GENERALIZATION OF SOME UNITARILY INVARIANT NORM INEQUALITIES FOR MATRICES

AHMAD AL-NATOOR, MOHAMMAD A. AMLEH, BAHAA’ ABUGHAZALEH AND ALIAA BURQAN

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Abstract. In this paper, we prove new unitarily invariant norm inequalities for positive semidefinite matrices. Some of these inequalities represents a generalization of earlier work due to Kittaneh who refines an inequality due to Davidson and Power which is useful in best approximation of \( C^* \)-algebras.

1. Introduction

Let \( M_n(\mathbb{C}) \) denote the space of all \( n \times n \) complex matrices. A matrix norm \( ||| \cdot ||| \) on \( M_n(\mathbb{C}) \) is called unitarily invariant if \( |||UAV||| = |||A||| \) for all \( A, U, V \in M_n(\mathbb{C}) \) with \( U \) and \( V \) are unitary matrices. An important example of unitarily invariant norms is the spectral norm of \( A \in M_n(\mathbb{C}) \), which is denoted by \( |||A||| \).

The block diagonal matrix \( \begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix} \) can be denoted by \( A \oplus B \), which is known as the direct sum of \( A \) and \( B \). For \( A \in M_n(\mathbb{C}) \) and \( B \in M_m(\mathbb{C}) \), the inequality \( |||A||| \leq |||B||| \) is equivalent to \( |||A \oplus 0||| \leq |||B||| \).

In [1], Al-Natoor, Benzamia and Kittaneh proved that if \( A, B \in M_n(\mathbb{C}) \), then

\[
\|AB - BA\| \leq 2\|A\|\|B\| + \frac{1}{2}\|A^*B - BA^*\|, \tag{1.1}
\]

which refines the inequality

\[
\|AB - BA\| \leq 2\|A\|\|B\|.
\]

Related to the inequality (1.1), Al-Natoor and Kittaneh [6] have proved that if \( A, B, C \in M_n(\mathbb{C}) \), then

\[
\|AB + BC\| \leq \max(\|A\|,\|C\|)\|B\| + \frac{1}{2}\|A^*B + BC^*\|.
\]


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In particular, if \( C = A \), we have
\[
\|AB + BA\| \leq \|A\| \|B\| + \frac{1}{2} \|A^*B + BA^*\|. \tag{1.2}
\]

In [4], Al-Natoor, Hirzallah, and Kittaneh proved a related inequality to the inequality (1.2). In fact, they proved that if \( A, B \in \mathbb{M}_n(\mathbb{C}) \), then
\[
\|AB^* + BA^*\| \leq \|A\| \|B\| + \frac{1}{2} \|A^*B + BA^*\|.
\]

For more results involving norm inequalities of matrices, we refer the reader to [2] and [9].

In this paper, we give new unitarily invariant norm inequalities for matrices. Some of our results gives a generalization of the inequalities (1.1) and (1.2). Other generalizations of norm inequalities will be given.

### 2. Main results

We start with the following theorem, which is based on the following two lemmas. The first lemma can be found in [8], while the second lemma is well-known fact (see, e.g., [7, p. 75]).

**Lemma 2.1.** Let \( A, B \in \mathbb{M}_n(\mathbb{C}) \). Then
\[
\|\|AB\|^2\| \leq \frac{1}{2} \|\|A\|^2 + \|B\|^2\|.
\]

**Lemma 2.2.** Let \( A, B, X \in \mathbb{M}_n(\mathbb{C}) \). Then
\[
\|\|AXB\|^2\| \leq \|\|A\|\|\|B\|\|\|X\|^2\|.
\]

**Theorem 2.3.** Let \( A, B, C, D, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C}) \) be such that \( A, B, C, \) and \( D \) are positive semidefinite. Then
\[
\|ZA^{1/2}C^{1/2}X - YB^{1/2}D^{1/2}W\| \\
\leq \frac{1}{2} \|A + C\|\|\|(|Z|^2) \oplus (X^*|^2)\|| + \frac{1}{2} \|B + D\|\|\|(|Y|^2) \oplus (W^*|^2)\|| \\
+ \frac{1}{2} \|\|\|\|A^{1/2}Z^*YB^{1/2} - C^{1/2}XW^*D^{1/2}\|\|\|\|\| + \frac{1}{2} \|\|\|\|A^{1/2}Z^*YB^{1/2} - C^{1/2}XW^*D^{1/2}\|\|\|\|\|\|\|\|\|\|. \tag{2.1}
\]
Similarly, Now, the result follows from the inequalities (2.3), (2.4), and (2.5).

