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NONEXISTENCE AND EXISTENCE OF POSITIVE GROUND STATE SOLUTIONS FOR GENERALIZED QUASILINEAR SCHRÖDINGER EQUATIONS

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Abstract. This paper is concerned with a class of generalized quasilinear Schrödinger equations which have appeared from plasma physics, as well as high-power ultrashort laser in matter. Combining the variable replacement and the Schauder-Tychonoff fixed point theorem, we establish the nonexistence and existence of positive radial ground state solutions for this problem.

1. Introduction and main results

In this paper, we consider the nonexistence and existence of positive radial ground state solutions for the generalized quasilinear Schrödinger equations

$$-\operatorname{div}(h^p(u)|\nabla u|^{p-2}\nabla u)+h^{p-1}(u)h'(u)|\nabla u|^p=f(x)|u|^{q-2}u+g(x)|u|^{m-2}u,\ \ x\in\mathbb{R}^N, \tag{1.1}$$

where $N\geqslant 3,\ 1<p<N,\ q>1,\ m>1,\ h\in C^1(\mathbb{R},\mathbb{R}^+)$ is an even function, $f(x),g(x)$ are positive radial continuous functions in $\mathbb{R}^N$.

In the case $p=2$, the study of (1.1) is related to standing wave solutions for quasilinear Schrödinger equations

$$i\partial_t\phi=-\Delta\phi+W(x)\phi-\widetilde{h}(|\phi|^2)\phi-\kappa\Delta(k(|\phi|^2))k'(|\phi|^2)\phi,\ x\in\mathbb{R}^N, \tag{1.2}$$

where $\phi: R\times\mathbb{R}^N\to\mathbb{C},\ W:\mathbb{R}^N\to\mathbb{R}$ is a given potential, $\kappa>0$ is a constant, $\widetilde{h}$ and k are real functions. Quasilinear equations such as (1.2) arise from more naturally mathematical physics and have been accepted as models of several physical phenomena corresponding to different types of k. The case $k(s)=s$, which was used, for instance, for the superfluid film equation in plasma physics by Kurihara in [11]. For the case $k(s)=(1+s)^{1/2}$, Ritchie [22] used (1.2) to model the self-channeling of a high-power

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ultrashort laser in matter. For more physical background and applications on (1.2), we refer the reader to [21, 18, 5] and their references.

Putting $\phi(t,x) = \exp(-iEt)u(x)$ in (1.2), where $E \in \mathbb{R}$ and u is a real function, (1.2) can be reduced to the following elliptic form

$$-\Delta u + V(x)u - \kappa\Delta(k(u^2))k'(u^2)u = \widetilde{h}(u^2)u, \quad x \in \mathbb{R}^N. \tag{1.3}$$

Particularly, take $h^p(u) = 1 + \frac{|(k(u^2))'|^p}{p}$, then (1.1) turns into

$$-\Delta_p u - \Delta_p(k(u^2))k'(u^2)\frac{2u}{p} = l(x,u), \quad x \in \mathbb{R}^N. \tag{1.4}$$

Clearly, when $p = 2$, equation (1.4) turns into (1.3) with $V(x) \equiv 0$, $\kappa = 1$. We note that equation (1.1) also arises in biological models and propagation of laser beams when $h(u)$ is a positive constant, see [10, 7]. If we set $h^p(u) = 1 + \frac{(2\alpha)^p}{p}|u|^{p(2\alpha-1)}$, i.e., $k(s) = s^\alpha$ with $\alpha > 1/2$, we get the superfluid film equation in plasma physics:

$$-\Delta_p u - \Delta_p(|u|^{2\alpha})\frac{2\alpha|u|^{2\alpha-2}u}{p} = l(x,u), \quad x \in \mathbb{R}^N. \tag{1.5}$$

If we let $h^p(u) = 1 + \frac{\alpha^p}{p}(1+u^2)^{(\frac{\alpha}{2}-1)p}|u|^p$, i.e., $k(s) = (1+s)^{\frac{\alpha}{2}}$ with $\alpha \geqslant 1$, then we conclude

$$-\Delta_p u - \left(\Delta_p(1+u^2)^{\alpha/2}\right)\frac{\alpha u}{p(1+u^2)^{(2-\alpha)/2}} = l(x,u), \quad x \in \mathbb{R}^N, \tag{1.6}$$

which models the self-channeling of a high-power ultrashort laser in matter.

