then

$$W_A^* W_A W_A^* = W_A^* W_A W_A^* W_B W_B^*,$$

i.e., $W_A^* = W_A^* W_B W_B^*$. With (30) we get

$$W_A \Lambda_A^{\frac{1}{2}} W_A^* \overset{\#}{\leqslant} W_B \Lambda_B^{\frac{1}{2}} W_B^*, \tag{32}$$

i.e., $H_A^{\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\frac{1}{2}}$.

On the contrary, applying $H_A^{\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\frac{1}{2}}$ and $E_A \overset{*}{\leqslant} E_B$, we can obtain $G_A^{\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\frac{1}{2}}$. $\square$

According to Theorem 8 we can obtain the following corollary.

COROLLARY 1. *Let* $A, B \in \mathbb{C}_{m,n}$,

$$A \overset{WC}{\leqslant} B \Leftrightarrow G_A^{\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\frac{1}{2}}, E_A \overset{*}{\leqslant} E_B, \tag{33}$$

$$\Leftrightarrow H_A^{\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\frac{1}{2}}, E_A \overset{*}{\leqslant} E_B. \tag{34}$$

It is well known that the star partial order is preserved for the Moore-Penrose inverse, that is,

$$A \overset{*}{\leqslant} B \Leftrightarrow A^\dagger \overset{*}{\leqslant} B^\dagger. \tag{35}$$

However, we note that core partial order is not necessarily preserved for the Moore-Penrose inverse, i.e., let $A, B \in \mathbb{C}_n^{CM}$, then

$$A \overset{\#}{\leqslant} B \not\Leftrightarrow A^\dagger \overset{\#}{\leqslant} B^\dagger, \tag{36}$$

the following example can illustrate it.

EXAMPLE 1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, it is obviously that $A, B \in \mathbb{C}_n^{CM}$. We can calculate that

$$A^{\#} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

with the definition of core partial order, we have $A \overset{\#}{\leqslant} B$. However, with calculating that

$$A^{\dagger} = \begin{bmatrix} \frac{1}{2} & 0 \\ \frac{1}{2} & 0 \end{bmatrix}, \quad B^{\dagger} = \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, \quad (A^{\#})^{\dagger} = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix},$$

and applying the definition of core partial order, we obtain $A^{\dagger}(A^{\#})^{\dagger} \neq B^{\dagger}(A^{\#})^{\dagger}$, i.e., $A \overset{\#}{\leqslant} B \not\Leftrightarrow A^{\dagger} \overset{\#}{\leqslant} B^{\dagger}$.

It follows from Lemma 8 that we drive the following Theorem.

THEOREM 9. *Let $A, B \in \mathbb{C}_{m,n}$, $A = G_A^{\frac{1}{2}} E_A H_A^{\frac{1}{2}}$ and $B = G_B^{\frac{1}{2}} E_B H_B^{\frac{1}{2}}$ be their polar-like decompositions. Then*

$$A^{\dagger} \overset{WC}{\leqslant} B^{\dagger} \Leftrightarrow G_A^{\dagger\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\dagger\frac{1}{2}}, E_A \overset{*}{\leqslant} E_B, \tag{37}$$

$$\Leftrightarrow H_A^{\dagger\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\dagger\frac{1}{2}}, E_A \overset{*}{\leqslant} E_B. \tag{38}$$

Next, we exemplify the relationships of the CS and WC partial orders from the minus, Löwner, GL and CL partial orders.

EXAMPLE 2. Let $A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}$, $B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}$. Then $rk(A) = 1$, $rk(B) = 2$ and

$$G_A = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix}, \quad E_A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix},$$

$$G_B = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix}, \quad E_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

With Lemma 1, we obtain

$$A^{\#} = \begin{bmatrix} 0.04 & 0.08 \\ 0.08 & 0.16 \end{bmatrix}, \quad B^{\#} = \begin{bmatrix} 0.2 & 0 \\ 0 & 0.2 \end{bmatrix}, \quad G_A^{\#} = \begin{bmatrix} 0.04 & 0.08 \\ 0.08 & 0.16 \end{bmatrix}.$$

By calculating, we find $G_A \overset{\#}{\leqslant} G_B$ and $E_A \overset{*}{\leqslant} E_B$, i.e., $A \overset{CS}{\leqslant} B$.

