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BLOCK DECOMPOSITION FOR HERZ SPACES ASSOCIATED WITH BALL BANACH FUNCTION SPACES

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Abstract. In this paper, we establish the block decomposition for Herz spaces associated with ball Banach function spaces. By using this decomposition, we obtain the boundedness for a class of sublinear operators on Herz spaces associated with ball Banach function spaces.

1. Introduction

The classical Herz spaces, initially introduced by Herz [6] to study the Fourier series and Fourier transform, are important extensions of Lebesgue spaces. Nowadays, Herz spaces have been widely used in harmonic analysis and PDE. For instance, many important operators in harmonic analysis, such as Hardy-Littlewood maximal operator and singular integral operators, were proved to be bounded on Herz spaces, see [1, 9, 11, 17, 19, 21, 23, 24, 26]. We also refer the readers to [3, 20, 25, 30] for some applications of Herz spaces in PDE.

In 1995, Lu and Yang [22] proved the block decomposition for classical Herz spaces, from which the boundedness of a class of sublinear operator satisfying some size conditions was established. For the block decomposition of some other block spaces and its applications, we refer the readers to [18, 24, 34, 33].

As is well-known, Banach function spaces, initially introduced by Bennett [2], unify various function spaces such as Lebesgue spaces, Orlicz spaces and Lorentz spaces in harmonic analysis. Recently, Sawano et al. [28] extended Banach function spaces to ball Banach function spaces, and proved that many function spaces, such as Morrey spaces, mixed-norm Lebesgue spaces and Orlicz-slice spaces are all ball Banach function spaces, which may not be Banach function spaces. We refer the readers to [4, 10, 16, 32, 38, 43, 44] for more studies on ball Banach function spaces.

By combining Herz spaces with ball Banach function spaces, Wei [41] introduced Herz spaces associated with ball Banach function spaces, which are extensions of Herz spaces with variable exponent [5, 12, 13, 15, 35, 36, 37, 42], mixed Herz spaces [39, 40] and some other Herz-type spaces. Inspired by [22, 40], we will establish the block decomposition for Herz spaces associated ball Banach function spaces in this paper. As

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an application of the decomposition, we further obtain the boundedness for a class of sublinear operators satisfying some size conditions.

Throughout the paper, we use the following notations.

For any $r>0$ and $x\in\mathbb{R}^n$, let $B(x,r)=\{y:|y-x|<r\}$ be the ball centered at x with radius r. Let $\mathbb{B}=\{B(x,r):x\in\mathbb{R}^n, r>0\}$ be the set of all such balls. We use χ_E and $|E|$ to denote the characteristic function and the Lebesgue measure of a measurable set E. Let $\mathscr{M}(E)$ be the class of Lebesgue measurable functions on E. For any quasi-Banach space X, the space $L^X_{\mathrm{loc}}(E)$ consists of all functions $f\in\mathscr{M}(E)$ such that $f\chi_F\in X$ for all compact subsets $F\subseteq E$. We denote the set of all non-negative integers, all integers and all complex numbers by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{C}$, respectively. By $A\lesssim B$, we mean that $A\leqslant CB$ for some constant $C>0$, and $A\sim B$ means that $A\lesssim B$ and $B\lesssim A$. For all $k\in\mathbb{Z}$, let $B_k=B(0,2^k)$, $C_k=B_k\backslash B_{k-1}$. Denote $\chi_k=\chi_{C_k}$ for $k\in\mathbb{Z}$, $\tilde{\chi}_k=\chi_{C_k}$ if $k\in\mathbb{N}^+$ and $\tilde{\chi}_0=\chi_{B_0}$.

This paper is organized as follows. Some definitions and preliminaries are presented in Sect. 2. The block decomposition for Herz spaces associated with ball Banach function spaces is established in Sect 3. By applying the decomposition, we obtain the boundedness for a class of sublinear operators satisfying some size conditions on Herz spaces associated with ball Banach function spaces in Sect. 4.

