SHARP INEQUALITIES OF IYENGAR–MADHAVA
RAO–NANJUNDIAH TYPE INCLUDING $\cos \left( \frac{x}{\sqrt{3}} + ax^r \right)$

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Abstract. In this paper, for $0 < x < \frac{\pi}{2}$ and $r > 0$, we consider the following Iyengar-Madhava Rao-Nanjundiah type inequality:

$$\cos \left( \frac{x}{\sqrt{3}} + ax^r \right) < \frac{\sin x}{x} < \cos \left( \frac{x}{\sqrt{3}} + bx^r \right).$$

Our main theorems show that $\alpha$ and $\beta$ depend on $r > 0$, and if $0 < r < 3$ then

$$\beta = \left( \frac{2}{\pi} \right)^r \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right)$$

and if $r > 4$ then

$$\alpha = \left( \frac{2}{\pi} \right)^r \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right).$$

1. Introduction

Mitrinović [2] in p.236, presented that the inequality

$$1 \geq \cos \frac{x}{2} \geq \cos ax \geq \frac{\sin x}{x} \geq \cos bx \geq \sqrt[3]{\cos x} \geq \cos x$$

(1)

holds for $0 < x < \frac{\pi}{2}$, where the constants $a = \frac{2}{\pi} \arccos \frac{2}{\pi} \approx 0.560664$ and $b = \frac{1}{\sqrt{3}} \approx 0.57735$ are the best possible. Especially, for $0 < x < \frac{\pi}{2}$, the inequality

$$\cos ax > \frac{\sin x}{x} > \cos bx$$

(2)

is known as Iyengar-Madhava Rao-Nanjundiah inequality [1]. Recently, Sándor [8] gave a new proof different from Iyengar’s one and proved the hyperbolic version. Also, for $0 < x < \frac{\pi}{2}$, the inequality

$$\frac{\sin x}{x} > \sqrt[3]{\cos x}$$

(3)

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is known as Mitrinović-Adamovic inequality and many known studies of Mitrinović-Adamovic type inequality. Recently, there are the following paper; [2], [3], [6], [7], [9]. On the other hand, little is known about Iyengar-Madhava Rao-Nanjundiah inequality (2). In this paper, we consider the improved Iyengar-Madhava Rao-Nanjundiah inequality and our main theorems are the followings.

**THEOREM 1.** For $0 < x < \frac{\pi}{2}$, we have
\[
\cos\left(\frac{x}{\sqrt{3}} + \alpha x^3\right) < \frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}} + \beta x^3\right),
\]
where the constants $\alpha = -\frac{1}{90\sqrt{3}} \approx -0.006415$ and $\beta = \frac{8}{\pi^3}\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right) \approx -0.0067626$ are the best possible.

**THEOREM 2.** For $0 < x < \frac{\pi}{2}$ and $0 < r < 3$, we have
\[
\cos\left(\frac{x}{\sqrt{3}} + \alpha x^r\right) < \frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}} + \beta x^r\right),
\]
where the constant $\alpha = 0$ is the best possible and $\beta_r = \left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right)$.

**THEOREM 3.** For $0 < x < \frac{\pi}{2}$ and $r > 4$, we have
\[
\frac{\sin x}{x} > \cos\left(\frac{x}{\sqrt{3}} + \alpha_r x^r\right),
\]
where $\alpha_r = \left(\frac{2}{\pi}\right)^r\left(-\frac{\pi}{2\sqrt{3}} + \arccos\frac{2}{\pi}\right)$ and the constant $\beta$ such that inequality
\[
\frac{\sin x}{x} < \cos\left(\frac{x}{\sqrt{3}} + \beta x^r\right)
\]
holds doesn’t exist.

Sándor [8] gave a new proof using the monotonicity of functions and L’Hospital’s rule (see e.g. [4], [5]). In this paper, we show the monotonicity of functions using computations by Mathematica software and, like Sándor, prove the our main theorems using the monotonicity of functions and L’Hospital’s rule.

### 2. Proof of Theorem 1

We may show some lemmas required for the proof of Theorem 1.