(1) Since $rk(B-A) = 1 = rk(B) - rk(A)$, A is below B under the minus partial order;

(2) Since $B - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ is a positive semidefinite matrix, A is below B under the Löwner partial order;

(3) Since

$$AB^* = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix},\ AA^* = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix},\ BB^* = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix},$$

and

$$AB^* = (AA^*)^{\frac{1}{2}}\,(BB^*)^{\frac{1}{2}},$$

A is below B under the GL partial order;

(4) Since $B^2B^{\#} - A^2A^{\#} = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix}$ is a positive semidefinite matrix, we have $A^2A^{\#} \overset{L}{\leqslant} B^2B^{\#}$. And $A^{\#}A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, $B^{\#}B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $(A^{\#}A)^{\#} = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, we can show that $A^{\#}A \overset{\#}{\leqslant} B^{\#}B$, so A is below B under the CL partial order.

EXAMPLE 3. Let $A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix}$, $B = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}$. Then $rk(A) = 1$, $rk(B) = 2$ and

$$G_A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix},\ E_A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix},\ H_A = \begin{bmatrix} 5 & 10 \\ 10 & 20 \end{bmatrix};$$

$$G_B = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix},\ E_B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},\ H_B = \begin{bmatrix} 25 & 0 \\ 0 & 25 \end{bmatrix}.$$

We can calculate that

$$G_A^{\frac{1}{2}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},\ H_A^{\frac{1}{2}} = \begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix},\ G_B^{\frac{1}{2}} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix},\ H_B^{\frac{1}{2}} = \begin{bmatrix} 5 & 0 \\ 0 & 5 \end{bmatrix};$$

$$A^{\#} = \begin{bmatrix} 0.008 & 0.016 \\ 0.016 & 0.032 \end{bmatrix},\ B^{\#} = \begin{bmatrix} 0.04 & 0 \\ 0 & 0.04 \end{bmatrix},\ G_A^{\#} = \begin{bmatrix} 0.008 & 0.016 \\ 0.016 & 0.032 \end{bmatrix},\ G_A^{\frac{1}{2}\#} = \begin{bmatrix} 0.04 & 0.08 \\ 0.08 & 0.16 \end{bmatrix}.$$

By calculating, we find $G_A^{\frac{1}{2}} \overset{\#}{\leqslant} G_B^{\frac{1}{2}}$, $E_A \overset{*}{\leqslant} E_B$, $H_A^{\frac{1}{2}} \overset{\#}{\leqslant} H_B^{\frac{1}{2}}$, i.e., $A \overset{WC}{\leqslant} B$.

(1) Since $rk(B-A) = 1 = rk(B) - rk(A)$, A is below B under the minus partial order;

(2) Since $B - A = \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}$ is a positive semidefinite matrix, A is below B under the Löwner partial order;

(3) Since

$$AB^* = \begin{bmatrix} 125 & 250 \\ 250 & 500 \end{bmatrix},\ AA^* = \begin{bmatrix} 125 & 250 \\ 250 & 500 \end{bmatrix},\ BB^* = \begin{bmatrix} 625 & 0 \\ 0 & 625 \end{bmatrix},$$

and

$$AB^* = (AA^*)^{\frac{1}{2}}\,(BB^*)^{\frac{1}{2}},$$

A is below B under the GL partial order;

(4) Since $B^2B^{\textcircled{\#}} - A^2A^{\textcircled{\#}} = \begin{bmatrix} 20 & -10 \\ -10 & 5 \end{bmatrix}$ is a positive semidefinite matrix, we have $A^2A^{\textcircled{\#}} \overset{L}{\leqslant} B^2B^{\textcircled{\#}}$. And $A^{\textcircled{\#}}A = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, $B^{\textcircled{\#}}B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$, $(A^{\textcircled{\#}}A)^{\textcircled{\#}} = \begin{bmatrix} 0.2 & 0.4 \\ 0.4 & 0.8 \end{bmatrix}$, we can show that $A^{\textcircled{\#}}A \overset{\textcircled{\#}}{\leqslant} B^{\textcircled{\#}}B$, so A is below B under the CL partial order.

From the Example 2 and 3 above, we can see that the CS and WC partial orders can imply the minus, Löwner, GL and CL partial orders in some special cases. The following remark is discussed whether the CS partial order can imply the WC partial order.