Recently, the issues about existence and multiplicity of solutions for the above equations have been extensively studied by different mathematical methods such as constrained minimization argument, change of variables, Nehari method, perturbation method, iterative techniques, see [20, 8, 16, 2, 19, 13, 34, 14] and references cited therein. In particular, Liu et al. [17] firstly studied the soliton solutions for the form (1.3) with $k(s) = s$ by introducing a new method called dual approach. Deng and Huang [4] studied equation (1.3) for the case $k(s) = (1+s)^{\frac{1}{2}}$ with critical growth. By utilizing dual approach and an abstract result developed by Jeanjean in [9], they obtained the existence of positive ground state solutions. Li and Wang [12] investigated (1.6) with $p = 2$ and $l(x,u) = \alpha h(x)|u|^{q-1}u + \beta H(x)|u|^{m-1}u$. Relying on some special arguments and fixed point theorem, nonexistence criteria, existence of positive ground state solutions were established under suitable assumptions on q, m, h and H. Shen and Wang [27] considered the following generalized quasilinear Schrödinger equation

$$-\operatorname{div}(h^2(u)\nabla u) + h(u)h'(u)|\nabla u|^2 + V(x)u = a(u). \tag{1.7}$$

By introducing a new variable replacement and using minimax methods, the existence of positive solution for the equation (1.7) is established. Similar works can be found in [3, 6, 25, 26, 24, 23, 15, 28, 33] and reference therein.

On the other hand, with the help of a dual approach and some new iterative techniques, Zhang et al. in [32] studied the existence and nonexistence of entire blow-up solutions for the following quasilinear elliptic equation with non-square diffusion term

$$\begin{cases} \Delta_p u + \Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u = f(x)a(u), \\ u > 0 \text{ in } \mathbb{R}^N, \ \lim\limits_{|x|\to\infty} u(x) = \infty. \end{cases} \tag{1.8}$$

In [1], Chen studied the existence of multiple solutions to a class of quasilinear Schrödinger equation

$$-\Delta_p u - \Delta_p(|u|^{2\alpha})|u|^{2\alpha-2}u + V(x)|u|^{p-2}u = a(u). \tag{1.9}$$

By using symmetric mountain pass lemma, the result of infinitely many solutions to equation (1.9) is established. For more related work on quasilinear Schrödinger equations with p-Laplacian operator, we refer the reader to [30, 29, 31] and their references.

However, it is worth stressing that there are relatively few results on nonexistence and existence of positive solutions to quasilinear Schrödinger equations. The aim of this paper is to extend such results to generalized quasilinear Schrödinger equations. As far as we are aware, there are no results dealing with this subject except for the case $p = 2$ or the special $h(t)$ ([12, 27, 32]).

Motivated by the aforementioned works, we consider the nonexistence and existence of positive radial ground state solutions for (1.1).

A function $u : \mathbb{R}^N \to \mathbb{R}$ is called a ground state solution of (1.1) if $u \in C^1(\mathbb{R}^N)$ that satisfies $u \to 0$ as $|x| \to \infty$ and for all $\varphi \in C_0^1(\mathbb{R}^N)$ it holds

$$\begin{aligned} &\int_{\mathbb{R}^N} \left[h^{p-1}(u)h'(u)|\nabla u|^p\varphi + h^p(u)|\nabla u|^{p-2}\nabla u\nabla\varphi\right]dx \\ =&\int_{\mathbb{R}^N} \left(f(x)|u|^{q-2}u + g(x)|u|^{m-2}u\right)\varphi dx. \end{aligned} \tag{1.10}$$

Notice that we cannot apply directly variational methods since it is hard to construct a suitable space such that the energy functional or integral operator is well defined and belongs to C^1-class. To solve this obstacle, inspired by [17, 2], we introduce another change of variables due to [27] as

$$v = H(u) = \int_0^u h(t)dt. \tag{1.11}$$

Making the change of variables $v = H(u)$ or $u = H^{-1}(v)$, (1.1) can be transformed into

$$-\Delta_p v = \frac{f(x)|H^{-1}(v)|^{q-2}H^{-1}(v) + g(x)|H^{-1}(v)|^{m-2}H^{-1}(v)}{h(H^{-1}(v))}, \quad x \in \mathbb{R}^N. \tag{1.12}$$

Since $f(x)$ and $g(x)$ are radial functions, we deduce that equation (1.12) is equivalent to the problem

$$\begin{aligned} &-\left(r^{N-1}|v'|^{p-2}v'\right)' \\ =&\frac{r^{N-1}\left(f(r)|H^{-1}(v)|^{q-2}H^{-1}(v) + g(r)|H^{-1}(v)|^{m-2}H^{-1}(v)\right)}{h(H^{-1}(v))}, \quad r > 0. \end{aligned} \tag{1.13}$$

Using the forthcoming Lemma 2.1, it is easy to verify that $u(x)$ is the positive ground state solution of (1.1) iff $v(x)$ is the positive ground state solution of (1.12). Remember that equation (1.12) is equivalent to problem (1.13). Hence, the purpose of our article is to establish the nonexistence and existence of positive radial ground state solutions of (1.1) via (1.13).

