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COMPLETE CONVERGENCE AND COMPLETE MOMENT CONVERGENCE FOR WEIGHTED SUMS OF m–EXTENDED NEGATIVELY DEPENDENT RANDOM VARIABLES

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Abstract. The authors study the complete convergence and complete moment convergence for weighted sums of m-extended negatively dependent (m-END) random variables. The results obtained in this paper extend and improve the corresponding results of Wu, Zhai and Peng [Y. F. Wu, M. Q. Zhai and J. Y. Peng, On the complete convergence for weighted sums of extended negatively dependent random variables, Journal of Mathematical Inequalities, 13 (1) (2019), 251–260] and Zarei and Jabbari [H. Zarei and H. Jabbari, Complete convergence of weighted sums under negative dependence, Statistical Papers, 52 (2011), 413–418].

1. Introduction

It is known that many elegant limit theorems were extended from independent random variables to dependent random variables since the independent assumption is rigorous in many realistic applications. Firstly, we will recall some concepts of dependent random variables. The first one is the concept of negatively associated (NA) random variables, which was studied in Joag-Dev and Proschan [1] as follows.

DEFINITION 1.1. The random variables $X_1, X_2, \ldots, X_n$ are said to be NA if for every pair of disjoint subsets A and B of $\{1, 2, \cdots, n\}$ and any real coordinatewise nondecreasing (or nonincreasing) functions f_1 on $\mathbb{R}^{|A|}$ and f_2 on $\mathbb{R}^{|B|}$,

$$Cov(f_1(X_i, i \in A), f_2(X_j, j \in B)) \leqslant 0,$$

whenever the covariance above exists, where $|A|$ and $|B|$ stand for the cardinalities of A and B, respectively.

The concept of negatively orthant dependent (NOD) random variables was introduced by Ebrahimi and Ghosh [2] as follows.

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DEFINITION 1.2. The random variables $X_1, X_2, \cdots, X_n$ are said to be negatively upper orthant dependent (NUOD) if for all real numbers $x_1, x_2, \cdots, x_n$,

$$P(X_i > x_i, i = 1, 2, \cdots, n) \leqslant \prod_{i=1}^{n} P(X_i > x_i),$$

and negatively lower orthant dependent (NLOD) if

$$P(X_i \leqslant x_i, i = 1, 2, \cdots, n) \leqslant \prod_{i=1}^{n} P(X_i \leqslant x_i).$$

Random variables $X_1, X_2, \ldots, X_n$ are said to be NOD if they are both NUOD and NLOD.

Joag-Dev and Proschan [1] mentioned that the NA structure must be NOD, but NOD is not necessarily NA. Liu [3] extended the NOD structure to extended negatively dependent (END) structure. The concept of END random variables is stated as follows.

DEFINITION 1.3. The random variables $X_1, X_2, \cdots, X_n$ are said to be END if for all real numbers $x_1, x_2, \cdots, x_n$, there exists a constant $M > 0$ such that both

$$P(X_i > x_i, i = 1, 2, \cdots, n) \leqslant M \prod_{i=1}^{n} P(X_i > x_i),$$

and

$$P(X_i \leqslant x_i, i = 1, 2, \cdots, n) \leqslant M \prod_{i=1}^{n} P(X_i \leqslant x_i)$$

hold.

Liu [3] gave an example of END random variables according to the multivariate copula function. She also pointed out that the END random variables can be taken as negatively or positively dependent random variables. Further, taking $M = 1$, then END is NOD. Since NA is NOD, it is also END. Hence, END contains independent, NA, NOD and some other dependent structures and it is more interesting to investigate the limit theorems for END random variables. There have already some scholars studying the limit behaviors for END random variables. For example, Chen et al. [4] investigated the strong law of large numbers for END random variables, Shen [5] established some probability inequalities for END random variables, Wu and Guan [6] studied some convergence properties for the partial sums of END random variables, Wang and Wang [7] investigated a general precise large deviation result for random sums of END real-valued random variables in the presence of consistent variation, Wang et al. [8] studied the complete consistency for the estimator of nonparametric regression models based on END errors, Shen and Volodin [9] obtained the L_r convergence, weak and strong laws of large numbers for END random variables, Yi et al. [10] got the complete moment convergence for weighted sums of END random variables, Wang et al. [11] investigated the mean consistency, weak consistency, strong consistency, complete consistency and strong convergence rate of the wavelet estimator in nonparametric regression model with repeated measurements based on END errors, and so on.

