Journal of
Mathematical
Inequalities
Volume 17, Number 2 (2023), 581–589 doi:10.7153/jmi-2023-17-37

GENERALIZATION OF SOME UNITARILY INVARIANT NORM INEQUALITIES FOR MATRICES

AHMAD AL-NATOOR, MOHAMMAD A. AMLEH, BAHA' ABUGHAZALEH AND ALIAA BURQAN

(*Communicated by M. Kian*)

Abstract. In this paper, we prove new unitarily invariant norm inequalities for positive semidefinite matrices. Some of these inequalities represents a generalization of earlier work due to Kittaneh who refines an inequality due to Davidson and Power which is useful in best approximation of C^*-algebras.

1. Introduction

Let $M_n(\mathbb{C})$ denote the space of all $n \times n$ complex matrices. A matrix norm $|||\cdot|||$ on $M_n(\mathbb{C})$ is called unitarily invariant if $|||UAV||| = |||A|||$ for all $A,U,V \in M_n(\mathbb{C})$ with U and V are unitary matrices. An important example of unitarily invariant norms is the spectral norm of $A \in \mathbb{M}_n(\mathbb{C})$, which is denoted by $||A||$.

The block diagonal matrix $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ can be denoted by $A \oplus B$, which is known as the direct sum of A and B. For $A \in M_n(\mathbb{C})$ and $B \in M_{2n}(\mathbb{C})$, the inequality $|||A||| \leqslant |||B|||$ is equivalent to $|||A \oplus 0||| \leqslant |||B|||$.

In [1], Al-Natoor, Benzamia and Kittaneh proved that if $A,B \in \mathbb{M}_n(\mathbb{C})$. Then

$$\|AB - BA\| \leqslant \|A\| \, \|B\| + \frac{1}{2} \|A^*B - BA^*\| , \tag{1.1}$$

which refines the inequality

$$\|AB - BA\| \leqslant 2\|A\| \|B\|.$$

Related to the inequality (1.1), Al-Natoor and Kittaneh [6] have proved that if $A,B,C \in \mathbb{M}_n(\mathbb{C})$, then

$$\|AB + BC\| \leqslant \max(\|A\| , \|C\|) \, \|B\| + \frac{1}{2} \|A^*B + BC^*\| .$$

Mathematics subject classification (2020): 15A18, 15A42, 15A60, 15B57, 47B30.
Keywords and phrases: Spectral norm, unitarily invariant norm, positive semidefinite matrix, inequality.

Paper JMI-17-37

In particular, if $C = A$, we have

$$\|AB + BA\| \leqslant \|A\| \, \|B\| + \frac{1}{2} \|A^*B + BA^*\| . \tag{1.2}$$

In [4], Al-Natoor, Hirzallah, and Kittaneh proved a related inequality to the inequality (1.2). In fact, they proved that if $A, B \in \mathbb{M}_n(\mathbb{C})$, then

$$\|AB^* + BA^*\| \leqslant \|A\| \, \|B\| + \frac{1}{2} \|A^*B + BA^*\| .$$

For more results involving norm inequalities of matrices, we refer the reader to [2] and [9].

In this paper, we give new unitarily invariant norm inequalities for matrices. Some of our results gives a generalization of the inequalities (1.1) and (1.2). Other generalizations of norm inequalities will be given.

2. Main results

We start with the following theorem, which is based on the following two lemmas. The first lemma can be found in [8], while the second lemma is well-known fact (see, e.g., [7, p. 75]).