In this article, we assume $h(t)$ satisfies the following

$(\mathbf{H_1})$ $h(t) \in C^1(\mathbb{R}, \mathbb{R}^+)$ is even and $h'(t) \geqslant 0$ for all $t \geqslant 0$.

$(\mathbf{H_2})$ There exists $\mu > 0$ such that $th'(t) \leqslant \mu h(t)$, for all $t > 0$.

We also suppose that $f(t)$ and $g(t)$ satisfy

$(\mathbf{A_1})$ $\displaystyle\int_a^{\infty} t^{p-1}\Big(t^{-\rho(q-p)}f(t) + t^{-\rho(m-p)}g(t)\Big)dt = \infty$, where $a > 0$ is some given constant, $\rho = \frac{N-p}{p-1}$.

$(\mathbf{A_2})$ $\displaystyle\int_0^{\infty}\Big(s^{1-N}\int_0^s t^{N-1}\big(f(t)+g(t)\big)dt\Big)^{\frac{1}{p-1}}ds = \infty.$

$(\mathbf{A_3})$ $\big(t^N f(t)\big)' \leqslant \dfrac{(N-p)q}{p(\mu+1)}t^{N-1}f(t)$, $\big(t^N g(t)\big)' \leqslant \dfrac{(N-p)m}{p(\mu+1)}t^{N-1}g(t)$, for all $t > 0$.

Now we may state our main results.

THEOREM 1.1. (Nonexistence) *Let $1 < p < N$ and $f(x), g(x)$ are positive radial continuous functions in $\mathbb{R}^N$. Assume that $(\mathbf{H_1})-(\mathbf{H_2})$ and $(\mathbf{A_1})$ hold. If $2 \leqslant p < \min\{q,m\}$. Then (1.1) has no positive radial ground state solutions.*

THEOREM 1.2. (Existence) *Let $1 < p < N$, $\min\{q,m\} > 1$ and $f(x), g(x)$ are positive radial continuous functions in $\mathbb{R}^N$. Assume that $(\mathbf{H_1})-(\mathbf{H_2})$ and $(\mathbf{A_2})-(\mathbf{A_3})$ hold. If $1 < p < \max\{q,m\}$. Then (1.1) has infinitely many positive radial ground state solutions.*

REMARK 1.3. If we let $h(t) = \sqrt[p]{1 + \frac{(2\alpha)^p}{p}|t|^{p(2\alpha-1)}}$ with $\alpha > 1/2$, then $h(t) \in C^1(\mathbb{R}, \mathbb{R}^+)$ is an even function and satisfies $(\mathbf{H_1})-(\mathbf{H_2})$. Thus the results of Theorems 1.1 and 1.2 are established for (1.5) with $l(x,u) = f(x)|u|^{q-2}u + g(x)|u|^{m-2}u$.

REMARK 1.4. If we let $h(t) = \sqrt[p]{1 + \frac{\alpha^p}{p}(1+t^2)^{(\frac{\alpha}{2}-1)p}|t|^p}$ with $\alpha \geqslant 1$, then $h(t) \in C^1(\mathbb{R}, \mathbb{R}^+)$ is an even function and satisfies $(\mathbf{H_1})-(\mathbf{H_2})$. The conclusions of Theorems 1.1 and 1.2 are also established for (1.6) with $l(x,u) = f(x)|u|^{q-2}u + g(x)|u|^{m-2}u$. Which recover the known results for the $p = 2$ in Li and Wang [12, Theorems 1.1 and 1.2].

This paper is organized as follows. In Section 2, we give some preliminary results and then prove Theorem 1.1. In the final Section, we will establish and prove the existence of infinitely positive ground state solutions of (1.13). Besides, we denote by C a positive constant, which may vary from line to line.

2. Nonexistence of ground state solutions

We first give some properties of $H(t)$ and $H^{-1}(t)$.