Wang et al. [12] introduced the following concept of m-extended negatively dependent (m-END) random variables.

DEFINITION 1.4. Let $m \geqslant 1$ be a fixed integer. A sequence $\{X_n, n \geqslant 1\}$ of random variables is said to be m-END if for any $n \geqslant 2$ and any $i_1, i_2, \cdots, i_n$ such that $|i_k - i_j| \geqslant m$ for all $1 \leqslant k \neq j \leqslant n$, we have that $X_{i_1}, X_{i_2}, \cdots, X_{i_n}$ are END.

Obviously, END is m-END with $m = 1$, and that is to say, the m-END structure is a natural extension of END structure. Therefore, it is of general interest to investigate the limit behaviors for m-END random variables. Since the concept of m-END random variables was introduced, some related results have already been established. For example, Xu et al. [13] studied the mean consistency of the weighted estimator in a nonparametric regression model based on m-END random errors, Wang et al. [14] obtained the complete and complete moment convergence for partial sums of m-END random variables and gave their applications to the EV regression model, Wu and Wang [15] studied the complete convergence and strong law of large numbers for weighted sums of m-END random variables and gave the applications to multiple linear regression models, conditional value-at-risk estimator and the quasi-renewal counting process.

The concept of complete convergence was first introduced by Hsu and Robbins [16] as follows:

DEFINITION 1.5. A sequence of random variables $\{U_n, n \geqslant 1\}$ is said to converge completely to a constant θ if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|U_n - \theta| > \varepsilon) < \infty.$$

By the Borel-Cantelli lemma, it is clear that the complete convergence result above implies $U_n \to \theta$ almost surely. Hence, the complete convergence is an important tool in studying some strong convergence of partial sums or weighted sums of random variables.

Chow [17] introduced the concept of complete moment convergence as follows, which is much stronger than complete convergence.

DEFINITION 1.6. Let $\{U_n, n \geqslant 1\}$ be a sequence of random variables and $a_n > 0, b_n > 0$. If for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} a_n E(b_n^{-1}|U_n| - \varepsilon)^+ < \infty,$$

then the result above is defined as complete moment convergence.

The motivation of this paper originates from a result on complete convergence for weighted sums of NOD random variables, which was obtained by Zarei and Jabbari [18] as follows.

THEOREM A. *Let $\{X_n, n \geqslant 1\}$ be a sequence of NOD and identically distributed random variables with $EX_1 = 0$ and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying $A_n = \sum_{k=1}^{n} a_{nk}^2 \leqslant Cn^{-\alpha}$, $|a_{nk}| \leqslant CA_n$ for some $0 < C < \infty$ and $0 < \alpha < 1$. If $E|X_1|^{2/\alpha} < \infty$, then*

$$\sum_{n=1}^{\infty} P\left(\left|\sum_{k=1}^{n} a_{nk}X_k\right| \geqslant \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

Recently, Wu et al. [19] extended and improved Theorem A for NOD random variables to END random variables as follows.