\[ \|Z A^{1/2} C^{1/2} X - Y B^{1/2} D^{1/2} W \| \]
\[ = \| K_1 K_2^* \| \]
\[ \leq \frac{1}{2} \| K_1^* K_1 + K_2^* K_2 \| \quad \text{(by Lemma 2.1)} \]
\[ = \frac{1}{2} \left\| \begin{bmatrix} A^{1/2} |Z|^2 A^{1/2} + C^{1/2} |X^*|^2 C^{1/2} & A^{1/2} Z Y B^{1/2} - C^{1/2} X W^* D^{1/2} \\ B^{1/2} Y^* Z A^{1/2} - D^{1/2} W X^* C^{1/2} & B^{1/2} |Y|^2 B^{1/2} + D^{1/2} |W^*|^2 D^{1/2} \end{bmatrix} \right\| . \]
(2.2)

Now, by applying the triangle inequality, we get
\[ \| Z A^{1/2} C^{1/2} X - Y B^{1/2} D^{1/2} W \| \]
\[ \leq \frac{1}{2} \left\| \begin{bmatrix} A^{1/2} |Z|^2 A^{1/2} + C^{1/2} |X^*|^2 C^{1/2} & 0 \\ 0 & 0 \end{bmatrix} \right\| 
+ \frac{1}{2} \left\| \begin{bmatrix} 0 & 0 \\ 0 & B^{1/2} |Y|^2 B^{1/2} + D^{1/2} |W^*|^2 D^{1/2} \end{bmatrix} \right\| 
+ \frac{1}{2} \left\| \begin{bmatrix} 0 & A^{1/2} Z^* Y B^{1/2} - C^{1/2} X W^* D^{1/2} \\ B^{1/2} Y^* Z A^{1/2} - D^{1/2} W X^* C^{1/2} & 0 \end{bmatrix} \right\|. \]
(2.3)

But,
\[ \| (A^{1/2} |Z|^2 A^{1/2} + C^{1/2} |X^*|^2 C^{1/2}) \oplus 0 \| \]
\[ = \| \begin{bmatrix} A^{1/2} C^{1/2} & |Z|^2 0 \\ 0 0 & |X^*|^2 \end{bmatrix} \begin{bmatrix} A^{1/2} 0 \\ 0 C^{1/2} \end{bmatrix} \| 
\leq \| A^{1/2} C^{1/2} \| \| A^{1/2} 0 \| \| |Z|^2 0 \| \| 0 |X^*|^2 \| \quad \text{(by Lemma 2.2)} \]
\[ = \| A + C \| \| (|Z|^2) \oplus (|X^*|^2) \| . \]
(2.4)

Similarly,
\[ \| (B^{1/2} |Y|^2 B^{1/2} + D^{1/2} |W^*|^2 D^{1/2}) \oplus 0 \| \leq \| B + D \| \| (|Y|^2) \oplus (|W^*|^2) \|. \]
(2.5)

Now, the result follows from the inequalities (2.3), (2.4), and (2.5). \[ \Box \]
COROLLARY 2.4. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$ be such that $A$ and $B$ are positive semidefinite. Then
\begin{equation}
|||ZAX - YBW|||
\leq \max (||X||, ||Y||) \max (||Z||, ||W||) ||A \oplus B|| + \frac{1}{2} \left( ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| + ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| \right).
\end{equation}

Proof. Let $C = A$ and $D = B$ in the inequality (2.2), we have
\begin{align}
|||ZAX - YBW|||
\leq & \frac{1}{2} \left( ||A^{1/2} |X|^2 A^{1/2} + A^{1/2} |X|^2 A^{1/2} - A^{1/2} ZYB^{1/2} - A^{1/2} XW^*B^{1/2}|| + ||B^{1/2} Y^*ZA^{1/2} - B^{1/2} WX^*A^{1/2}|| + ||B^{1/2} |Y|^2 B^{1/2} + B^{1/2} |W|^2 B^{1/2}|| \right) \\
\leq & \frac{1}{2} \left( ||A^{1/2} |X|^2 A^{1/2} + 0|| B^{1/2} |Y|^2 B^{1/2}|| + ||B^{1/2} Y^*ZA^{1/2} - B^{1/2} WX^*A^{1/2}|| + ||B^{1/2} |Y|^2 B^{1/2} + B^{1/2} |W|^2 B^{1/2}|| \right) \\
\leq & \frac{1}{2} \left( ||A^{1/2} |X|^2 A^{1/2} + 0|| B^{1/2} |Y|^2 B^{1/2}|| + ||B^{1/2} Y^*ZA^{1/2} - B^{1/2} WX^*A^{1/2}|| + ||B^{1/2} |Y|^2 B^{1/2} + B^{1/2} |W|^2 B^{1/2}|| \right) \\
\leq & \frac{1}{2} \left( ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| + ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| \right).
\end{align}