LEMMA 2.1. *The functions $H(t)$ and $H^{-1}(t)$ satisfy:*
(1) *$H(t)$, $H^{-1}(t)$ are odd, strictly increasing, and C^2 in $\mathbb{R}$.*
(2) $H(t) \geqslant h(0)t,\ \forall t \in [0,\infty);\ 0 < (H^{-1})'(t) \leqslant \frac{1}{h(0)},\ \forall t \in \mathbb{R};\ |H^{-1}(t)| \leqslant \frac{1}{h(0)}|t|,$ $\forall t \in \mathbb{R}$.
(3) $\lim\limits_{t\to 0} \frac{H^{-1}(t)}{t} = \frac{1}{h(0)};\ \lim\limits_{t\to\infty} \frac{H^{-1}(t)}{t} = \begin{cases} \frac{1}{h(\infty)}, & \textit{if } h \textit{ is bounded},\\ 0, & \textit{if } h \textit{ is unbounded}. \end{cases}$
(4) $h(H^{-1}(t))H^{-1}(t) \leqslant (\mu+1)t \leqslant (\mu+1)h(H^{-1}(t))H^{-1}(t),\ \forall t \geqslant 0.$
(5) *There exists a positive constant C such that $|H^{-1}(t)| \geqslant C|t|,\ \forall |t| \leqslant 1$.*

Proof. Utilizing the definition of $h(t)$, (1.11) and assumptions $(\mathbf{H_1})-(\mathbf{H_2})$, we may argue similarly as in the proof of Lemma 2.1 in [6] to prove our results and is hence omitted. □

Define the following operator

$$L(v)(r) = -\left(r^{N-1}|v'|^{p-2}v'\right)',\ r > 0.$$

We have the following result.

LEMMA 2.2. *Suppose that $v(r) \in C^2(0,\infty)$ is a positive solution of (1.13) and $1 < p < N$. If*

$$\begin{cases} L(v)(r) \geqslant 0,\ r > 0,\\ v'(0) = 0. \end{cases} \tag{2.1}$$

Then for all $r > 0$, the function $S(r) := r^{\rho}v(r)$ is nondecreasing, here $\rho > 0$ is given in $(\mathbf{A1})$.

Proof. Integrating (1.13) from 0 to r, we conclude that (2.1) holds and $v'(r) < 0$ $(r > 0)$. Since

$$\begin{aligned} L(v)(r) &= -\left(r^{N-1}|v'|^{p-2}v'\right)' = -(p-1)r^{N-2}|v'|^{p-2}(rv'' + \rho_1 v')\\ &= -(p-1)r^{N-2}|v'|^{p-2}(rv' + \rho v)' = -(p-1)r^{N-2}|v'|^{p-2}\lambda'(r), \end{aligned} \tag{2.2}$$

where $\rho_1 = \frac{N-1}{p-1}$, $\lambda(r) = rv' + \rho v$. Thus, $\lambda'(r) \leqslant 0\ (r > 0)$.

We claim that for all $r > 0$, $\lambda(r) \geqslant 0$. If this is not true, then there is $r_0 > 0$ such that $\lambda(r_0) < 0$. Remembering $\lambda'(r) \leqslant 0\ (r > 0)$, we get $\lambda(r)$ is nonincreasing for all $r > 0$. So,

$$rv' \leqslant rv' + \rho v = \lambda(r) \leqslant \lambda(r_0),\ \text{for all } r \in (r_0,\infty),$$

that is,

$$v'(r) \leqslant \frac{\lambda(r_0)}{r}, \text{ for all } r \in (r_0, \infty).$$

Integrating the above inequality from r_0 to r, we deduce that

$$v(r) \leqslant v(r_0) + \lambda(r_0) \ln \frac{r}{r_0} \to -\infty, \text{ when } r \to \infty.$$

This is a contradiction to the positive solution $v(r)$. Consequently, the claim is proved and

$$\lambda(r) = rv' + \rho v = r^{1-\rho}(r^\rho v)' = r^{1-\rho} S'(r) \geqslant 0, \text{ for all } r \in (0, \infty). \tag{2.3}$$

This shows that $S(r)$ is nondecreasing for all $r > 0$. The proof is complete. $\square$

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. The proof of Theorem 1.1 is by contradiction. We assume that (1.13) has a positive ground state solution v, then Lemma 2.2 holds. Moreover, from (2.2), we have

$$\begin{aligned} L(v)(r) &= -(p-1)r^{N-2}|v'|^{p-2}\lambda'(r) \\ &= r^{N-1} \cdot \frac{f(r)(H^{-1}(v))^{q-1} + g(r)(H^{-1}(v))^{m-1}}{h(H^{-1}(v))}, \end{aligned} \tag{2.4}$$

which yields

$$-\lambda'(r) = r|v'|^{2-p} \cdot \frac{f(r)(H^{-1}(v))^{q-1} + g(r)(H^{-1}(v))^{m-1}}{(p-1)h(H^{-1}(v))}. \tag{2.5}$$

Note that v is positive ground state solution, it follows from Lemma 2.1-(3),(4) that there are positive constants $a_0 \geqslant a$, C such that

$$\begin{aligned} -\lambda'(r) &= r|v'|^{2-p} \cdot \frac{f(r)(H^{-1}(v))^{q-1} + g(r)(H^{-1}(v))^{m-1}}{(p-1)h(H^{-1}(v))} \\ &\geqslant Cr|v'|^{2-p}\big(f(r)v^{q-1} + g(r)v^{m-1}\big), \text{ for all } r \in [a_0, \infty), \end{aligned} \tag{2.6}$$

where a is given in $(\mathbf{A1})$.