THEOREM B. *Let $\{X_n, n \geqslant 1\}$ be a sequence of END and identically distributed random variables with $EX_1 = 0$ and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying*

$$\sum_{k=1}^{n} a_{nk}^2 = O(n^{-\alpha}) \tag{1.1}$$

and

$$\max_{1 \leqslant k \leqslant n} |a_{nk}| = O(n^{-\alpha}) \tag{1.2}$$

for some $1/p \leqslant \alpha < 1$ and $p \geqslant 2$. If $E|X_1|^p < \infty$, then

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1 \leqslant j \leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

In this work, the authors will further investigate the complete convergence for maximal weighted sums of m-END random variables. The result not only extends Theorem A and Theorem B from NOD random variables and END random variables respectively to m-END random variables, but also improves the conditions of Theorem A and Theorem B. Moreover, we also obtain the complete moment convergence for maximal weighted sums of m-END random variables, which is much stronger than complete convergence.

Throughout the current paper, C will be used to represent various positive constants, which may differ from one place to another. The symbol $I(A)$ will stand for the indicator function of A. Let $\log x = \ln\max(x, e)$. $a_n = O(b_n)$ implies that $\limsup_{n\to\infty} a_n/b_n < \infty$. $x^+ = \max\{x, 0\}$ and $x^- = \max\{-x, 0\}$.

2. Main results

Now we state our main results. The proofs will be presented in the next section.

THEOREM 2.1. *Let $p \geqslant 1$ and $\alpha p > 1$. Let $\{X_n, n \geqslant 1\}$ be a sequence of m-END and identically distributed random variables with $EX_1 = 0$ and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying*

$$\sum_{k=1}^{n} |a_{nk}|^q = O(n^{1-\alpha q}) \tag{2.1}$$

for some $q > p$. Assume further that

$$\sum_{k=1}^{n} a_{nk}^2 = O(n^{-\delta}) \tag{2.2}$$

for some $\delta > 0$ if $p \geqslant 2$. Then $E|X_1|^p < \infty$ implies

$$\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

REMARK 2.1. Comparing Theorem 2.1 with Theorem B, we can find the following improvements or extensions: First, Theorem B is extended for END random variables to m-END random variables; Second, $p \geqslant 2$ in Theorem B is improved to $p \geqslant 1$; Third, if (1.2) holds, then (2.1) also holds automatically, i.e., (2.1) is weaker than (1.2); Fourth, if $p \geqslant 2$, (1.1) is improved to (2.2). To sum up, Theorem 2.1 improves and extends Theorem B to m-END random variables.

REMARK 2.2. For $p \geqslant 2$, if $\alpha > 1/2$, we have by (2.1) and Hölder's inequality that

$$\sum_{k=1}^{n} a_{nk}^2 \leqslant \left(\sum_{k=1}^{n} |a_{nk}|^q\right)^{2/q} \cdot n^{1-2/q} = O(n^{1-2\alpha}).$$

That is to say, (2.2) holds with $\delta = 2\alpha - 1$.

If we take $p = 2/\alpha$, we can obtain the following conclusion.

COROLLARY 2.1. *Let $0 < \alpha \leqslant 2$. Let $\{X_n, n \geqslant 1\}$ be a sequence of m-END and identically distributed random variables with $EX_1 = 0$ and let $\{a_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ be an array of real numbers satisfying (2.1) for some $q > 2/\alpha$. Assume further (2.2) holds for some $\delta > 0$ if $\alpha \leqslant 1$. If $E|X_1|^{2/\alpha} < \infty$, then*

$$\sum_{n=1}^{\infty} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) < \infty \text{ for all } \varepsilon > 0.$$

REMARK 2.3. Since the corresponding conditions on the weights in Theorem A are improved to (2.1) and (2.2), $0 < \alpha < 1$ is improved to $0 < \alpha \leqslant 2$, and normal weighted sums are replaced by maximal weighted sums, Corollary 2.1 improves and extends Theorem A from NOD random variables to m-END random variables.

If we replace $p \geqslant 1$ in Theorem 2.1 by a little stronger condition $p > 1$, we have the following result of complete moment convergence.