LEMMA 2.1. *Let $A, B \in M_n(\mathbb{C})$. Then*

$$|||AB^*||| \leqslant \frac{1}{2} \left|\left|\left| |A|^2 + |B|^2 \right|\right|\right| .$$

LEMMA 2.2. *Let $A, B, X \in M_n(\mathbb{C})$. Then*

$$|||AXB||| \leqslant \|A\| \|B\| \, |||X||| .$$

THEOREM 2.3. *Let $A, B, C, D, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$ be such that $A, B, C,$ and D are positive semidefinite. Then*

$$\begin{aligned} &\left|\left|\left| ZA^{1/2}C^{1/2}X - YB^{1/2}D^{1/2}W \right|\right|\right| \\ &\leqslant \frac{1}{2} \|A + C\| \left|\left|\left| \left(|Z|^2\right) \oplus (|X^*|^2) \right|\right|\right| + \frac{1}{2} \|B + D\| \left|\left|\left| \left(|Y|^2\right) \oplus (|W^*|^2) \right|\right|\right| \\ &\quad + \frac{1}{2} \left|\left|\left| \left(A^{1/2}Z^*YB^{1/2} - C^{1/2}XW^*D^{1/2}\right) \oplus \left(A^{1/2}Z^*YB^{1/2} - C^{1/2}XW^*D^{1/2}\right) \right|\right|\right| . \end{aligned} \tag{2.1}$$

Proof. Let $K_1 = \begin{bmatrix} ZA^{1/2} & YB^{1/2} \\ 0 & 0 \end{bmatrix}$, $K_2^* = \begin{bmatrix} C^{1/2}X & 0 \\ -D^{1/2}W & 0 \end{bmatrix}$. Then

$$\begin{aligned}
&\left|\left|\left|ZA^{1/2}C^{1/2}X - YB^{1/2}D^{1/2}W\right|\right|\right| \\
&= |||K_1K_2^*||| \\
&\leqslant \frac{1}{2}\,|||K_1^*K_1 + K_2^*K_2||| \quad \text{(by Lemma 2.1)} \\
&= \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{1/2}|Z|^2A^{1/2} + C^{1/2}|X^*|^2C^{1/2} & A^{1/2}Z^*YB^{1/2} - C^{1/2}XW^*D^{1/2} \\ B^{1/2}Y^*ZA^{1/2} - D^{1/2}WX^*C^{1/2} & B^{1/2}|Y|^2B^{1/2} + D^{1/2}|W^*|^2D^{1/2} \end{bmatrix}\right|\right|\right|.
\end{aligned} \tag{2.2}$$

Now, by applying the triangle inequality, we get

$$\begin{aligned}
&\left|\left|\left|ZA^{1/2}C^{1/2}X - YB^{1/2}D^{1/2}W\right|\right|\right| \\
&\leqslant \frac{1}{2}\left|\left|\left|\begin{bmatrix} A^{1/2}|Z|^2A^{1/2} + C^{1/2}|X^*|^2C^{1/2} & 0 \\ 0 & 0 \end{bmatrix}\right|\right|\right| \\
&\quad + \frac{1}{2}\left|\left|\left|\begin{bmatrix} 0 & 0 \\ 0 & B^{1/2}|Y|^2B^{1/2} + D^{1/2}|W^*|^2D^{1/2} \end{bmatrix}\right|\right|\right| \\
&\quad + \frac{1}{2}\left|\left|\left|\begin{bmatrix} 0 & A^{1/2}Z^*YB^{1/2} - C^{1/2}XW^*D^{1/2} \\ B^{1/2}Y^*ZA^{1/2} - D^{1/2}WX^*C^{1/2} & 0 \end{bmatrix}\right|\right|\right|.
\end{aligned} \tag{2.3}$$

But,

$$\begin{aligned}
&\left|\left|\left|\left(A^{1/2}|Z|^2A^{1/2} + C^{1/2}|X^*|^2C^{1/2}\right) \oplus 0\right|\right|\right| \\
&= \left|\left|\left|\begin{bmatrix} A^{1/2} & C^{1/2} \\ 0 & 0 \end{bmatrix}\begin{bmatrix} |Z|^2 & 0 \\ 0 & |X^*|^2 \end{bmatrix}\begin{bmatrix} A^{1/2} & 0 \\ C^{1/2} & 0 \end{bmatrix}\right|\right|\right| \\
&\leqslant \left\|\begin{bmatrix} A^{1/2} & C^{1/2} \\ 0 & 0 \end{bmatrix}\right\|\left\|\begin{bmatrix} A^{1/2} & 0 \\ C^{1/2} & 0 \end{bmatrix}\right\|\left|\left|\left|\begin{bmatrix} |Z|^2 & 0 \\ 0 & |X^*|^2 \end{bmatrix}\right|\right|\right| \quad \text{(by Lemma 2.2)} \\
&= \left\|\begin{bmatrix} A^{1/2} & C^{1/2} \\ 0 & 0 \end{bmatrix}\right\|^2\left|\left|\left|\begin{bmatrix} |Z|^2 & 0 \\ 0 & |X^*|^2 \end{bmatrix}\right|\right|\right| \\
&= \|A + C\|\left|\left|\left|\left(|Z|^2\right) \oplus (|X^*|^2)\right|\right|\right|.
\end{aligned} \tag{2.4}$$