Recall that $\lambda(r) = rv' + \rho v \geqslant 0$ $(r > 0)$, we have $-\frac{\rho v}{r} \leqslant v' < 0$ $(r > 0)$. Consequently, $|v'| = -v' \leqslant \frac{\rho v}{r}$ and

$$|v'|^{2-p} \geqslant \rho^{2-p} v^{2-p} r^{p-2}, \text{ for all } r \in (0, \infty), \tag{2.7}$$

here we have used $p \geqslant 2$. By virtue of (2.6) and (2.7), we find a constant $C > 0$ such that

$$-\lambda'(r) \geqslant Cr^{p-1}\big(f(r)v^{q+1-p} + g(r)v^{m+1-p}\big), \text{ for all } r \in [a_0, \infty).$$

Integrating the above inequality from a_0 to r, we obtain

$$-\lambda(r)+\lambda(a_0)\geqslant C\int_{a_0}^{r}t^{p-1}\big(f(t)v^{q+1-p}(t)+g(t)v^{m+1-p}(t)\big)dt.$$

Since $S(r)$ is nondecreasing in $(0,\infty)$ and $\lambda(r)\geqslant 0$ $(r>0)$, we get

$$\begin{aligned}\lambda(a_0)&\geqslant C\int_{a_0}^{r}t^{p-1}\big(f(t)v^{q+1-p}(t)+g(t)v^{m+1-p}(t)\big)dt\\&=C\int_{a_0}^{r}t^{p-1}\big(f(t)S^{q+1-p}(t)t^{-\rho(q+1-p)}+g(t)S^{m+1-p}(t)t^{-\rho(m+1-p)}\big)dt\qquad(2.8)\\&\geqslant CS^{q+1-p}(a_0)\int_{a_0}^{r}f(t)t^{p-1-\rho(q+1-p)}dt+CS^{m+1-p}(a_0)\int_{a_0}^{r}g(t)t^{p-1-\rho(m+1-p)}dt\\&\geqslant C\min\{S^{q+1-p}(a_0),S^{m+1-p}(a_0)\}\int_{a_0}^{r}t^{p-1}\big(f(t)t^{-\rho(q+1-p)}+g(t)t^{-\rho(m+1-p)}\big)dt.\end{aligned}$$

Now we divide the rest proof into two cases.

Case 1. If $\int_{a_0}^{\infty}t^{p-1}\big(f(t)t^{-\rho(q+1-p)}+g(t)t^{-\rho(m+1-p)}\big)dt=\infty$, then we have a contradiction by letting $r\to\infty$ in (2.8). Hence (1.13) has no positive ground state solution.

Case 2. If $\int_{a_0}^{\infty}t^{p-1}\big(f(t)t^{-\rho(q+1-p)}+g(t)t^{-\rho(m+1-p)}\big)dt<\infty$. Set

$$A(s):=\int_{s}^{\infty}t^{p-1}\big(f(t)t^{-\rho(q+1-p)}+g(t)t^{-\rho(m+1-p)}\big)dt,\ s\in[a_0,\infty).\qquad(2.9)$$

Then $A(s)$ is bounded for $s\geqslant a_0$. Moreover, we have

$$A'(s)=-s^{p-1}\big(f(s)s^{-\rho(q+1-p)}+g(s)s^{-\rho(m+1-p)}\big).$$

From (2.3), we get $S'(r)=r^{\rho-1}\lambda(r)$. Combining this with (2.8) and (2.9), we conclude that for all $s\geqslant a_0$,

$$S'(s)\geqslant Cs^{\rho-1}\min\big\{S^{q+1-p}(s),S^{m+1-p}(s)\big\}A(s),\qquad\text{as } r\to\infty.$$

If $\min\big\{S^{q+1-p}(s),S^{m+1-p}(s)\big\}=S^{q+1-p}(s)$, then we can find

$$\frac{S'(s)}{S^{q+1-p}(s)}\geqslant Cs^{\rho-1}A(s),\ \ s\in[a_0,\infty).\qquad(2.10)$$