THEOREM 2.2. *Assume that the conditions of Theorem 2.1 hold for some $p > 1$, then*

$$\sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| - \varepsilon\right)^{+} < \infty \text{ for all } \varepsilon > 0.$$

REMARK 2.4. It is easy to check that

$$\begin{aligned}
\infty &> \sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| - \varepsilon\right)^{+} \\
&\geqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \int_0^{\varepsilon} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| - \varepsilon > t\right) dt \\
&\geqslant \varepsilon \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| > 2\varepsilon\right).
\end{aligned}$$

That is to say, the result on complete moment convergence in Theorem 2.2 is much stronger than complete convergence. Hence, Theorem 2.2 improves and extends the corresponding results of Theorem A and Theorem B from complete convergence for NOD random variables and END random variables respectively to complete moment convergence for m-END random variables.

3. The proofs

LEMMA 3.1. (Wang et al. [12]) *Let $\{X_k, k\geqslant 1\}$ be a sequence of m-END random variables. If $f_k(\cdot), k\geqslant 1$ are all nondecreasing (or nonincreasing) functions, then $\{f_k(X_k), k\geqslant 1\}$ is still a sequence of m-END random variables.*

LEMMA 3.2. (Xu et al. [13]) *Let $\{X_k, k\geqslant 1\}$ be a sequence of m-END random variables with $EX_k = 0$ and $E|X_k|^s < \infty$ for all $k\geqslant 1$ with some $s\geqslant 2$. Then there exists a positive constant $C_{m,s}$ depending only on m and s such that*

$$E\left|\sum_{k=1}^{n} X_k\right|^s \leqslant C_{m,s}\left\{\sum_{k=1}^{n} E|X_k|^s + \left(\sum_{k=1}^{n} EX_k^2\right)^{s/2}\right\}.$$

LEMMA 3.3. (Wu and Wang [15]) *Let $\{X_k, k\geqslant 1\}$ be a sequence of m-END random variables with $EX_k = 0$ and $E|X_k|^s < \infty$ for all $k\geqslant 1$ and some $s\geqslant 2$. Then there exists a positive constant $C_{m,s}$ depending only on m and s such that*

$$E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} X_k\right|^s\right) \leqslant C_{m,s}(\log n)^s\left\{\sum_{k=1}^{n} E|X_k|^s + \left(\sum_{k=1}^{n} EX_k^2\right)^{s/2}\right\}.$$

LEMMA 3.4. (Sung [20]) *Let Y and Z be two random variables. Then for any $s>1$, $\varepsilon>0$ and $a>0$, we have*

$$E(|Y+Z| - \varepsilon a)^{+} \leqslant \left(\varepsilon^{-s} + (s-1)^{-1}\right) a^{1-s} E|Y|^s + E|Z|.$$

Now we give the proofs of the main results as follows.

Proof of Theorem 2.1. Denote for each $n \geqslant 1$, $1 \leqslant k \leqslant n$ that

$$\begin{aligned} Y_{nk} &= -I(a_{nk}X_k < -1) + a_{nk}X_k I(|a_{nk}X_k| \leqslant 1) + I(a_{nk}X_k > 1), \\ Z_{nk} &= (a_{nk}X_k + 1)I(a_{nk}X_k < -1) + (a_{nk}X_k - 1)I(a_{nk}X_k > 1). \end{aligned}$$

Then $a_{nk}X_k = Y_{nk} + Z_{nk}$, where $\{Y_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ and $\{Z_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ are both m-END by Lemma 3.1. Hence,

$$\begin{aligned} &\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| \geqslant \varepsilon\right) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|a_{nk}X_k| > 1) + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} Y_{nk}\right| \geqslant \varepsilon\right) \\ &=: I_1 + I_2. \end{aligned}$$