2. Definitions and preliminaries

In this section, we give the defnitions and some basic properties of ball Banach function spaces and Herz spaces associated with ball Banach function spaces.

We first recall the definition of ball Banach function spaces.

DEFINITION 2.1. A Banach space $X\subseteq\mathscr{M}(\mathbb{R}^n)$ is called a ball Banach function space if it satisfies

(i) $\|f\|_X=0$ implies that $f=0$ almost everywhere;

(ii) $|g|\leqslant|f|$ almost everywhere implies that $\|g\|_X\leqslant\|f\|_X$;

(iii) $0\leqslant f_m\uparrow f$ almost everywhere implies that $0\leqslant\|f_m\|_X\uparrow\|f\|_X$;

(iv) $B\in\mathbb{B}$ implies that $\chi_B\in X$;

(v) for any $B\in\mathbb{B}$, there exists a positive constant $C(B)$, depending on B, such that, for any $f\in X$,

$$\int_B f(x)dx\leqslant C(B)\|f\|_X.$$

REMARK 2.1. (i) In Definition 2.1, if we replace the ball B by any bounded measurable set E, then the definitions are mutually equivalent.

(ii) From the definition, one can see that every Banach function space is a ball Banach function space, and the converse is not necessary to be true. For instance, it was shown in [8] that mixed-norm spaces are ball Banach function spaces, but not Banach function spaces.

Now we give the definition of the associate space of a ball Banach function space, see, for instance, [2, Chapter 1, Definitions 2.1 and 2.3].

DEFINITION 2.2. For any ball Banach function space X, the associate space (also called the Köthe dual) X' is defined by setting

$$X' := \{f \in \mathscr{M}(\mathbb{R}^n) : \|f\|_{X'} = \sup_{g \in X,\, \|g\|_X \leqslant 1} \|fg\|_{L^1} < \infty\}, \tag{1}$$

where $\|\cdot\|_{X'}$ is called the associate norm of $\|\cdot\|_X$.

REMARK 2.2. From [28, Proposition 2.3], we know that, if X is a ball Banach function space, then its associate space X' is also a ball Banach function space. We also point out that X' is different form X^*, the dual space of X. However, the condition $X' = X^*$ may hold for a large class of ball Banach function spaces. In fact, we have $X' = X^*$ as long as X has an absolutely continuous norm (see [2, Chapter 1, Corollary 4.3]). As we know, many function spaces appeared in harmonic analysis have absolutely continuous norms under some mild conditions, and therefore, they satisfy $X' = X^*$.

The Hölder's inequality for ball Banach function spaces can be deduced from Definition 2.1 and (1), see also [2, Theorem 2.4], for the proof.

LEMMA 2.1. *Let X be a ball Banach function space, and X' its associate space. If $f \in X$ and $g \in X'$, then fg is integrable and*

$$\int_{\mathbb{R}^n} |f(x)g(x)|dx \leqslant \|f\|_X \|g\|_{X'}.$$

The Hardy-Littlewood maximal operator M is defined by setting, for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $x \in \mathbb{R}^n$,

$$M(f)(x) := \sup_{B \ni x} \frac{1}{|B|} \int_B |f(y)|dy, \tag{2}$$

where the supremum is taken over all balls $B \in \mathbb{B}$.

If we impose the boundedness of M on the associate space of the ball Banach function space X, then the norm $\|\cdot\|_X$ has properties similar to the classical Muckenhoupt weights.