**LEMMA 1.** For $0 < x < \frac{\pi}{2}$, we have
\[
\frac{2x}{\sqrt{3}} - 3\arccos\left(\frac{\sin x}{x}\right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} < 0.
\]
Proof. We set
\[ f(x) = \frac{2x}{\sqrt{3}} - 3 \arccos \left( \frac{\sin x}{x} \right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} \]
and the derivative of \( f(x) \) is
\[ f'(x) = \frac{2}{\sqrt{3}} + \frac{4x^3 \cos x - 5x^2 \sin x + x^4 \sin x - 2x \cos x \sin^2 x + 3 \sin^3 x}{x^4 \left( \frac{x^2 - \sin^2 x}{x^2} \right)^{\frac{3}{2}}} . \]
For \( 0 < x < \frac{\pi}{2} \) and non-negative integers \( m \) and \( n \), we have
\[ u(x, 2n + 1) < \sin x < u(x, 2n) \quad \text{and} \quad v(x, 2m + 1) < \cos x < v(x, 2m) , \]
where
\[ u(x, p) = \sum_{k=0}^{p} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad v(x, q) = \sum_{k=0}^{q} \frac{(-1)^k x^{2k}}{(2k)!} \]
are truncations of
\[ \sin x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k+1}}{(2k+1)!} \quad \text{and} \quad \cos x = \sum_{k=0}^{\infty} \frac{(-1)^k x^{2k}}{(2k)!} . \]
We have
\[ f'(x) < \frac{2}{\sqrt{3}} + \frac{4x^3 v(x, 4) - 5x^2 u(x, 3) + x^4 u(x, 4) - 2x v(x, 5) (u(x, 3))^2 + 3 (u(x, 4))^3}{x \left( x^2 - (u(x, 3))^2 \right)^{\frac{3}{2}}}
= g(x) \]
and the derivative of \( g(x) \) is
\[ g'(x) = \frac{x^3 h(x)}{622080(8467200 - 1128960x^2 + 80640x^4 - 3444x^6 + 84x^8 - x^{10})^{\frac{5}{2}}} . \]
Here, \( h(x) \) is negative as in Appendix A in the final part of this paper, therefore \( g(x) \) is strictly decreasing for \( 0 < x < \frac{\pi}{2} \). From \( \lim_{x \to 0} g(x) = \frac{2}{\sqrt{3}} - \frac{17698046607360000}{622080(8467200)^{\frac{5}{2}}} = 0 \), we obtain \( g(x) < 0 \) for \( 0 < x < \frac{\pi}{2} \) and \( f(x) \) is strictly decreasing for \( 0 < x < \frac{\pi}{2} \). From L’Hospital’s rule, we have
\[ \lim_{x \to 0} \sqrt{\frac{x^2 - \sin^2 x}{x^4}} = \lim_{x \to 0} \sqrt{\frac{x + \sin x}{x}} \cdot \sqrt{\frac{x - \sin x}{x^3}} = \sqrt{2} \cdot \frac{1}{\sqrt{6}} = \frac{1}{\sqrt{3}} . \]
and

\[
\lim_{x \to +0} \frac{x \cos x - \sin x}{x^2} = \lim_{x \to +0} \frac{-x \sin x}{2x} = 0.
\]

Hence, we can get

\[
\lim_{x \to +0} \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} = \lim_{x \to +0} \frac{\cos x - \frac{\sin x}{x}}{x \sqrt{\frac{x^2 - \sin^2 x}{x^4}}} = \lim_{x \to +0} \frac{1}{\sqrt{\frac{x^2 - \sin^2 x}{x^4}}} \frac{x \cos x - \sin x}{x^2} = \sqrt{3} - 0 = 0
\]

and \( \lim_{x \to +0} f(x) = 0 \) and \( f(x) < 0 \) for \( 0 < x < \frac{\pi}{2} \). \( \square \)

**Lemma 2.** We have

\[
\lim_{x \to +0} -\sqrt{3}x \cos x + \sqrt{3} \sin x - x \sqrt{x^2 - \sin^2 x} \cdot x^5 = -\frac{1}{30 \sqrt{3}}.
\]

**Proof.** Since we have \( u(x, 3) < \sin x < u(x, 2) \) and \( v(x, 3) < \cos x < v(x, 2) \) for \( 0 < x < \frac{\pi}{2} \), we have

\[
-\sqrt{3} x v(x, 2) + \sqrt{3} u(x, 3) - x \sqrt{x^2 - (u(x, 3))^2}
\]

\[
< -\sqrt{3} x \cos x + \sqrt{3} \sin x - x \sqrt{x^2 - \sin^2 x}
\]

\[
< -\sqrt{3} x v(x, 3) + \sqrt{3} u(x, 2) - x \sqrt{x^2 - (u(x, 2))^2}
\]