For I_1, we can easily get by (2.1) that

$$\begin{aligned} I_1 &= \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|a_{nk}X_k| > 1, |X_k| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|a_{nk}X_k| > 1, |X_k| \leqslant n^{\alpha}) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} P(|X_k| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|a_{nk}X_k|^q I(|X_k| \leqslant n^{\alpha}) \\ &\leqslant \sum_{n=1}^{\infty} n^{\alpha p-1} P(|X_1| > n^{\alpha}) + C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} E|X_1|^q I(|X_1| \leqslant n^{\alpha}) \\ &\leqslant CE|X_1|^p + C\sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} \sum_{i=1}^{n} E|X_1|^q I((i-1)^{\alpha} < |X_1| \leqslant i^{\alpha}) \\ &= CE|X_1|^p + C\sum_{i=1}^{\infty} E|X_1|^q I((i-1)^{\alpha} < |X_1| \leqslant i^{\alpha}) \sum_{n=i}^{\infty} n^{\alpha p-1-\alpha q} \\ &\leqslant CE|X_1|^p + C\sum_{i=1}^{\infty} i^{\alpha p-\alpha q} E|X_1|^q I((i-1)^{\alpha} < |X_1| \leqslant i^{\alpha}) \\ &\leqslant CE|X_1|^p < \infty. \end{aligned}$$

Now we prove $I_2 < \infty$. Take $t > 0$ such that $(p-1)t < \alpha p - 1$ and denote

$$\begin{aligned} \xi_{nk} &= -n^{-t} I(a_{nk}X_k < -n^{-t}) + a_{nk}X_k I(|a_{nk}X_k| \leqslant n^{-t}) + n^{-t} I(a_{nk}X_k > n^{-t}), \\ \eta_{nk} &= (a_{nk}X_k - n^{-t}) I(n^{-t} < a_{nk}X_k \leqslant 1) + (1 - n^{-t}) I(a_{nk}X_k > 1), \\ \gamma_{nk} &= (a_{nk}X_k + n^{-t}) I(-1 \leqslant a_{nk}X_k < -n^{-t}) + (-1 + n^{-t}) I(a_{nk}X_k < -1). \end{aligned}$$

Then $Y_{nk} = \xi_{nk} + \eta_{nk} + \gamma_{nk}$ with $\{\xi_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$, $\{\eta_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ and $\{\gamma_{nk}, 1 \leqslant k \leqslant n, n \geqslant 1\}$ being all m-END by Lemma 3.1. By (2.1) and Hölder's inequality we have that for all $0 < r < q$,

$$\sum_{k=1}^{n} |a_{nk}|^r \leqslant \left(\sum_{k=1}^{n} |a_{nk}|^q\right)^{r/q} \cdot n^{1-r/q} = O(n^{1-\alpha r}). \tag{3.1}$$

Hence, by $EX_1 = 0$ and (3.1) we have that

$$\begin{aligned}\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} E\xi_{nk}\right| &= \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} E(a_{nk}X_k - \xi_{nk})\right| \\ &\leqslant \sum_{k=1}^{n} E|a_{nk}X_k| I(|a_{nk}X_k| > n^{-t}) \\ &\leqslant n^{-t(1-p)}\sum_{k=1}^{n}|a_{nk}|^p E|X_1|^p \leqslant Cn^{t(p-1)+1-\alpha p}E|X_1|^p \to 0.\end{aligned} \tag{3.2}$$

Similar to (3.2), we also have

$$\begin{aligned}&\left|\sum_{k=1}^{n} E\eta_{nk}\right| = \sum_{k=1}^{n} E\eta_{nk} \\ &\leqslant \sum_{k=1}^{n} E[(a_{nk}X_k - n^{-t})I(n^{-t} < a_{nk}X_k \leqslant 1) + (a_{nk}X_k - n^{-t})I(a_{nk}X_k > 1)] \\ &= \sum_{k=1}^{n} E(a_{nk}X_k - n^{-t})I(a_{nk}X_k > n^{-t}) \\ &\leqslant \sum_{k=1}^{n} E|a_{nk}X_k| I(|a_{nk}X_k| > n^{-t}) \to 0,\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}\left|\sum_{k=1}^{n} E\gamma_{nk}\right| &= -\sum_{k=1}^{n} E\gamma_{nk} \\ &\leqslant -\sum_{k=1}^{n} E(a_{nk}X_k + n^{-t})I(a_{nk}X_k < -n^{-t}) \\ &\leqslant \sum_{k=1}^{n} E|a_{nk}X_k| I(|a_{nk}X_k| > n^{-t}) \to 0.\end{aligned} \tag{3.4}$$