Integrating the above inequality from a_0 to r, thanks to $p<\min\{q,m\}$, we deduce

$$\frac{1}{p-q}\big[S^{p-q}(r)-S^{p-q}(a_0)\big]\geqslant C\Big[r^{\rho}A(r)-a_0^{\rho}A(a_0)\\+\int_{a_0}^{r}t^{p-1}\big(f(t)t^{-\rho(q-p)}+g(t)t^{-\rho(m-p)}\big)dt\Big].$$

As a consequence,

$$\frac{1}{q-p}S^{p-q}(a_0) \geqslant C\Big[\int_{a_0}^{r} t^{p-1}\big(f(t)t^{-\rho(q-p)}+g(t)t^{-\rho(m-p)}\big)dt - s^{\rho}A(a_0)\Big].$$

Combining this and the hypothesis $(\mathbf{A_1})$, we can derive

$$\begin{aligned}&0<S(a_0)\\ &\leqslant C\Big[\int_{a_0}^{r} t^{p-1}\big(f(t)t^{-\rho(q-p)}+g(t)t^{-\rho(m-p)}\big)dt - a_0^{\rho}A(a_0)\Big]^{\frac{1}{p-q}} \to 0, \ \text{ as } \ r\to\infty,\end{aligned}$$

which is a contradiction.

If $\min\{S^{q+1-p}(s), S^{m+1-p}(s)\} = S^{m+1-p}(s)$. Same as arguments in the above paragraph, we also get a contradiction. So, (1.13) has no positive ground state solution and the proof is finished. $\square$

REMARK 2.3. For all $s \geqslant 1$, we have

$$\begin{aligned}&\int_{s}^{r} t^{p-1}\big(f(t)t^{-\rho(q+1-p)}+g(t)t^{-\rho(m+1-p)}\big)dt\\ \leqslant &\int_{s}^{r} t^{p-1}\big(f(t)t^{-\rho(q-p)}+g(t)t^{-\rho(m-p)}\big)dt.\end{aligned}$$

Then

$$\begin{aligned}&\int_{s}^{\infty} t^{p-1}\big(f(t)t^{-\rho(q+1-p)}+g(t)t^{-\rho(m+1-p)}\big)dt=\infty\\ \Longrightarrow &\int_{s}^{\infty} t^{p-1}\big(f(t)t^{-\rho(q-p)}+g(t)t^{-\rho(m-p)}\big)dt=\infty.\end{aligned}$$

3. Existence of ground state solutions

In this section, we will prove the existence of positive radial ground state solutions for (1.13) by using the Schauder-Tychonoff fixed point theorem.

We still consider (1.13), namely

$$\begin{cases} -\big(r^{N-1}|v'|^{p-2}v'\big)' = \dfrac{r^{N-1}\big(f(r)|H^{-1}(v)|^{q-2}H^{-1}(v)+g(r)|H^{-1}(v)|^{m-2}H^{-1}(v)\big)}{h(H^{-1}(v))}, \quad r>0,\\ v_0=v(0)>0, \quad v(r)\to 0, \quad \text{as} \quad r\to\infty,\end{cases}$$

where $\min\{q,m\}>1,\ 1<p<N$.

Proof of Theorem 1.2. We divide the proof into two parts.

Step 1. In this step, we will show that for each $v_0>0$, there exists $\delta>0$ and $v=v(r)$ such that

$$\begin{aligned}&-\big(r^{N-1}|v'|^{p-2}v'\big)'\\ =&\frac{r^{N-1}\big(f(r)|H^{-1}(v)|^{q-2}H^{-1}(v)+g(r)|H^{-1}(v)|^{m-2}H^{-1}(v)\big)}{h(H^{-1}(v))}, \quad r\in(0,\delta),\end{aligned} \tag{3.1}$$

with $\frac{v_0}{2} \leqslant v(r) \leqslant v_0,\ r \in [0,\delta];\ v'(r) < 0, r \in (0,\delta)$.

By (1.11), we have $H^{-1}(v) > 0$. Thus, (3.1) becomes

$$-\left(r^{N-1}|v'|^{p-2}v'\right)' = \frac{r^{N-1}\left(f(r)(H^{-1}(v))^{q-1} + g(r)(H^{-1}(v))^{m-1}\right)}{h(H^{-1}(v))}, \quad r \in (0,\delta). \tag{3.2}$$

Remember that $f(t), g(t)$ are positive continuous functions, we see that

$$\lim_{r\to 0^+} \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} f(t)dt \right)^{\frac{1}{p-1}} ds = 0, \quad \lim_{r\to 0^+} \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} g(t)dt \right)^{\frac{1}{p-1}} ds = 0.$$