But, by using the fact that $|||C^{1/2} |D|^2 C^{1/2}|| = |||DC^*D||$ for any matrices $C$ and $D$ such that $C$ is positive semidefinite, the inequality (2.7) becomes
\begin{align}
|||ZAX - YBW|||
= & \frac{1}{2} \left( |||X^*AX||| + |||YBY^*||| + |||ZAZ^*||| + |||W^*BW||| \right) \\
+ & \frac{1}{2} \left( ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| + ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| \right).
\end{align}

Replace $X$ by $tX$ and $Y$ by $tY$ and take the minimum over $t > 0$, we have
\begin{align}
|||ZAX - YBW|||
\leq & \sqrt{|||X^*AX||| + |||YBY^*||| + |||ZAZ^*||| + |||W^*BW|||} \leq \sqrt{\max (||X||, ||Y||)^2 (\max (||Z||, ||W||))^2 ||A \oplus B||^2} \\
+ & \frac{1}{2} \left( ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| + ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| \right) \\
\leq & \max (||X||, ||Y||) \max (||Z||, ||W||) ||A \oplus B|| \leq \frac{1}{2} \left( ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| + ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| \right) \leq \max (||X||, ||Y||) \max (||Z||, ||W||) ||A \oplus B|| \\
\leq & \max (||X||, ||Y||) \max (||Z||, ||W||) ||A \oplus B|| \leq \frac{1}{2} \left( ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| + ||A^{1/2} (Z^*Y - XW^*) B^{1/2}|| \right),
\end{align}
as required. □

Specifying Corollary 2.4 to the spectral norm, we have the following corollary.
COROLLARY 2.5. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$ be such that $A$ and $B$ are positive semidefinite. Then

$$\|ZAX - YBW\| \leq \max (\|X\|, \|Y\|) \max (\|Z\|, \|W\|) \max (\|A\|, \|B\|) + \frac{1}{2} \left\| A^{1/2} (Z^* Y - XW^*) B^{1/2} \right\|.$$ 

In particular, if $A = B = I$, then

$$\|ZX - YW\| \leq \max (\|X\|, \|Y\|) \max (\|Z\|, \|W\|) + \frac{1}{2} \|Z^* Y - XW^*\|. \quad (2.8)$$

The inequality (2.8) gives a general version of the inequality (1.1). In fact, the inequality (1.1) can be retained from the inequality (2.8) by letting $Z = W = A$, $X = Y = B$.

Replace the matrix $W$ by $-W$ in the inequality (2.6), we have the following corollary.

COROLLARY 2.6. Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$ be such that $A$ and $B$ are positive semidefinite. Then

$$\|ZAX + YBW\| \leq \max (\|X\|, \|Y\|) \max (\|Z\|, \|W\|) \max (\|A\|, \|B\|) + \frac{1}{2} \left\| A^{1/2} (Z^* Y + XW^*) B^{1/2} \right\|.$$ 

(2.9)

Letting $Z = W = I$ and $Y = X$ in the inequality (2.9), we have

$$\|AX + XB\| \leq \|X\| \max (\|A\|, \|B\|) + \left\| A^{1/2} X B^{1/2} \right\| \left\| A^{1/2} X B^{1/2} \right\| , \quad (2.10)$$

which has been obtained recently in [5]. Specifying the inequality (2.10) to the spectral norm and letting $X = I$, we have

$$\|A + B\| \leq \max (\|A\|, \|B\|) + \left\| A^{1/2} B^{1/2} \right\| \quad (2.11)$$

which is obtained in [12]. It should be mentioned here that the inequality (2.11) improves an inequality due to Davidson and Power which is useful in best approximation of $C^*$-algebras. A refinement of the inequality (2.11) has been proved in [13] so that

$$\|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4 \|A^{1/2} B^{1/2}\|^2} \right). \quad (2.12)$$

The following theorem gives a general version of the inequality (2.12). To see this, we need the following lemma which can be found in [11].