Hence, by (3.2)–(3.4) we have that

$$\begin{aligned}I_2 \leqslant\ & \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\xi_{nk}\right| \geqslant \varepsilon/3\right) + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\eta_{nk}\right| \geqslant \varepsilon/3\right) \\ & + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\gamma_{nk}\right| \geqslant \varepsilon/3\right) \\ =\ & \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\xi_{nk}\right| \geqslant \varepsilon/3\right) + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{k=1}^{n}\eta_{nk}\right| \geqslant \varepsilon/3\right) \\ & + \sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{k=1}^{n}\gamma_{nk}\right| \geqslant \varepsilon/3\right)\end{aligned}$$

$$
\begin{aligned}
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})\right|\geqslant \varepsilon/6\right)\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{k=1}^{n}(\eta_{nk}-E\eta_{nk})\right|\geqslant \varepsilon/6\right)\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2} P\left(\left|\sum_{k=1}^{n}(\gamma_{nk}-E\gamma_{nk})\right|\geqslant \varepsilon/6\right)\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})\right|^{s}\right)+C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left|\sum_{k=1}^{n}(\eta_{nk}-E\eta_{nk})\right|^{s}\\
&\quad +C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left|\sum_{k=1}^{n}(\gamma_{nk}-E\gamma_{nk})\right|^{s}\\
&=: I_{21}+I_{22}+I_{23}.
\end{aligned}
$$

Take $s>\max\{2,q,\frac{2\alpha p-2}{\delta}\}$. By Markov inequality, Lemma 3.3, and (3.1) we have that if $1\leqslant p<2$,

$$
\begin{aligned}
I_{21} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^{s}\left\{\sum_{k=1}^{n}E|\xi_{nk}|^{s}+\left(\sum_{k=1}^{n}E\xi_{nk}^{2}\right)^{s/2}\right\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^{s}\left\{n^{-t(s-p)}\sum_{k=1}^{n}|a_{nk}|^{p}E|X_1|^{p}+\left(n^{-t(2-p)}\sum_{k=1}^{n}|a_{nk}|^{p}E|X_1|^{p}\right)^{s/2}\right\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-t(s-p)}(\log n)^{s}+C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^{s}\left(\sum_{k=1}^{n}|a_{nk}|^{p}E|X_1|^{p}\right)^{s/2}\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-t(s-p)}(\log n)^{s}+C\sum_{n=1}^{\infty} n^{-1+(\alpha p-1)(1-s/2)}(\log n)^{s}\\
&<\infty,
\end{aligned}
$$

and if $p\geqslant 2$,

$$
\begin{aligned}
I_{21} &\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^{s}\left\{\sum_{k=1}^{n}E|\xi_{nk}|^{s}+\left(\sum_{k=1}^{n}E\xi_{nk}^{2}\right)^{s/2}\right\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2}(\log n)^{s}\left\{n^{-t(s-p)}\sum_{k=1}^{n}|a_{nk}|^{p}E|X_1|^{p}+\left(\sum_{k=1}^{n}a_{nk}^{2}EX_1^{2}\right)^{s/2}\right\}\\
&\leqslant C\sum_{n=1}^{\infty} n^{-1-t(s-p)}(\log n)^{s}+C\sum_{n=1}^{\infty} n^{\alpha p-2-\delta s/2}(\log n)^{s}\\
&<\infty.
\end{aligned}
$$