LEMMA 2.2. *Let X be a ball Banach function space. Suppose that the Hardy-Littlewood maximal operator M is bounded on X'. Then for all balls $B \in \mathbb{B}$, we have*

$$\|\chi_B\|_X \|\chi_B\|_{X'} \lesssim |B|. \tag{3}$$

LEMMA 2.3. *Let X be a ball Banach function space. Suppose that M is bounded on the associate space X'. Then there exists a constant $0 < \delta_X < 1$, such that for all balls $B \subseteq \mathbb{B}$ and all measurable sets $E \subseteq B$, we have*

$$\frac{\|\chi_E\|_X}{\|\chi_B\|_X} \lesssim \left(\frac{|E|}{|B|}\right)^{\delta_X}. \tag{4}$$

The proofs of Lemma 2.2 and Lemma 2.3 can be found in [14]. Although these lemmas were proved only for Banach function spaces, the results also hold for ball Banach function spaces by checking the proofs carefully. One can see that in the particular case $X = L^q(\mathbb{R}^n)$ $(1 \leqslant q < \infty)$, $\delta_X = 1/q$. Throughout this paper, for any ball Banach function spaces X, δ_X is the same as in Lemma 2.3.

The definitions of the classical Herz spaces are as follows, see, for instance [22, 24].

DEFINITION 2.3. Let $\alpha \in \mathbb{R}$, $0 < p, q \leqslant \infty$.

(i) The homogeneous Herz space $\dot{K}_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{K}_q^{\alpha,p}} := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_q^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_q^{\alpha,p}} := \left\{ \sum_{k\in\mathbb{Z}} 2^{kp\alpha} \|f\chi_k\|_{L^q}^p \right\}^{1/p}.$$

If $p = \infty$ or $q = \infty$, then we have to make appropriate modifications.

(ii) The non-homogeneous Herz space $K_q^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\|f\|_{K_q^{\alpha,p}} := \left\{ f \in L^q_{\mathrm{loc}}(\mathbb{R}^n) : \|f\|_{K_q^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{K_q^{\alpha,p}} := \left\{ \sum_{k\in\mathbb{N}} 2^{kp\alpha} \|f\tilde{\chi}_k\|_{L^q}^p \right\}^{1/p}.$$

If $p = \infty$ or $q = \infty$, then we have to make appropriate modifications.

By replacing $L^q(\mathbb{R}^n)$ with some ball Banach function space X, Wei [41] introduced Herz spaces associated with ball Banach function spaces.

DEFINITION 2.4. Let $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and X be a ball quasi-Banach function space.

(i) The homogeneous Herz-type space associated with ball quasi-Banach function spaces $\dot{K}_X^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\|f\|_{\dot{K}_X^{\alpha,p}} := \left\{ f \in L^X_{\mathrm{loc}}(\mathbb{R}^n \setminus \{0\}) : \|f\|_{\dot{K}_X^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{\dot{K}_X^{\alpha,p}} := \left\{ \sum_{k\in\mathbb{Z}} 2^{kp\alpha} \|f\chi_k\|_X^p \right\}^{1/p}.$$

If $p = \infty$, then we have to make appropriate modifications.

(ii) The non-homogeneous Herz-type space $K_X^{\alpha,p}(\mathbb{R}^n)$ is defined by

$$\|f\|_{K_X^{\alpha,p}} := \left\{ f \in L_{\mathrm{loc}}^X(\mathbb{R}^n) : \|f\|_{K_X^{\alpha,p}} < \infty \right\},$$

where

$$\|f\|_{K_X^{\alpha,p}} := \left\{ \sum_{k\in\mathbb{N}} 2^{kp\alpha} \|f\tilde{\chi}_k\|_X^p \right\}^{1/p}.$$

If $p = \infty$, then we have to make appropriate modifications.

For $\alpha \in \mathbb{R}$, $0 < p \leqslant \infty$ and any ball Banach function space X, $\dot{K}_X^{\alpha,p}(\mathbb{R}^n)$ and $K_X^{\alpha,p}(\mathbb{R}^n)$ are quasi-Banach spaces, and if further $p \geqslant 1$, then they are Banach spaces. Obviously, Herz spaces associated with ball Banach function spaces are extensions of classical Herz spaces. If $X = L^{q(\cdot)}(\mathbb{R}^n)$, the variable exponent Lebesgue spaces, we recover the Herz spaces with variable exponent studied in [12]. Moreover, by taking different ball Banach function spaces, we can define some concrete Herz-type spaces such as mixed Herz spaces and Herz-Lorentz spaces, see [39, 40, 41]. For more about Herz spaces associated with ball Banach function spaces, the readers are referred to [41].