Similarly,

$$\left|\left|\left|\left(B^{1/2}|Y|^2B^{1/2} + D^{1/2}|W^*|^2D^{1/2}\right) \oplus 0\right|\right|\right| \leqslant \|B + D\|\left|\left|\left|\left(|Y|^2\right) \oplus (|W^*|^2)\right|\right|\right|. \tag{2.5}$$

Now, the result follows from the inequalities (2.3), (2.4), and (2.5). □

COROLLARY 2.4. *Let* $A,B,X,Y,Z,W \in \mathbb{M}_n(\mathbb{C})$ *be such that* A *and* B *are positive semidefinite. Then*

$$\begin{aligned}&|||ZAX-YBW|||\\ &\leqslant \max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|)\,|||A\oplus B|||\\ &\quad+\frac{1}{2}\left|\left|\left|\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\oplus\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\right|\right|\right|. \end{aligned}\tag{2.6}$$

Proof. Let $C=A$ and $D=B$ in the inequality (2.2), we have

$$\begin{aligned}&|||ZAX-YBW|||\\ &\leqslant \frac{1}{2}\left|\left|\left|\begin{bmatrix}A^{1/2}|Z|^2A^{1/2}+A^{1/2}|X^*|^2A^{1/2} & A^{1/2}Z^*YB^{1/2}-A^{1/2}XW^*B^{1/2}\\ B^{1/2}Y^*ZA^{1/2}-B^{1/2}WX^*A^{1/2} & B^{1/2}|Y|^2B^{1/2}+B^{1/2}|W^*|^2B^{1/2}\end{bmatrix}\right|\right|\right|\\ &\leqslant \frac{1}{2}\left|\left|\left|\begin{bmatrix}A^{1/2}|X^*|^2A^{1/2} & 0\\ 0 & B^{1/2}|Y|^2B^{1/2}\end{bmatrix}\right|\right|\right|\\ &\quad+\frac{1}{2}\left|\left|\left|\begin{bmatrix}A^{1/2}|Z|^2A^{1/2} & 0\\ 0 & B^{1/2}|W^*|^2B^{1/2}\end{bmatrix}\right|\right|\right|\\ &\quad+\frac{1}{2}\left|\left|\left|\begin{bmatrix}0 & A^{1/2}(Z^*Y-XW^*)B^{1/2}\\ B^{1/2}(Y^*Z-WX^*)A^{1/2} & 0\end{bmatrix}\right|\right|\right|. \end{aligned}\tag{2.7}$$

But, by using the fact that $\left|\left|\left|C^{1/2}|D|^2C^{1/2}\right|\right|\right| = |||DCD^*|||$ for any matrices C and D such that C is positive semidefinite, the inequality (2.7) becomes

$$\begin{aligned}&|||ZAX-YBW|||\\ &=\frac{1}{2}|||(X^*AX)\oplus(YBY^*)|||+\frac{1}{2}|||(ZAZ^*)\oplus(W^*BW)|||\\ &\quad+\frac{1}{2}\left|\left|\left|\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\oplus\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\right|\right|\right|.\end{aligned}$$