Now we prove $I_{22}<\infty$. Observe that $|\eta_{nk}|\leqslant |a_{nk}X_k|I(|a_{nk}X_k|\leqslant 1)+I(|a_{nk}X_k|>1)$. Hence, by Lemma 3.2 we have that for $s>\max\{2,q,\frac{2\alpha p-2}{\delta}\}$,

$$\begin{aligned}
I_{22}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}\left\{\sum_{k=1}^{n}E|\eta_{nk}|^s+\left(\sum_{k=1}^{n}E\eta_{nk}^2\right)^{s/2}\right\}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{k=1}^{n}E|a_{nk}X_k|^sI(|a_{nk}X_k|\leqslant 1)+C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{k=1}^{n}P(|a_{nk}X_k|>1)\\
&\quad+C\sum_{n=1}^{\infty}n^{\alpha p-2}\left(\sum_{k=1}^{n}E\eta_{nk}^2\right)^{s/2}\\
&=: I_{221}+I_{222}+I_{223}.
\end{aligned}$$

According to $I_1<\infty$, we have $I_{222}<\infty$. Furthermore, similar to the proof of $I_1<\infty$, we also have

$$\begin{aligned}
I_{221}&=C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{k=1}^{n}E|a_{nk}X_k|^sI(|a_{nk}X_k|\leqslant 1,|X_k|>n^{\alpha})\\
&\quad+C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{k=1}^{n}E|a_{nk}X_k|^sI(|a_{nk}X_k|\leqslant 1,|X_k|\leqslant n^{\alpha})\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{k=1}^{n}P(|X_k|>n^{\alpha})+C\sum_{n=1}^{\infty}n^{\alpha p-2}\sum_{k=1}^{n}E|a_{nk}X_k|^qI(|X_k|\leqslant n^{\alpha})\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-1}P(|X_1|>n^{\alpha})+C\sum_{n=1}^{\infty}n^{\alpha p-1-\alpha q}E|X_1|^qI(|X_1|\leqslant n^{\alpha})\\
&\leqslant CE|X_1|^p<\infty.
\end{aligned}$$

For I_{223}, we also obtain by $|\eta_{nk}|\leqslant\min\{|a_{nk}X_k|,1\}$ that if $1\leqslant p<2$,

$$\begin{aligned}
I_{223}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}\left(\sum_{k=1}^{n}|a_{nk}|^pE|X_k|^p\right)^{s/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{-1+(\alpha p-1)(1-s/2)}<\infty,
\end{aligned}$$

and if $p\geqslant 2$,

$$\begin{aligned}
I_{223}&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2}\left(\sum_{k=1}^{n}a_{nk}^2EX_k^2\right)^{s/2}\\
&\leqslant C\sum_{n=1}^{\infty}n^{\alpha p-2-\delta s/2}<\infty
\end{aligned}$$

according to $s>\max\{2,q,\frac{2\alpha p-2}{\delta}\}$. Hence, we have proved $I_{22}<\infty$. Similar to the proof of $I_{22}<\infty$, we can also obtain $I_{23}<\infty$. The proof is completed. $\square$

Proof of Theorem 2.2. Observe that $a_{nk}X_k = \xi_{nk} + \eta_{nk} + \gamma_{nk} + Z_{nk}$, where ξ_{nk}, η_{nk}, γ_{nk} and Z_{nk} are defined in the proof of Theorem 2.1. Hence, we have by (3.2)–(3.4) that for all n sufficiently large,

$$
\begin{aligned}
\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| &\leqslant \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\xi_{nk}\right| + \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\eta_{nk}\right| \\
&\quad + \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\gamma_{nk}\right| + \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}Z_{nk}\right| \\
&= \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}\xi_{nk}\right| + \left|\sum_{k=1}^{n}\eta_{nk}\right| + \left|\sum_{k=1}^{n}\gamma_{nk}\right| + \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}Z_{nk}\right| \qquad (3.5) \\
&\leqslant \max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})\right| + \left|\sum_{k=1}^{n}(\eta_{nk}-E\eta_{nk})\right| \\
&\quad + \left|\sum_{k=1}^{n}(\gamma_{nk}-E\gamma_{nk})\right| + \sum_{k=1}^{n}|Z_{nk}| + \frac{\varepsilon}{2}.
\end{aligned}
$$