and

\[
\frac{1680 \sqrt{3} - 168 \sqrt{3} x^2 - \sqrt{3} x^4 - \sqrt{8467200 - 1128960 x^2 + 80640 x^4 - 3444 x^6 + 84 x^8 - x^{10}}}{5040 x^2}
\]

\[
< \frac{-\sqrt{3} x \cos x + \sqrt{3} \sin x - x \sqrt{x^2 - \sin^2 x} \cdot x^5}{x^5}
\]

\[
< \frac{240 \sqrt{3} - 24 \sqrt{3} x^2 + \sqrt{3} x^4 - 6 \sqrt{4800 - 640 x^2 + 40 x^4 - x^6}}{720 x^2}.
\]

Since we have

\[
\lim_{x \to +0} \frac{1680 \sqrt{3} - 168 \sqrt{3} x^2 - \sqrt{3} x^4 - \sqrt{8467200 - 1128960 x^2 + 80640 x^4 - 3444 x^6 + 84 x^8 - x^{10}}}{5040 x^2}
\]

\[
= \lim_{x \to +0} \frac{-336 \sqrt{3} x + 4 \sqrt{3} x^3 - \frac{-2257920 x + 322560 x^3 - 20664 x^5 + 672 x^7 - 10 x^9}{2 \sqrt{8467200 - 1128960 x^2 + 80640 x^4 - 3444 x^6 + 84 x^8 - x^{10}}}}{10080 x}
\]

\[
= \frac{-336 \sqrt{3} - \frac{-2257920}{2 \sqrt{8467200}}}{10080} = -\frac{1}{30 \sqrt{3}}
\]
and
\[
\lim_{x \to 0} \frac{240 \sqrt{3} - 24 \sqrt{3} x^2 + \sqrt{3} x^4 - 6 \sqrt{4800 - 640 x^2 + 40 x^4 - x^6}}{720 x^2} = \lim_{x \to 0} \frac{-48 \sqrt{3} x + 4 \sqrt{3} x^3 - \frac{3(-1280 x + 160 x^3 - 6 x^5)}{\sqrt{4800 - 640 x^2 + 40 x^4 - x^6}}}{1440 x} = \lim_{x \to 0} \frac{-48 \sqrt{3} - \frac{-3 \cdot 1280}{\sqrt{4800}}}{1440} = -\frac{1}{30 \sqrt{3}},
\]
we can get
\[
\lim_{x \to 0} \frac{-\sqrt{3} \cos x + \sqrt{3} \sin x - x \sqrt{x^2 - \sin^2 x}}{x^5} = -\frac{1}{30 \sqrt{3}}. \quad \Box
\]

**Lemma 3.** We have
\[
\lim_{x \to 0} \left( \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^3} - \frac{1}{\sqrt{3} x^2} \right) = -\frac{1}{90 \sqrt{3}}.
\]

**Proof.** From Lemma 2 and L’Hospital’s rule, we have
\[
\lim_{x \to 0} \left( \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^3} - \frac{1}{\sqrt{3} x^2} \right) = \lim_{x \to 0} \frac{\sqrt{3} \arccos \left( \frac{\sin x}{x} \right) - x}{\sqrt{3} x^3} = \lim_{x \to 0} \frac{-1 - \frac{\sqrt{3} (\cos x - \sin x)}{x^2}}{3 \sqrt{3} x^2} = \lim_{x \to 0} \frac{-\sqrt{3} \cos x + \sqrt{3} \sin x - x \sqrt{x^2 - \sin^2 x}}{3 \sqrt{3} x^5 \sqrt{\frac{x^2 - \sin^2 x}{x^4}}} = \lim_{x \to 0} \frac{1}{3 \sqrt{3} \sqrt{\frac{x^2 - \sin^2 x}{x^4}}} = \frac{1}{3} \left( -\frac{1}{30 \sqrt{3}} \right) = -\frac{1}{90 \sqrt{3}}. \quad \Box
\]

**Proof of Theorem 1.** We consider the equation
\[
\frac{x}{\sqrt{3}} + ax^3 = \arccos \left( \frac{\sin x}{x} \right)
\]
and we have
\[
a = \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^3} - \frac{1}{\sqrt{3} x^2} = f(x).
\]