3. Block decomposition for Herz spaces associated with ball Banach function spaces

As is known, the classical Herz spaces possess block decomposition, see [22, 24]. Besides, block-type spaces can characterize various function spaces. For instance, the pre-dual of various Morrey-type spaces can be identified with block spaces, see [7, 27, 29]. This section is devoted to establishing the block decomposition for Herz spaces associated with ball Banach function spaces.

We first give the definition of central X-block.

DEFINITION 3.1. Let $0 < \alpha < \infty$, $0 < p < \infty$ and X be a ball Banach function space.

(I) A function b on $\mathbb{R}^n$ is said to be a central X-block if

(a) $\operatorname{supp} b \in B(0,r)$ for some $r > 0$.

(b) $\|b\|_X \leqslant r^{-\alpha}$.

(II) A function b on $\mathbb{R}^n$ is said to be a central X-block of restricted type if

(c) $\operatorname{supp} b \in B(0,r)$ for some $r \geqslant 1$.

(d) $\|b\|_X \leqslant r^{-\alpha}$.

If $r = 2^i$ for some $i \in \mathbb{Z}$ in Definition 3.1, then the central X-block is also called a dyadic central X-block.

Our first main result in this section is as follows.

THEOREM 3.1. *Suppose $0<\alpha<\infty$, $0<p<\infty$ and X be a ball Banach function space. Then the following facts are equivalent.*

(i) $f \in \dot{K}_X^{\alpha,p}(\mathbb{R}^n)$.

(ii) *f can be represented as*

$$f(x)=\sum_{i\in\mathbb{Z}}\lambda_i b_i(x),$$

where each b_i is a dyadic central X-block with the support B_i, and $\sum_{i\in\mathbb{Z}}|\lambda_i|^p<\infty$.

Proof. We first prove (i) implies (ii). For $f \in \dot{K}_X^{\alpha,p}(\mathbb{R}^n)$, rewrite f as

$$f(x)=\sum_{i\in\mathbb{Z}}f(x)\chi_i(x)=\sum_{i\in\mathbb{Z}}2^{i\alpha}\|f\chi_i\|_X\frac{f(x)\chi_i(x)}{2^{i\alpha}\|f\chi_i\|_X}=\sum_{i\in\mathbb{Z}}\lambda_i b_i,$$

where $\lambda_i=2^{i\alpha}\|f\chi_i\|_X$ and $b_i=\frac{f(x)\chi_i(x)}{2^{i\alpha}\|f\chi_i\|_X}$.

It is obvious that $\operatorname{supp} b_i \in B_i$ and $\|b_i\|_X=2^{-i\alpha}$. Therefore each b_i is a dyadic central X-block with support B_i and

$$\sum_{i\in\mathbb{Z}}|\lambda_i|^p=\sum_{i\in\mathbb{Z}}2^{ip\alpha}\|f\chi_i\|_X^p=\|f\|^p_{\dot{K}_X^{\alpha,p}(\mathbb{R}^n)}<\infty.$$

Next we prove that (ii) implies (i). Assume $f(x)=\sum_{i\in\mathbb{Z}}\lambda_i b_i(x)$ be a decomposition of f which satisfies the hypothesis (ii).

For any $i\in\mathbb{Z}$, using Minkowski's inequality on ball Banach function spaces (see [31, Lemma 3.4]), we obtain

$$\|f\chi_i\|_X \leqslant \sum_{j=i}^{\infty}|\lambda_j|\,\|b_j\|_X. \tag{5}$$

Now we consider two different cases for the index p.