Hence, using Lemma 3.4 with $Y = \max_{1\leqslant j\leqslant n}|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})| + |\sum_{k=1}^{n}(\eta_{nk}-E\eta_{nk})| + |\sum_{k=1}^{n}(\gamma_{nk}-E\gamma_{nk})|$, $Z = \sum_{k=1}^{n}|Z_{nk}|$ and $a = 1/2$, we have by (3.5) and C_r inequality that for $s > \max\{2, q, \frac{2\alpha p-2}{\delta}\}$,

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j} a_{nk}X_k\right| - \varepsilon\right)^{+} \\
&\leqslant C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left(\max_{1\leqslant j\leqslant n}\left|\sum_{k=1}^{j}(\xi_{nk}-E\xi_{nk})\right|^{s}\right) \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left|\sum_{k=1}^{n}(\eta_{nk}-E\eta_{nk})\right|^{s} \\
&\quad + C\sum_{n=1}^{\infty} n^{\alpha p-2} E\left|\sum_{k=1}^{n}(\gamma_{nk}-E\gamma_{nk})\right|^{s} + C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{k=1}^{n} E|Z_{nk}| \\
&= C(I_{21}+I_{22}+I_{23}) + C\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{k=1}^{n} E|Z_{nk}|,
\end{aligned}
$$

where I_{21}, I_{22} and I_{23} are the same as those in the proof of Theorem 2.1. By the proof of Theorem 2.1, we have that $I_{21}+I_{22}+I_{23} < \infty$. Now we only need to show

$$
\sum_{n=1}^{\infty} n^{\alpha p-2}\sum_{k=1}^{n} E|Z_{nk}| < \infty. \qquad (3.6)
$$

Actually, by $|Z_{nk}| \leqslant |a_{nk}X_k|I(|a_{nk}X_k| > 1)$ and (3.1) we have that

$$
\begin{aligned}
&\sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|Z_{nk}| \leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|a_{nk}X_k|I(|a_{nk}X_k| > 1) \\
&= \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|a_{nk}X_k|I(|a_{nk}X_k| > 1, |X_k| > n^{\alpha}) \\
&\quad + \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|a_{nk}X_k|I(|a_{nk}X_k| > 1, |X_k| \leqslant n^{\alpha}) \\
&\leqslant \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|a_{nk}X_k|I(|X_k| > n^{\alpha}) + \sum_{n=1}^{\infty} n^{\alpha p-2} \sum_{k=1}^{n} E|a_{nk}X_k|^q I(|X_k| \leqslant n^{\alpha}) \\
&\leqslant C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha} E|X_1|I(|X_1| > n^{\alpha}) + C \sum_{n=1}^{\infty} n^{\alpha p-1-\alpha q} E|X_1|^q I(|X_1| \leqslant n^{\alpha}) \\
&= C \sum_{i=1}^{\infty} E|X_1|I(i^{\alpha} < |X_1| \leqslant (i+1)^{\alpha}) \sum_{n=1}^{i} n^{\alpha p-1-\alpha} \\
&\quad + C \sum_{i=1}^{\infty} E|X_1|^q I((i-1)^{\alpha} < |X_1| \leqslant i^{\alpha}) \sum_{n=i}^{\infty} n^{\alpha p-1-\alpha q} \\
&\leqslant C \sum_{i=1}^{\infty} i^{\alpha p-\alpha} E|X_1|I(i^{\alpha} < |X_1| \leqslant (i+1)^{\alpha}) \\
&\quad + C \sum_{i=1}^{\infty} i^{\alpha p-\alpha q} E|X_1|^q I((i-1)^{\alpha} < |X_1| \leqslant i^{\alpha}) \\
&\leqslant CE|X_1|^p < \infty.
\end{aligned}
$$

The proof is completed. $\square$

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