Replace X by tX and Y by tY and take the minimum over $t>0,$ we have

$$\begin{aligned}&|||ZAX-YBW|||\\ &\leqslant \sqrt{|||(X^*AX)\oplus(YBY^*)|||\ \ |||(ZAZ^*)\oplus(W^*BW^*)|||}\\ &\quad+\frac{1}{2}\left|\left|\left|\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\oplus\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\right|\right|\right|\\ &\leqslant \sqrt{(\max(\|X\|,\|Y\|))^2(\max(\|Z\|,\|W\|))^2\,|||A\oplus B|||^2}\\ &\quad+\frac{1}{2}\left|\left|\left|\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\oplus\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\right|\right|\right| \quad \text{(by Lemma 2.2)}\\ &=\max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|)\,|||A\oplus B|||\\ &\quad+\frac{1}{2}\left|\left|\left|\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\oplus\left(A^{1/2}(Z^*Y-XW^*)B^{1/2}\right)\right|\right|\right|,\end{aligned}$$

as required. □

Specifying Corollary 2.4 to the spectral norm, we have the following corollary.

COROLLARY 2.5. *Let $A,B,X,Y,Z,W \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\begin{aligned}\|ZAX - YBW\| \leqslant\ & \max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|)\max(\|A\|,\|B\|)\\ &+\frac{1}{2}\left\|A^{1/2}(Z^*Y - XW^*)B^{1/2}\right\|.\end{aligned}$$

In particular, if $A = B = I$, then

$$\|ZX - YW\| \leqslant \max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|) + \frac{1}{2}\|Z^*Y - XW^*\|. \tag{2.8}$$

The inequality (2.8) gives a general version of the inequality (1.1). In fact, the inequality (1.1) can be retained from the inequality (2.8) by letting $Z = W = A$, $X = Y = B$.

Replace the matrix W by $-W$ in the inequality (2.6), we have the following corollary.

COROLLARY 2.6. *Let $A,B,X,Y,Z,W \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\begin{aligned}&|||ZAX + YBW|||\\ &\leqslant \max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|)\,|||A\oplus B|||\\ &\quad+\frac{1}{2}\left|\left|\left|\left(A^{1/2}(Z^*Y + XW^*)B^{1/2}\right)\oplus A^{1/2}(Z^*Y + XW^*)B^{1/2}\right|\right|\right|.\end{aligned} \tag{2.9}$$

Letting $Z = W = I$ and $Y = X$ in the inequality (2.9), we have

$$|||AX + XB||| \leqslant \|X\|\ \ |||A\oplus B||| + \left|\left|\left|\left(A^{1/2}XB^{1/2}\right)\oplus\left(A^{1/2}XB^{1/2}\right)\right|\right|\right|, \tag{2.10}$$

which has been obtained recently in [5]. Specifying the inequality (2.10) to the spectral norm and letting $X = I$, we have

$$\|A + B\| \leqslant \max(\|A\|,\|B\|) + \left\|A^{1/2}B^{1/2}\right\| \tag{2.11}$$

which is obtained in [12]. It should be mentioned here that the inequality (2.11) improves an inequality due to Davidson and Power which is useful in best approximation of C^*-algebras. A refinement of the inequality (2.11) has been proved in [13] so that

$$\|A + B\| \leqslant \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{1/2}B^{1/2}\right\|^2}\right). \tag{2.12}$$

The following theorem gives a general version of the inequality (2.12). To see this, we need the following lemma which can be found in [11].

LEMMA 2.7. *Let $A,B,C,D \in M_n(\mathbb{C})$. Then*

$$\left\|\begin{bmatrix}A & B\\ C & D\end{bmatrix}\right\| \leqslant \left\|\begin{bmatrix}\|A\| & \|B\|\\ \|C\| & \|D\|\end{bmatrix}\right\|.$$

THEOREM 2.8. *Let $A,B,X,Y,Z,W \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\begin{aligned} &\|ZAX+YBW\| \\ &\leqslant \frac{1}{4}\left(\begin{array}{c} \left(\|Z\|^2+\|X\|^2\right)\|A\| + \left(\|Y\|^2+\|W\|^2\right)\|B\| \\ +\sqrt{\begin{array}{c} \left(\left\|A^{1/2}(|Z|^2+|X^*|^2)A^{1/2}\right\| - \left\|B^{1/2}(|Y|^2+|W^*|^2)B^{1/2}\right\|\right)^2 \\ +4\left\|A^{1/2}(Z^*Y+XW^*)B^{1/2}\right\|^2 \end{array}} \end{array}\right). \end{aligned} \tag{2.13}$$