The derivative of \( f(x) \) is
\[
f'(x) = \frac{1}{x^4} \left( \frac{2x}{\sqrt{3}} - 3 \arccos \left( \frac{\sin x}{x} \right) - \frac{\cos x \cdot \frac{\sin x}{x} - \arccos \left( \frac{\sin x}{x} \right) \cdot \sqrt{1 - \frac{\sin^2 x}{x^2}}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} \right).
\]
By Lemma 1, we have \( f'(x) < 0 \) for \( 0 < x < \frac{\pi}{2} \). Since \( f(x) \) is strictly decreasing for \( 0 < x < \frac{\pi}{2} \) and by Lemma 3, we have

\[
\lim_{x \to -\frac{\pi}{2}} f(x) < f(x) < \lim_{x \to +0} f(x)
\]

and

\[
\frac{8}{\pi^3} \left(-\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi}\right) < f(x) < -\frac{1}{90\sqrt{3}}
\]

for \( 0 < x < \frac{\pi}{2} \). Hence, the proof of Theorem 1 is complete. \( \square \)

3. Proof of Theorems 2 and 3

We may show some lemmas required for the proof of Theorems 2 and 3.

**Lemma 4.** For \( 0 < x < \frac{\pi}{2} \), we have

\[
\frac{3x}{\sqrt{3}} - 4 \arccos \left(\frac{\sin x}{x}\right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}} > 0.
\]

**Proof.** We set

\[
f(x) = \frac{3x}{\sqrt{3}} - 4 \arccos \left(\frac{\sin x}{x}\right) - \frac{\cos x - \frac{\sin x}{x}}{\sqrt{1 - \frac{\sin^2 x}{x^2}}}
\]

and the derivative of \( f(x) \) is

\[
f'(x) = \sqrt{3} + \frac{5x^3 \cos x - 6x^2 \sin x + x^4 \sin x - 3x \cos x \sin^2 x + 4 \sin^3 x}{x^4 \left(\frac{x^2 - \sin^2 x}{x^2}\right)^{\frac{3}{2}}}
\]

\[
> \sqrt{3} + \frac{5x^3 v(x, 3) - 6x^2 u(x, 4) + x^4 u(x, 3) - 3x v(x, 4)(u(x, 2))^2 + 4(u(x, 3))^3}{x^4 \left(x^2 - \frac{(u(x, 2))^2}{x^2}\right)^{\frac{3}{2}}}
\]

\[
= j(x)
\]

and the derivative of \( j(x) \) is

\[
j'(x) = \frac{xk(x)}{148176(4800 - 640x^2 + 40x^4 - x^6)^{\frac{3}{2}}}.
\]

Here, \( k(x) \) is positive as in Appendix A in the final part of this paper, therefore \( j(x) \) is strictly increasing for \( 0 < x < \frac{\pi}{2} \). From \( \lim_{x \to +0} j(x) = \sqrt{3} - \frac{8534937600}{148176(4800)^{\frac{3}{2}}} = 0 \), we
obtain \( j(x) > 0 \) for \( 0 < x < \frac{\pi}{2} \) and \( f(x) \) is strictly increasing for \( 0 < x < \frac{\pi}{2} \). From \( \lim_{x \to +0} \frac{\cos x - \sin x}{\sqrt{1 - \sin^2 x}} = 0 \) (see the proof of Lemma 1), we can get \( \lim_{x \to +0} f(x) = 0 \) and \( f(x) > 0 \) for \( 0 < x < \frac{\pi}{2} \). □

**Lemma 5.** We have

\[
\lim_{x \to +0} \left( \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}} \right) = 0
\]

for \( 0 < r < 3 \).

**Proof.** We set

\[
f(x, r) = \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}}.
\]

From \( \cos \left( \frac{x}{\sqrt{3}} \right) < \frac{\sin x}{x} \) for \( 0 < x < \frac{\pi}{2} \), we have \( \arccos \left( \frac{\sin x}{x} \right) < \frac{x}{\sqrt{3}} \) and

\[
f(x, r) = \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}} < \frac{x}{\sqrt{3} x^r} - \frac{x^{1-r}}{\sqrt{3}} = 0
\]

for \( 0 < x < \frac{\pi}{2} \) and \( r > 0 \). Also, by Theorem 1, we have

\[
\frac{8}{\pi^3} \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right) < \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^3} - \frac{x^{-2}}{\sqrt{3}}
\]

and

\[
x^{3-r} \cdot \frac{8}{\pi^3} \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right) < f(x, r)
\]

for \( 0 < x < \frac{\pi}{2} \) and \( 0 < r < 3 \). From \( \lim_{x \to +0} \frac{8x^{3-r}}{\pi^3} \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right) = 0 \) for \( 0 < r < 3 \), we can get \( \lim_{x \to +0} f(x, r) = 0 \) for \( 0 < x < \frac{\pi}{2} \) and \( 0 < r < 3 \). □

**Lemma 6.** We have

\[
\lim_{x \to +0} \left( \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}} \right) = -\infty
\]

for \( r > 4 \).