For $0<p\leqslant 1$, the inequality (5) yields that

$$\begin{aligned}
\|f\|^p_{\dot{K}_X^{\alpha,p}} &= \sum_{i\in\mathbb{Z}}2^{ip\alpha}\|f\chi_i\|_X^p\\
&\leqslant \sum_{i\in\mathbb{Z}}2^{ip\alpha}\left(\sum_{j\geqslant i}|\lambda_j|^p\|b_j\|_X^p\right)\\
&\leqslant \sum_{i\in\mathbb{Z}}2^{ip\alpha}\left(\sum_{j\geqslant i}|\lambda_j|^p 2^{-jp\alpha}\right)\\
&= \sum_{j\in\mathbb{Z}}|\lambda_j|^p\sum_{i\leqslant j}2^{(i-j)p\alpha}\lesssim \sum_{j\in\mathbb{Z}}|\lambda_j|^p,
\end{aligned}$$

since $\alpha>0$.

When $1<p<\infty$, by using (5) and Hölder's inequality, we get

$$\begin{aligned}\|f\chi_i\|_X &\leqslant \sum_{j\geqslant i}|\lambda_j|\|b_j\|_X^{1/2}\|b_j\|_X^{1/2}\\ &\leqslant \left(\sum_{j\geqslant i}|\lambda_j|^p\|b_j\|_X^{p/2}\right)^{1/p}\left(\sum_{j\geqslant i}\|b_j\|_X^{p'/2}\right)^{1/p'}\\ &\leqslant \left(\sum_{j\geqslant i}|\lambda_j|^p 2^{-jp\alpha/2}\right)^{1/p}\left(\sum_{j\geqslant i}2^{-jp'\alpha/2}\right)^{1/p'}.\end{aligned}$$

Therefore,

$$\begin{aligned}\|f\|^p_{\dot{K}^{\alpha,p}_X} &\lesssim \sum_{i\in\mathbb{Z}}2^{ip\alpha}\left(\sum_{j\geqslant i}|\lambda_j|^p 2^{-jp\alpha/2}\right)\left(\sum_{j\geqslant i}2^{-jp'\alpha/2}\right)^{p/p'}\\ &\lesssim \sum_{j\in\mathbb{Z}}|\lambda_j|^p\left(\sum_{i\leqslant j}2^{(i-j)p\alpha/2}\right)\lesssim \sum_{j\in\mathbb{Z}}|\lambda_j|^p.\end{aligned}$$

As a consequence, we have $f\in\dot{K}^{\alpha,p}_X(\mathbb{R}^n)$, which finishes the proof. □

REMARK 3.1. (i) One can see from the proof of Theorem 3.1 that if $f\in\dot{K}^{\alpha,p}_X(\mathbb{R}^n)$ and $f=\sum_{i\in\mathbb{Z}}\lambda_i b_i$ be a dyadic central X-block decomposition, then

$$\|f\|_{\dot{K}^{\alpha,p}_X}\sim\left(\sum_{i\in\mathbb{Z}}|\lambda_i|^p\right)^{1/p}.$$

(ii) Theorem 3.1 extends the results in [22, 34], where the block decomposition for some particular homogeneous Herz spaces associated with ball Banach function spaces was build.

By an argument similar to the proof of Theorem 3.1, we have the block decomposition for non-homogeneous Herz spaces associated with ball Banach function spaces.

THEOREM 3.2. *Suppose $0<\alpha<\infty$, $0<p<\infty$ and X be a ball Banach function space. Then the following facts are equivalent.*

(i) $f\in K^{\alpha,p}_X(\mathbb{R}^n)$.

(ii) *f can be represented as*

$$f(x)=\sum_{i\in\mathbb{N}}\lambda_i b_i(x), \tag{6}$$

where each b_i is a dyadic central X-block of restricted type with support contained B_i, and $\sum_{i=0}^{\infty}|\lambda_i|^p<\infty$.