In particular, if $A=B=I$, then

$$\|ZX+YW\| \leqslant \frac{1}{4}\left(\begin{array}{c} \left(\|Z\|^2+\|X\|^2\right) + \left(\|Y\|^2+\|W\|^2\right) \\ +\sqrt{\left(\left\||Z|^2+|X^*|^2\right\| - \left\||Y|^2+|W^*|^2\right\|\right)^2 + 4\|Z^*Y+XW^*\|^2} \end{array}\right).$$

Proof. Specifying the inequality (2.2) to the spectral norm, replacing the matrix W by $-W$, and letting $C=A$ and $D=B$, we have

$$\begin{aligned} &\|ZAX+YBW\| \\ &= \frac{1}{2}\left\| \begin{bmatrix} A^{1/2}|Z|^2A^{1/2} + A^{1/2}|X^*|^2A^{1/2} & A^{1/2}Z^*YB^{1/2} + A^{1/2}XW^*B^{1/2} \\ B^{1/2}Y^*ZA^{1/2} + B^{1/2}WX^*A^{1/2} & B^{1/2}|Y|^2B^{1/2} + B^{1/2}|W^*|^2B^{1/2} \end{bmatrix} \right\| \\ &= \frac{1}{2}\left\| \begin{bmatrix} A^{1/2}(|Z|^2+|X^*|^2)A^{1/2} & A^{1/2}(Z^*Y+XW^*)B^{1/2} \\ B^{1/2}(Y^*Z+WX^*)A^{1/2} & B^{1/2}(|Y|^2+|W^*|^2)B^{1/2} \end{bmatrix} \right\| \\ &\leqslant \frac{1}{2}\left\| \begin{bmatrix} \left\|A^{1/2}(|Z|^2+|X^*|^2)A^{1/2}\right\| & \left\|A^{1/2}(Z^*Y+XW^*)B^{1/2}\right\| \\ \left\|B^{1/2}(Y^*Z+WX^*)A^{1/2}\right\| & \left\|B^{1/2}(|Y|^2+|W^*|^2)B^{1/2}\right\| \end{bmatrix} \right\|. \end{aligned}$$

Now, by calculating the spectral norm (which is the greatest singular value) of the matrix $\begin{bmatrix} \left\|A^{1/2}(|Z|^2+|X^*|^2)A^{1/2}\right\| & \left\|A^{1/2}(Z^*Y+XW^*)B^{1/2}\right\| \\ \left\|B^{1/2}(Y^*Z+WX^*)A^{1/2}\right\| & \left\|B^{1/2}(|Y|^2+|W^*|^2)B^{1/2}\right\| \end{bmatrix}$, we have

$$\begin{aligned} &\|ZAX+YBW\| \\ &\leqslant \frac{1}{4}\left(\begin{array}{c} \left\|A^{1/2}(|Z|^2+|X^*|^2)A^{1/2}\right\| + \left\|B^{1/2}(|Y|^2+|W^*|^2)B^{1/2}\right\| \\ +\sqrt{\begin{array}{c} \left(\left\|A^{1/2}(|Z|^2+|X^*|^2)A^{1/2}\right\| - \left\|B^{1/2}(|Y|^2+|W^*|^2)B^{1/2}\right\|\right)^2 \\ +4\left\|A^{1/2}(Z^*Y+XW^*)B^{1/2}\right\|^2 \end{array}} \end{array}\right) \\ &\leqslant \frac{1}{4}\left(\begin{array}{c} \left(\|Z\|^2+\|X\|^2\right)\|A\| + \left(\|Y\|^2+\|W\|^2\right)\|B\| \\ +\sqrt{\begin{array}{c} \left(\left\|A^{1/2}(|Z|^2+|X^*|^2)A^{1/2}\right\| - \left\|B^{1/2}(|Y|^2+|W^*|^2)B^{1/2}\right\|\right)^2 \\ +4\left\|A^{1/2}(Z^*Y+XW^*)B^{1/2}\right\|^2 \end{array}} \end{array}\right), \end{aligned}$$

as required. □

Letting $X = Y = Z = W = I$ in the inequality (2.13), we have

$$\|A+B\| \leqslant \frac{1}{2}\left(\|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\left\|A^{1/2}B^{1/2}\right\|^2}\right),$$

which is the inequality (2.12).