**Proof.** We set

\[
f(x, r) = \frac{\arccos \left( \frac{\sin x}{x} \right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}}
\]
and the derivative of $f(x,r)$ by $r$ is
\[
\frac{\partial f(x,r)}{\partial r} = \frac{x^{-r} \left(x - \sqrt{3} \arccos \left(\frac{\sin x}{x}\right)\right) \ln x}{\sqrt{3}}.
\]

From $\cos \left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$, we have $\arccos \left(\frac{\sin x}{x}\right) < \frac{x}{\sqrt{3}}$ and $\frac{\partial f(x,r)}{\partial r} < 0$ for $0 < x < 1$ and $r > 0$. Therefore, for $0 < x < 1$, $\frac{\partial f(x,r)}{\partial r} < 0$ and $f(x,r)$ is strictly decreasing for $r > 0$. Hence, we have
\[
f(x,r) < f(x,4) = \frac{\arccos \left(\frac{\sin x}{x}\right)}{4} - \frac{x^{-3}}{\sqrt{3}}
\]
for $0 < x < 1$ and $r > 4$. By Theorem 1, we have
\[
\frac{\arccos \left(\frac{\sin x}{x}\right)}{x^3} - \frac{x^{-2}}{\sqrt{3}} < -\frac{1}{90\sqrt{3}} \quad \text{and} \quad f(x,r) < f(x,4) < -\frac{1}{90\sqrt{3}x}.
\]
From $\lim_{x \to +0} -\frac{1}{90\sqrt{3}x} = -\infty$, we obtain $\lim_{x \to +0} f(x,r) = -\infty$ for $r > 4$. \(
\)
We consider the equation
\[
\frac{x}{\sqrt{3}} + ax' = \arccos \left(\frac{\sin x}{x}\right)
\]
and we have
\[
a = \frac{\arccos \left(\frac{\sin x}{x}\right)}{x^r} - \frac{x^{1-r}}{\sqrt{3}} = f(x,r).
\]
The derivative of $f(x,r)$ by $x$ is
\[
\frac{\partial f(x,r)}{\partial x} = x^{-1-r} \left(\frac{x(r-1)}{\sqrt{3}} - r \arccos \left(\frac{\sin x}{x}\right) - \frac{\cos x - x^{-1} \sin x}{\sqrt{1 - \frac{\sin^2 x}{x^2}}}\right) = x^{-1-r} g(x,r)
\]
and the derivative of $g(x,r)$ by $r$ is
\[
\frac{\partial g(x,r)}{\partial r} = \frac{x}{\sqrt{3}} - \arccos \left(\frac{\sin x}{x}\right).
\]
From $\cos \left(\frac{x}{\sqrt{3}}\right) < \frac{\sin x}{x}$ for $0 < x < \frac{\pi}{2}$, we have $\frac{\partial g(x,r)}{\partial r} > 0$ for $0 < x < \frac{\pi}{2}$ and $r > 0$. Hence, $g(x,r)$ is strictly increasing for $r > 0$.

**Proof of Theorem 2.** By Lemma 1, we have $g(x,r) < g(x,3) < 0$ for $0 < x < \frac{\pi}{2}$ and $0 < r < 3$. Thus, $f(x,r)$ is strictly decreasing for $0 < x < \frac{\pi}{2}$ and we have
\[
\lim_{x \to +\frac{\pi}{2}} f(x,r) < f(x,r) < \lim_{x \to +0} f(x,r)
\]
for \(0 < x < \frac{\pi}{2}\) and \(0 < r < 3\). By Lemma 5, we can get
\[
\left( \frac{2}{\pi} \right)^r \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right) < f(x,r) < 0
\]
for \(0 < x < \frac{\pi}{2}\) and \(0 < r < 3\). Hence, the proof of Theorem 2 is complete. \(\square\)