Choose $\delta > 0$ small enough so that

$$\begin{aligned} &\int_0^\delta \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} f(t)dt \right)^{\frac{1}{p-1}} ds \leqslant \frac{1}{4}(h(0))^{\frac{q}{p-1}}(v_0)^{\frac{p-q}{p-1}}, \\ &\int_0^\delta \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} g(t)dt \right)^{\frac{1}{p-1}} ds \leqslant \frac{1}{4}(h(0))^{\frac{m}{p-1}}(v_0)^{\frac{p-m}{p-1}}. \end{aligned} \tag{3.3}$$

Denote

$$X = \left\{ v \in C[0,\delta] \,\middle|\, \frac{v_0}{2} \leqslant v(r) \leqslant v_0,\ r \in [0,\delta] \right\},$$

and the operator $T : X \to C[0,\delta]$ with $\tilde{v}(r) = T(v)(r)$, where

$$\begin{aligned} \tilde{v}(r) = v_0 &- \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} f(t) \cdot \frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))} dt \right)^{\frac{1}{p-1}} ds \\ &- \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} g(t) \cdot \frac{(H^{-1}(v))^{m-1}}{h(H^{-1}(v))} dt \right)^{\frac{1}{p-1}} ds. \end{aligned} \tag{3.4}$$

Clearly, X is a nonempty closed convex set of $C[0,\delta]$. To exploit Schauder-Tychonoff fixed point theorem, we first show that T maps X into itself. It is clear that $\tilde{v}(r) \in C[0,\delta]$. Since $v(r) \in X$, it follows from (3.3) and Lemma 2.1-(2) that

$$\begin{aligned} 0 \leqslant &\int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} f(t) \cdot \frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))} dt \right)^{\frac{1}{p-1}} ds \\ &+ \int_0^r \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} g(t) \cdot \frac{(H^{-1}(v))^{m-1}}{h(H^{-1}(v))} dt \right)^{\frac{1}{p-1}} ds \\ \leqslant & h(0)^{\frac{q}{1-p}} v_0^{\frac{q-1}{p-1}} \int_0^\delta \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} f(t)dt \right)^{\frac{1}{p-1}} ds \\ &+ h(0)^{\frac{m}{1-p}} v_0^{\frac{m-1}{p-1}} \int_0^\delta \left(\int_0^s \left(\frac{t}{s}\right)^{N-1} g(t)dt \right)^{\frac{1}{p-1}} ds \\ \leqslant & \frac{v_0}{4} + \frac{v_0}{4} = \frac{v_0}{2}. \end{aligned}$$

Substituting the above inequality into (3.4), we conclude $\frac{v_0}{2} \leqslant \tilde{v}(r) \leqslant v_0,\ r \in [0,\delta]$. Hence, the desired result is obtained and $TX \subset X$.

Next, we prove that T is continuous. To this aim, we assume that $v_n(r)$ is a sequence in X which converges to $v(r)$ uniformly in $[0,\delta]$.

Set

$$\omega_n^1(s)=\int_0^s\Big(\frac{t}{s}\Big)^{N-1}f(t)\cdot\frac{(H^{-1}(v_n))^{q-1}}{h(H^{-1}(v_n))}dt,\ \omega^1(s)=\int_0^s\Big(\frac{t}{s}\Big)^{N-1}f(t)\cdot\frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))}dt,$$
$$\omega_n^2(s)=\int_0^s\Big(\frac{t}{s}\Big)^{N-1}g(t)\cdot\frac{(H^{-1}(v_n))^{q-1}}{h(H^{-1}(v_n))}dt,\ \omega^2(s)=\int_0^s\Big(\frac{t}{s}\Big)^{N-1}g(t)\cdot\frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))}dt.$$

Then we have

$$\begin{aligned}\big|\omega_n^1(s)-\omega^1(s)\big|&\leqslant\int_0^s f(t)\Big|\frac{(H^{-1}(v_n))^{q-1}}{h(H^{-1}(v_n))}-\frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))}\Big|dt,\\ \big|\omega_n^2(s)-\omega^2(s)\big|&\leqslant\int_0^s g(t)\Big|\frac{(H^{-1}(v_n))^{q-1}}{h(H^{-1}(v_n))}-\frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))}\Big|dt,\end{aligned}\tag{3.5}$$

and

$$\big|\tilde v_n(r)-\tilde v(r)\big|\leqslant\int_0^r\Big(\big|(\omega_n^1(s))^{\frac{1}{p-1}}-(\omega^1(s))^{\frac{1}{p-1}}\big|+\big|(\omega_n^2(s))^{\frac{1}{p-1}}-(\omega^2(s))^{\frac{1}{p-1}}\big|\Big)ds.\tag{3.6}$$