Moreover, the norms $\|f\|_{K^{\alpha,p}_X}$ and $\inf(\sum_{i\in\mathbb{N}}|\lambda_i|^p)^{1/p}$ are equivalent, where the infimum is taken over all the decomposition of f as in (6).

4. Boundedness for a class of sublinear operators on Herz spaces associated with ball Banach function spaces

In this section, we establish the boundedness for a class of sublinear operators on Herz spaces associated with ball Banach function spaces. We first recall the definition of sublinear operators.

Let V be a vector space. We say that an operator $T : V \to \mathscr{M}(\mathbb{R}^n)$ is a sublinear operator if for any $f,g \in V$ and $t \in \mathbb{C}$,

$$|T(f+g)| \leqslant |Tf| + |Tg|$$

and

$$|T(tf)| = |t|\, |Tf|.$$

Our main result in this section can be read as follows.

THEOREM 4.1. *Let $0 < \alpha < \infty$, $0 < p < \infty$ and X be a ball Banach function space such that $\alpha < n\delta_{X'}$, $X' = X^*$ and the Hardy-Littlewood maximal operator M is bounded on X. If a sublinear operator T initially defined on X satisfies*

$$|Tf(x)| \lesssim \|f\|_{L^1}/|x|^n \quad \text{if} \quad \operatorname{dist}(x, \operatorname{supp} f) > |x|/2, \tag{7}$$

for any $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ with a compact support and T is bounded on X, then T can be extended to a bounded operator on $\dot{K}_X^{\alpha,p}(\mathbb{R}^n)$ and $K_X^{\alpha,p}(\mathbb{R}^n)$.

Proof. It suffices to prove the boundedness of T on $\dot{K}_X^{\alpha,p}(\mathbb{R}^n)$. The non-homogeneous case can be proved in the similar way.

Assume $f \in \dot{K}_X^{\alpha,p}(\mathbb{R}^n)$ and $f \in C_c^\infty(\mathbb{R}^n)$, the set of smooth functions with compact support. For such a nice function f, Tf is well-defined since $C_c^\infty(\mathbb{R}^n) \subseteq X$. By virtue of Theorem 3.1, we may rewrite f as $f(x) = \sum_{i\in\mathbb{Z}} \lambda_i b_i(x)$, where each b_i is a dyadic central X-block with support contained in B_i and

$$\|f\|_{\dot{K}_X^{\alpha,p}} \sim \left(\sum_{i\in\mathbb{Z}} |\lambda_i|^p\right)^{1/p}.$$

Therefore, we have

$$\begin{aligned}
\|Tf\|^p_{\dot{K}_X^{\alpha,p}} &= \sum_{j\in\mathbb{Z}} 2^{jp\alpha} \|(Tf)\chi_j\|_X^p \\
&\lesssim \sum_{j\in\mathbb{Z}} 2^{jp\alpha} \left(\sum_{i=-\infty}^{j-2} |\lambda_i|\, \|(Tb_i)\chi_j\|_X\right)^p + \sum_{j\in\mathbb{Z}} 2^{jp\alpha} \left(\sum_{i=j-1}^{\infty} |\lambda_i|\, \|(Tb_i)\chi_j\|_X\right)^p \\
&=: I_1 + I_2.
\end{aligned}$$

Let us first estimate I_1. By the condition (7) and Lemma 2.1, for any $x \in C_j$, we have

$$|Tb_i(x)| \lesssim 2^{-jn} \int_{B_i} |b_i(x)|dx \lesssim 2^{-jn} \|b_i\|_X \|\chi_{B_i}\|_{X'}. \tag{8}$$