Specifying Corollary 2.6 to the spectral norm, we have the following corollary.

COROLLARY 2.9. *Let $A,B,X,Y,Z,W \in \mathbb{M}_n(\mathbb{C})$ be such that A and B are positive semidefinite. Then*

$$\begin{aligned}\|ZAX + YBW\| \leqslant{}& \max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|)\max(\|A\|,\|B\|)\\ &+\frac{1}{2}\left\|A^{1/2}(Z^*Y + XW^*)B^{1/2}\right\|.\end{aligned}$$

In particular, if $A = B = I$, then

$$\|ZX + YW\| \leqslant \max(\|X\|,\|Y\|)\ \max(\|Z\|,\|W\|) + \frac{1}{2}\|Z^*Y + XW^*\|. \tag{2.14}$$

Clearly, the inequality (2.14) gives a general version of the inequality (1.2). In fact, the inequality (1.2) can be retained by letting $Z = W = A,\ X = Y = B$ in the inequality (2.14).

To prove our next result, we need the following lemma which can be found in [14]. Recently, the authors in [3] gave a completely different proof of this lemma.

LEMMA 2.10. *Let $A,B,C \in M_n(\mathbb{C})$ be such that the block matrix $T = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive semidefinite. Then*

$$2s_j(C) \leqslant s_j(T)$$

for $j = 1,\ldots,n$, and so

$$2\,|||C||| \leqslant |||T|||.$$

THEOREM 2.11. *Let $A,B,C,D \in \mathbb{M}_n(\mathbb{C})$. Then*

$$\begin{aligned}2\,|||A^*B + C^*D||| \leqslant{}& |||(A^*A + C^*C) \oplus (B^*B + D^*D)|||\\ &+|||(A^*B + C^*D) \oplus (B^*A + D^*C)|||.\end{aligned} \tag{2.15}$$

In particular,

$$\|A^*B + C^*D\| \leqslant \max(\|A^*A + C^*C\|, \|B^*B + D^*D\|). \tag{2.16}$$

Proof. Let $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then

$$E^*E = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix}\begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^*A + C^*C & A^*B + C^*D \\ B^*A + D^*C & B^*B + D^*D \end{bmatrix}$$

is positive semidefinite. So, by Lemma 2.10, we have

$$\begin{aligned} &2\,|||A^*B+C^*D||| \\ &\leqslant \left|\left|\left|\begin{bmatrix} A^*A+C^*C & A^*B+C^*D \\ B^*A+D^*C & B^*B+D^*D \end{bmatrix}\right|\right|\right| \\ &\leqslant \left|\left|\left|\begin{bmatrix} A^*A+C^*C & 0 \\ 0 & B^*B+D^*D \end{bmatrix}+\begin{bmatrix} 0 & A^*B+C^*D \\ B^*A+D^*C & 0 \end{bmatrix}\right|\right|\right| \\ &= |||(A^*A+C^*C)\oplus(B^*B+D^*D)|||+|||(A^*B+C^*D)\oplus(B^*A+D^*C)|||, \end{aligned} \tag{2.17}$$

which proves the inequality (2.15). The inequality (2.16) follows by specifying the inequality (2.15) to the spectral norm. □

COROLLARY 2.12. *Let $A,B,C,D \in M_n(\mathbb{C})$. Then*

$$\|A^*B+C^*D\| \leqslant \frac{1}{2}\left(\begin{array}{c} \|A^*A+C^*C\|+\|B^*B+D^*D\| \\ +\sqrt{(\|A^*A+C^*C\|-\|B^*B+D^*D\|)^2+4\,\|A^*B+C^*D\|^2} \end{array}\right).$$