LEMMA 2.7. Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then

$$\left\| \begin{bmatrix} A & B \\ C & D \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} \|A\| & \|B\| \\ \|C\| & \|D\| \end{bmatrix} \right\|.$$
THEOREM 2.8. Let $A,B,X,Y,Z,W \in \mathbb{M}_n(\mathbb{C})$ be such that $A$ and $B$ are positive semidefinite. Then

$$
\|ZAX + YBW\| \\
\leq \frac{1}{4} \left( \left(\|Z\|^2 + \|X\|^2\right) \|A\| + \left(\|Y\|^2 + \|W\|^2\right) \|B\| + \sqrt{\left(\|A \|^2 (\|Z\|^2 + |X|^2) + \|B \|^2 (\|Y\|^2 + |W|^2)\right)^2} + 4 \|A \|^2 (Z^*Y + XW^*)B^{1/2}\|^2 \right).
$$

(2.13)

In particular, if $A = B = I$, then

$$
\|ZX + YW\| \leq \frac{1}{4} \left( \left(\|Z\|^2 + \|X\|^2\right) + \left(\|Y\|^2 + \|W\|^2\right) + \sqrt{\left(\|Z\|^2 + |X|^2\right) - \||Y\|^2 + |W|^2\|^2} + 4 \|Z^*Y + XW^*\|^2 \right).
$$

Proof. Specifying the inequality (2.2) to the spectral norm, replacing the matrix $W$ by $-W$, and letting $C = A$ and $D = B$, we have

$$
\|ZAX + YBW\| \\
= \frac{1}{2} \left\| A^{1/2} Z A^{1/2} + A^{1/2} X A^{1/2} + A^{1/2} Y B^{1/2} + A^{1/2} X W B^{1/2} + A^{1/2} X W B^{1/2} \right\| \\
= \frac{1}{2} \left\| A^{1/2} (\|Z\|^2 + |X|^2) A^{1/2} + A^{1/2} (Z^*Y + XW^*) B^{1/2} \right\| \\
\leq \frac{1}{2} \left\| A^{1/2} (\|Z\|^2 + |X|^2) A^{1/2} + A^{1/2} (Z^*Y + XW^*) B^{1/2} \right\|.
$$

Now, by calculating the spectral norm (which is the greatest singular value) of the matrix

$$
\begin{bmatrix}
\left| A^{1/2} (\|Z\|^2 + |X|^2) A^{1/2} \right| & \left| A^{1/2} (Z^*Y + XW^*) B^{1/2} \right| \\
\left| B^{1/2} (Y^*Z + WX^*) A^{1/2} \right| & \left| B^{1/2} (\|Y\|^2 + |W|^2) B^{1/2} \right|
\end{bmatrix},
$$

we have

$$
\|ZAX + YBW\| \\
\leq \frac{1}{4} \left( \left(\|Z\|^2 + \|X\|^2\right) \|A\| + \left(\|Y\|^2 + \|W\|^2\right) \|B\| + \sqrt{\left(\|A \|^2 (\|Z\|^2 + |X|^2) + \|B \|^2 (\|Y\|^2 + |W|^2)\right)^2} + 4 \|A \|^2 (Z^*Y + XW^*)B^{1/2}\|^2 \right)
$$

as required. □
Letting $X = Y = Z = W = I$ in the inequality (2.13), we have
\[ \|A + B\| \leq \frac{1}{2} \left( \|A\| + \|B\| + \sqrt{(\|A\| - \|B\|)^2 + 4\|A^{1/2}B^{1/2}\|^2} \right), \]
which is the inequality (2.12).

Specifying Corollary 2.6 to the spectral norm, we have the following corollary.