Therefore, from Lemma 2.2, Lemma 2.3 and the definition of dyadic central X-block, we obtain

$$\begin{aligned}\|(Tb_i)\chi_j\|_X &\lesssim 2^{-jn}\|b_i\|_X\|\chi_{B_i}\|_{X'}\|\chi_{B_j}\|_X\\ &\lesssim 2^{-jn}\|b_i\|_X\frac{\|\chi_{B_i}\|_{X'}}{\|\chi_{B_j}\|_{X'}}\|\chi_{B_j}\|_X\|\chi_{B_j}\|_{X'}\\ &\lesssim 2^{-i\alpha}2^{(i-j)n\delta_{X'}}.\end{aligned}$$

Consequently, when $0<p\leqslant 1$, according to $\alpha<n\delta_{X'}$, we get

$$\begin{aligned}I_1 &= \sum_{j\in\mathbb{Z}}2^{jp\alpha}\left(\sum_{i=-\infty}^{j-2}|\lambda_i|\;\|(Tb_i)\chi_j\|_X\right)^p\\ &\lesssim \sum_{j\in\mathbb{Z}}2^{jp\alpha}\left(\sum_{i=-\infty}^{j-2}|\lambda_i|^p\;2^{-ip\alpha}2^{(i-j)pn\delta_{X'}}\right)\\ &= \sum_{i\in\mathbb{Z}}|\lambda_i|^p\left(\sum_{j=i+2}^{\infty}2^{(i-j)p(n\delta_{X'}-\alpha)}\right)\\ &\lesssim \sum_{i\in\mathbb{Z}}|\lambda_i|^p\lesssim\|f\|^p_{\dot{K}_X^{\alpha,p}}.\end{aligned}$$

When $1<p<\infty$, by using $\alpha<n\delta_{X'}$ and Hölder's inequality, we have

$$\begin{aligned}I_1 &\lesssim \sum_{j\in\mathbb{Z}}2^{jp\alpha}\left(\sum_{i=-\infty}^{j-2}|\lambda_i|\;2^{-i\alpha}2^{(i-j)n\delta_{X'}}\right)^p\\ &= \sum_{j\in\mathbb{Z}}\left(\sum_{i=-\infty}^{j-2}|\lambda_i|\;2^{(i-j)(n\delta_{X'}-\alpha)}\right)^p\\ &\lesssim \sum_{j\in\mathbb{Z}}\left(\sum_{i=-\infty}^{j-2}|\lambda_i|^p\;2^{(i-j)p(n\delta_{X'}-\alpha)/2}\right)\left(\sum_{i=-\infty}^{j-2}2^{(i-j)p'(n\delta_{X'}-\alpha)/2}\right)^{p/p'}\\ &\lesssim \sum_{i\in\mathbb{Z}}|\lambda_i|^p\sum_{j=i+2}^{\infty}2^{(i-j)p(n\delta_{X'}-\alpha)/2}\\ &\lesssim \sum_{i\in\mathbb{Z}}|\lambda_i|^p\lesssim\|f\|^p_{\dot{K}_X^{\alpha,p}}.\end{aligned}$$

Let us now estimate I_2. By the boundedness of T on X and the definition of dyadic central X-block, we get

$$\|(Tb_i)\chi_j\|_X\lesssim\|b_i\|_X\lesssim 2^{-i\alpha}. \tag{9}$$

We also consider two cases for p. When $0 < p \leqslant 1$, in view of $\alpha > 0$, we obtain

$$\begin{aligned} I_2 &= \sum_{j\in\mathbb{Z}} 2^{jp\alpha}\left(\sum_{i=j-1}^{\infty} |\lambda_i| \, \|(Tb_i)\chi_j\|_X\right)^p \lesssim \sum_{j\in\mathbb{Z}} 2^{jp\alpha}\left(\sum_{i=j-1}^{\infty} |\lambda_i| \, 2^{-i\alpha}\right)^p \\ &= \sum_{j\in\mathbb{Z}}\left(\sum_{i=j-1}^{\infty} |\lambda_i| \, 2^{(j-i)\alpha}\right)^p \lesssim \sum_{j\in\mathbb{Z}}\left(\sum_{i=j-1}^{\infty} |\lambda_i|^p \, 2^{(j-i)p\alpha}\right) \\ &= \sum_{i\in\mathbb{Z}} |\lambda_i|^p \left(\sum_{j=-\infty}^{i+1} 2^{(j-i)p\alpha}\right) \lesssim \sum_{i\in\mathbb{Z}} |\lambda_i|^p \lesssim \|f\|^p_{\dot{K}^{\alpha,p}_X}. \end{aligned}$$