Proof. Specifying the inequality (2.17) to the spectral norm, we have

$$\begin{aligned} 2\,\|A^*B+C^*D\| &\leqslant \left\|\begin{bmatrix} A^*A+C^*C & A^*B+C^*D \\ B^*A+D^*C & B^*B+D^*D \end{bmatrix}\right\| \\ &\leqslant \left\|\begin{bmatrix} \|A^*A+C^*C\| & \|A^*B+C^*D\| \\ \|B^*A+D^*C\| & \|B^*B+D^*D\| \end{bmatrix}\right\| \quad \text{(by Lemma 2.7)} \\ &\leqslant \frac{1}{2}\left(\begin{array}{c} \|A^*A+C^*C\|+\|B^*B+D^*D\| \\ +\sqrt{(\|A^*A+C^*C\|-\|B^*B+D^*D\|)^2+4\,\|A^*B+C^*D\|^2} \end{array}\right), \end{aligned}$$

as required. □

REFERENCES

[1] A. AL-NATOOR, S. BENZAMIA AND F. KITTANEH, *Unitarily invariant norm inequalities for positive semidefinite matrices*, Linear Algebra Appl. 633 (2022), 303–315.

[2] A. AL-NATOOR, O. HIRZALLAH AND F. KITTANEH, *Interpolating inequalities for functions of positive semidefinite matrices*, Banach J. Math. Anal. 12 (2018), 955–969.

[3] A. AL-NATOOR, O. HIRZALLAH AND F. KITTANEH, *Singular value inequalities for convex functions of positive semidefinite matrices*, Ann. Funct. Anal. 14 (2023), Paper No. 7, 14 pp.

[4] A. AL-NATOOR, O. HIRZALLAH AND F. KITTANEH, *Singular value inequalities for product and sums of matrices*, preprint.

[5] A. AL-NATOOR AND F. KITTANEH, *Singular value and norm inequalities for positive semidefinite matrices*, Linear Multilinear Algebra, https://doi.org/10.1080/03081087.2021.1882373.

[6] A. AL-NATOOR AND F. KITTANEH, *Further unitarily invariant norm inequalities for positive semidefinite matrices*, Positivity, 26 (2022), Paper No. 8, 11 pp.

[7] R. BHATIA, *Matrix Analysis*, Springer-Verlag, New York, 1997.

[8] R. BHATIA AND F. KITTANEH, *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl. 11 (1990), 272–277.

[9] A. BURQAN AND F. KITTANEH, *Singular value and norm Inequalities associated with 2×2 positive semidefinite block matrices*, The Electronic Journal of Linear Algebra, 32 (2017), 116–124.

[10] K. DAVIDSON AND S. C. POWER, *Best approximation in C^*-algebras*, J. Reine Angew. Math. 368 (1986), 43–62.
[11] J. C. HOU, H. K. DU, *Norm inequalities of positive operator matrices*, Integral Equations Operator Theory 22 (1995), 281–294.
[12] F. KITTANEH, *Norm inequalities for certain operator sums*, J Funct Anal. 1997; 143: 337–348.
[13] F. KITTANEH, *Norm inequalities for sums of positive operators*, J. Operator Theory 48 (2002), 95–103.
[14] Y. TAO, *More results on singular value inequalities of matrices*, Linear Algebra Appl. 416 (2006) 724–729.

(Received August 28, 2022)

Ahmad Al-Natoor
Department of mathematics
Isra University
Amman, Jordan
e-mail: ahmad.alnatoor@iu.edu.jo

Mohammad A. Amleh
Department of mathematics
Zarqa University
Zarqa, Jordan
e-mail: malamleh@zu.edu.jo

Baha' Abughazaleh
Department of mathematics
Isra University
Amman, Jordan
e-mail: baha.abughazaleh@iu.edu.jo

Aliaa Burqan
Department of mathematics
Zarqa University
Zarqa, Jordan
e-mail: aliaaburqan@zu.edu.jo