**Corollary 2.9.** Let $A, B, X, Y, Z, W \in \mathbb{M}_n(\mathbb{C})$ be such that $A$ and $B$ are positive semidefinite. Then
\[ \|ZAX + YBW\| \leq \max (\|X\|, \|Y\|) \max (\|Z\|, \|W\|) \max (\|A\|, \|B\|) \]
\[ + \frac{1}{2} \left\| A^{1/2} (Z^* Y + XW^*) B^{1/2} \right\|. \]
In particular, if $A = B = I$, then
\[ \|ZX + YW\| \leq \max (\|X\|, \|Y\|) \max (\|Z\|, \|W\|) + \frac{1}{2} \left\| Z^* Y + XW^* \right\|. \] (2.14)

Clearly, the inequality (2.14) gives a general version of the inequality (1.2). In fact, the inequality (1.2) can be retained by letting $Z = W = A$, $X = Y = B$ in the inequality (2.14).

To prove our next result, we need the following lemma which can be found in [14]. Recently, the authors in [3] gave a completely different proof of this lemma.

**Lemma 2.10.** Let $A, B, C \in M_n(\mathbb{C})$ be such that the block matrix $T = \begin{bmatrix} A & C \\ C^* & B \end{bmatrix}$ is positive semidefinite. Then
\[ 2s_j(C) \leq s_j(T) \]
for $j = 1, \ldots, n$, and so
\[ 2 \|\|C\|\| \leq \|\|T\|\|. \]

**Theorem 2.11.** Let $A, B, C, D \in \mathbb{M}_n(\mathbb{C})$. Then
\[ 2 \|\|A^* B + C^* D\|\| \leq \|\|(A^* A + C^* C) \oplus (B^* B + D^* D)\|\|
\[ + \|\|((A^* B + C^* D) \oplus (B^* A + D^* C))\|\|. \] (2.15)
In particular,
\[ \|A^* B + C^* D\| \leq \max (\|A^* A + C^* C\|, \|B^* B + D^* D\|). \] (2.16)

**Proof.** Let $E = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$. Then
\[ E^* E = \begin{bmatrix} A^* & C^* \\ B^* & D^* \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A^* A + C^* C & A^* B + C^* D \\ B^* A + D^* C & B^* B + D^* D \end{bmatrix} \]
is positive semidefinite. So, by Lemma 2.10, we have

\[ 2 \left\| \begin{bmatrix} A^*B + C^*D \\ B^*A + D^*C \\ B^*B + D^*D \end{bmatrix} \right\| \leq \left\| \begin{bmatrix} A^*A + C^*C \\ B^*A + D^*C \\ B^*B + D^*D \end{bmatrix} \right\| \]

(2.17)

which proves the inequality (2.15). The inequality (2.16) follows by specifying the inequality (2.15) to the spectral norm. □

**Corollary 2.12.** Let \( A, B, C, D \in M_n(\mathbb{C}) \). Then

\[ \| A^*B + C^*D \| \leq \frac{1}{2} \left( \frac{\| A^*A + C^*C \| + \| B^*B + D^*D \|}{\sqrt{\left( \| A^*A + C^*C \| - \| B^*B + D^*D \| \right)^2 + 4 \| A^*B + C^*D \|^2}} \right). \]

**Proof.** Specifying the inequality (2.17) to the spectral norm, we have

\[ 2 \left\| A^*B + C^*D \right\| \leq \left\| \begin{bmatrix} A^*A + C^*C \\ B^*A + D^*C \\ B^*B + D^*D \end{bmatrix} \right\| \]

(by Lemma 2.7)

\[ \leq \left\| \begin{bmatrix} A^*A + C^*C \\ B^*A + D^*C \\ B^*B + D^*D \end{bmatrix} \right\| \]

\[ \leq \frac{1}{2} \left( \frac{\| A^*A + C^*C \| + \| B^*B + D^*D \|}{\sqrt{\left( \| A^*A + C^*C \| - \| B^*B + D^*D \| \right)^2 + 4 \| A^*B + C^*D \|^2}} \right), \]

as required. □

**References**


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Ahmad Al-Natoor  
Department of mathematics  
Isra University  
Amman, Jordan  
e-mail: ahmad.alnatoor@iu.edu.jo

Mohammad A. Amleh  
Department of mathematics  
Zarqa University  
Zarqa, Jordan  
e-mail: malamleh@zu.edu.jo

Baha’ Abughazaleh  
Department of mathematics  
Isra University  
Amman, Jordan  
e-mail: baha.abughazaleh@iu.edu.jo

Aliaa Burqan  
Department of mathematics  
Zarqa University  
Zarqa, Jordan  
e-mail: aliaaburqan@zu.edu.jo