REMARK 1. With Theorem 5 and Corollary 1, we only need to prove that

$$G_A \overset{\textcircled{\#}}{\leqslant} G_B \Longrightarrow G_A^{\frac{1}{2}} \overset{\textcircled{\#}}{\leqslant} G_B^{\frac{1}{2}},\ H_A \overset{\textcircled{\#}}{\leqslant} H_B \Longrightarrow H_A^{\frac{1}{2}} \overset{\textcircled{\#}}{\leqslant} H_B^{\frac{1}{2}}.$$

Let $A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*$, $B = U \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} U^*$, $A \neq B$, with the definition of core partial order, we have $A^{\textcircled{\#}} = U \begin{bmatrix} (\Sigma K)^{-1} & 0 \\ 0 & 0 \end{bmatrix} U^*$. If $A^2 \overset{\textcircled{\#}}{\leqslant} B^2$, we suppose that $A \overset{\textcircled{\#}}{\leqslant} B$. According to $A^{\textcircled{\#}}A = A^{\textcircled{\#}}B$, $AA^{\textcircled{\#}} = BA^{\textcircled{\#}}$ and Lemma 9, we have

$$W = \Sigma K,\ X = \Sigma L,\ Y = 0,\ Z = 0,$$

i.e., $B = A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*$, it is obviously contradicted with $A \neq B$. In other words, $A \overset{CS}{\leqslant} B \nRightarrow A \overset{WC}{\leqslant} B$ in the general case.

Next, we give another remark to discuss whether the CS partial order can be derived from the WC partial order.

REMARK 2. Analysis similar to Remark 1, we need to prove that

$$G_A^{\frac{1}{2}} \overset{\textcircled{\#}}{\leqslant} G_B^{\frac{1}{2}} \Longrightarrow G_A \overset{\textcircled{\#}}{\leqslant} G_B,\ H_A^{\frac{1}{2}} \overset{\textcircled{\#}}{\leqslant} H_B^{\frac{1}{2}} \Longrightarrow H_A \overset{\textcircled{\#}}{\leqslant} H_B.$$

If $A, B \in \mathbb{C}_n^{CM}$, $A \overset{\textcircled{\#}}{\leqslant} B$, with Lemma 2 and Lemma 3 we know $A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*$, $B = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix} U^*$. By taking $A = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & 0 \end{bmatrix} U^*$, $B = U \begin{bmatrix} \Sigma K & \Sigma L \\ 0 & Z \end{bmatrix} U^*$ into Lemma 9, and then we find the B is contradicted with the condition $WX + XZ = \Sigma K \Sigma L$ of Lemma 9 by calculating, i.e., $A \overset{\textcircled{\#}}{\leqslant} B \nRightarrow A^2 \overset{\textcircled{\#}}{\leqslant} B^2$. Therefore, we conclude that $A^{\frac{1}{2}} \overset{\textcircled{\#}}{\leqslant} B^{\frac{1}{2}} \nRightarrow A \overset{\textcircled{\#}}{\leqslant} B$. In other words, $A \overset{WC}{\leqslant} B \nRightarrow A \overset{CS}{\leqslant} B$.

4. Discussion

In quantum physics, we often discuss and describe the properties and qualities of observables. A supplementary point of view is obtained when we compare different observables. In ([9] chapter 3, section 5), there are three relations between observables and these relations can be used to make ststements like 'A is better than B' precise. Here we consider one pre-order which called state distinction.

DEFINITION 2. Let A and B be observables. If, for all states $\rho_1, \rho_2 \in S(H)$,

$$\Phi_A(\rho_1) = \Phi_A(\rho_2) \Rightarrow \Phi_B(\rho_1) = \Phi_B(\rho_2),$$

then we write $B \preccurlyeq_i A$ and say that the state-distinction power of A is greater than or equal to that of B. If $B \preccurlyeq_i A \preccurlyeq_i B$, we say that A and B are informationally equivalent, and write $A \sim_i B$.

According to the Definition 2, we can easily confirm that $\preccurlyeq_i$ is a preorder which satisfies the reflexivity and transitivity, and $\sim_i$ is an equivalence relation. With Example 2 and 3, we know that CS and WC partial order can lead to minus, Löwner, GL and CL partial orders in some special cases. However, in [16] the CL partial order cannot lead to minus, Löwner and GL partial orders. So we can say that CS and WC order are the better informationally equivalences than CL order. As the construction of partial order is still being studied, so we have not yet got a best or worst partial order to describe these relationships in quantum physics.

The more details about the relevant knowledge in quantum physics can be referred to [9].

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