When $1 < p < \infty$, we have

$$\begin{aligned} I_2 &\lesssim \sum_{j\in\mathbb{Z}}\left(\sum_{i=j-1}^{\infty} |\lambda_i| \, 2^{(j-i)\alpha}\right)^p \\ &\lesssim \sum_{j\in\mathbb{Z}}\left(\sum_{i=j-1}^{\infty} |\lambda_i|^p \, 2^{(j-i)p\alpha/2}\right)\left(\sum_{i=j-1}^{\infty} 2^{(j-i)p'\alpha/2}\right)^{p/p'} \\ &\lesssim \sum_{i\in\mathbb{Z}} |\lambda_i|^p \left(\sum_{j=-\infty}^{i+1} 2^{(j-i)p\alpha/2}\right) \lesssim \sum_{i\in\mathbb{Z}} |\lambda_i|^p \lesssim \|f\|^p_{\dot{K}^{\alpha,p}_X} \end{aligned}$$

since $\alpha > 0$.

Combining all the estimates for I_1 and I_2, we have for any $f \in \dot{K}^{\alpha,p}_X(\mathbb{R}^n) \cap C^\infty_{\mathrm{c}}(\mathbb{R}^n)$, $\|Tf\|_{\dot{K}^{\alpha,p}_X} \lesssim \|f\|_{\dot{K}^{\alpha,p}_X}$.

To finish the proof, we also need some denseness results. Fortunately, it was pointed out in [41, Proof of Theorem 4.1] that $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ is dense in $\dot{K}^{\alpha,p}_X(\mathbb{R}^n)$ if $X' = X^*$. One can also refer to [9, Proposition 4.2] for the detailed proof in the particular case $X = L^{q(\cdot)}(\mathbb{R}^n)$.

Since T is a sublinear operator on X, we have $|Tf - Tg| \leqslant |T(f-g)|$ for $f, g \in C^\infty_{\mathrm{c}}(\mathbb{R}^n) \subseteq \dot{K}^{\alpha,p}_X(\mathbb{R}^n)$. Consequently,

$$\|Tf - Tg\|_{\dot{K}^{\alpha,p}_X} \leqslant \|T(f-g)\|_{\dot{K}^{\alpha,p}_X} \lesssim \|f-g\|_{\dot{K}^{\alpha,p}_X}$$

for $f, g \in C^\infty_{\mathrm{c}}(\mathbb{R}^n)$.

The above inequalities, together with the denseness of $C^\infty_{\mathrm{c}}(\mathbb{R}^n)$ in $\dot{K}^{\alpha,p}_X(\mathbb{R}^n)$, guarantee that T can be extended to be a bounded operator on $\dot{K}^{\alpha,p}_X(\mathbb{R}^n)$. □

REMARK 4.1. (i) By taking $X = L^q(\mathbb{R}^n)$ in Theorem 4.1, we recover the results of [22, Theorem 1.3]. Moreover, if we choose $X = L^{q(\cdot)}(\mathbb{R}^n)$, then we obtain the result of [34, Theorem 4]. In general, we can obtain the boundedness for a class of sublinear operators satisfying condition (7) on various Herz-type spaces by taking concrete ball Banach function spaces in Theorem 4.1.

(ii) The size condition (7) is satisfied by many operators in harmonic analysis. For instance, the Hardy-Littlewood maximal operator and the Calderón-Zygmund singular integral operators fulfill the condition (7).

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