**Proof of Theorem 3.** By Lemma 4, we have \(g(x,r) > g(x,4) > 0\) for \(0 < x < \frac{\pi}{2}\) and \(r > 4\). Since \(f(x,r)\) is strictly increasing for \(0 < x < \frac{\pi}{2}\), we have
\[
\lim_{x \to +0} f(x,r) < f(x,r) < \lim_{x \to -\frac{\pi}{2}} f(x,r)
\]
for \(0 < x < \frac{\pi}{2}\) and \(r > 4\). By Lemma 6, we can get
\[
-\infty < f(x,r) < \left( \frac{2}{\pi} \right)^r \left( -\frac{\pi}{2\sqrt{3}} + \arccos \frac{2}{\pi} \right).
\]
for \(0 < x < \frac{\pi}{2}\) and \(r > 4\). Hence, the proof of Theorem 3 is complete. \(\square\)

4. Appendix A

Computations in this paper were made using Mathematica software.

\[
g(x) = \frac{2}{\sqrt{3}} + \frac{1}{622080(8467200 - 1128960x^2 + 80640x^4 - 3444x^6 + 84x^8 - x^{10})^{\frac{3}{2}}}
\times \left( -1769804660736000 + 3539609321472000x^2 - 389910089318400x^4
+ 2853107712000x^6 - 1412654100480x^8 + 48718817280x^{10}
- 1159401600x^{12} + 18418752x^{14} - 177552x^{16} + 648x^{18} + 5x^{20} \right).
\]

\[
h(x) = -646686623032934400000 + 199357876203945984000x^2
- 26263901165322240000x^4 + 2146275295911936000x^6
- 127829057745715200x^8 + 6144522998906880x^{10}
- 250109274562560x^{12} + 8492502412800x^{14} - 226881527040x^{16}
+ 4427664192x^{18} - 55679328x^{20} + 314724x^{22} + 1416x^{24} - 25x^{26}
< -646686623032934400000 + 199357876203945984000 \left( \frac{158}{100} \right)^2
- 26263901165322240000x^4 + 2146275295911936000x^6 \left( \frac{158}{100} \right)^2
- 127829057745715200x^8 + 6144522998906880x^8 \left( \frac{158}{100} \right)^2
\(-250109274562560x^{12} + 8492502412800x^{12}\left(\frac{158}{100}\right)^{2} - 226881527040x^{16}\)

\(+4427664192x^{16}\left(\frac{158}{100}\right)^{2} - 55679328x^{20} + 314724x^{20}\left(\frac{158}{100}\right)^{2}\)

\(+1416x^{20}\left(\frac{158}{100}\right)^{4} - 25x^{26}\)

\(= \frac{1}{781250}\left(-116413766310471598080000000 - 16332765247349752320000000x^{8} - 1788348371400360000000x^{12} - 168615864180540000x^{16}\right) < 0.\)

\(j(x) = \sqrt{3} + \frac{1}{148176(4800 - 640x^{2} + 40x^{4} - x^{6})^{\frac{3}{2}}} \times \left(-85349376000 + 17713382400x^{2} - 1844579520x^{4} + 106686720x^{6} - 3625776x^{8} + 64512x^{10} - 315x^{12} - 8x^{14}\right).\)

\(k(x) = 6177669120000 - 3595511808000x^{2} + 6508228608000x^{4} - 72508262400x^{6} + 54761616000x^{8} - 277159680x^{10} + 7972944x^{12} - 83792x^{14} - 1615x^{16} + 40x^{18}\)

\(> 6177669120000 - 3595511808000x^{2} + 6508228608000x^{4}\)

\(-72508262400x^{4}\left(\frac{158}{100}\right)^{2} + 54761616000x^{8} - 277159680x^{10}\left(\frac{158}{100}\right)^{2}\)

\(+7972944x^{12} - 83792x^{12}\left(\frac{158}{100}\right)^{4} - 1615x^{12}\left(\frac{158}{100}\right)^{4} + 40x^{18}\)

\(= 6177669120000 - 3595511808000x^{2} + \frac{11745330863616x^{4}}{25} + \frac{598032521856x^{8}}{125} + \frac{9692126187837x^{12}}{1250000} + 40x^{18}\)

\(> 6177669120000 - 3595511808000x^{2} + \frac{11745330863616x^{4}}{25}\)

\(= -\frac{9460645253283840000}{13486243} + \frac{11745330863616}{25}\left(x^{2} - \frac{464450000}{121376187}\right)^{2}\)

\(= -\frac{9460645253283840000}{13486243} + \frac{11745330863616}{25}\left(\frac{158}{100} - \frac{464450000}{121376187}\right)^{2}\)

\(= \frac{1266753993852697056}{9765625} > 0.\)
REFERENCES


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