Hence, by (3.5), we deduce that as $n\to\infty$, $\{\omega_n^1(s)\}$ and $\{\omega_n^2(s)\}$ converges to $\{\omega^1(s)\}$ and $\{\omega^2(s)\}$ uniformly in $[0,\delta]$, respectively. As a result, $\{(\omega_n^1(s))^{\frac{1}{p-1}}\}$ and $\{(\omega_n^2(s))^{\frac{1}{p-1}}\}$ converges to $(\{\omega^1(s))^{\frac{1}{p-1}}\}$ and $\{(\omega^2(s))^{\frac{1}{p-1}}\}$ uniformly in $[0,\delta]$, respectively. Combining this with (3.6), we obtain $\tilde v_n(r)$ converges to $v(r)$ in $C[0,\delta]$. Namely, T is continuous.

At last, we will verify that $T(X)$ is relatively compact. Since $v(r)\in X$, it follows from Lemma 2.1-(2) and (3.4) that

$$\begin{aligned}\tilde v'(r)&=\Big(\int_0^r\big(\frac{t}{r}\big)^{N-1}f(t)\cdot\frac{(H^{-1}(v))^{q-1}}{h(H^{-1}(v))}dt\Big)^{\frac{1}{p-1}}+\Big(\int_0^r\big(\frac{t}{r}\big)^{N-1}g(t)\cdot\frac{(H^{-1}(v))^{m-1}}{h(H^{-1}(v))}dt\Big)^{\frac{1}{p-1}}\\&\leqslant h(0)^{\frac{q}{1-p}}v_0^{q-1}\big(\int_0^\delta f(t)dt\big)^{\frac{1}{p-1}}+h(0)^{\frac{m}{1-p}}v_0^{m-1}\big(\int_0^\delta g(t)dt\big)^{\frac{1}{p-1}},\ \text{ for all } r\in[0,\delta],\end{aligned}$$

which implies $\{\tilde v(r)\big|v\in X\}$ is bounded. From the Arzela-Ascoli theorem, we get that $T(X)$ is relatively compact.

Thus, applying the Schauder-Tychonoff fixed point theorem, we can see that there exists a $v(r)\in X$ such that $Tv=v$. Moreover, the $v(r)$ can be extended and satisfies $v'(r)<0$ provided $v(r)>0$.

Indeed, let

$$U=\big\{\sigma\geqslant 0\big|v(r)>0,0\leqslant r<\sigma\big\}.$$

We claim that $U=[0,\infty)$. On the contrary we assume that $U\neq[0,\infty)$, then there is $\sigma_0>0$ such that

$$v(r)>0,\ \ 0\leqslant r<\sigma_0;\ v(\sigma_0)=0;\ \ v'(r)<0,\ \ 0<t\leqslant\sigma_0.$$

Multiplying (3.2) by $-rv'(r)$ and $v(r)$, respectively, then integrating the results from 0 to σ_0, it follows that

$$\frac{p-1}{p}\sigma_0^N(-v'(\sigma_0))^p+\int_0^{\sigma_0}\Big[\frac{N-p}{p}r^{N-1}(-v')^p+\frac{r^Nf(r)((H^{-1}(v))^q)'}{q}\\ +\frac{r^Ng(r)((H^{-1}(v))^m)'}{m}\Big]dr=0, \tag{3.7}$$

and

$$\int_0^{\sigma_0}r^{N-1}(-v')^pdr=-\big[r^{N-1}(-v')^{p-1}v\big]_0^{\sigma_0}+\int_0^{\sigma_0}v\big(r^{N-1}(-v')^{p-1}\big)'dr\\ =\int_0^{\sigma_0}\frac{r^{N-1}\big(f(r)(H^{-1}(v))^{q-1}+g(r)(H^{-1}(v))^{m-1}\big)v}{h(H^{-1}(v))}dr. \tag{3.8}$$

Note that

$$\int_0^{\sigma_0}r^Nf(r)((H^{-1}(v))^q)'dr=\big[r^Nf(r)(H^{-1}(v))^q\big]_0^{\sigma_0}-\int_0^{\sigma_0}(H^{-1}(v))^q\big(r^Nf(r)\big)'dr\\ =-\int_0^{\sigma_0}(H^{-1}(v))^q\big(r^Nf(r)\big)'dr, \tag{3.9}$$

and

$$\int_0^{\sigma_0}r^Ng(r)((H^{-1}(v))^m)'dr=\big[r^Ng(r)(H^{-1}(v))^m\big]_0^{\sigma_0}-\int_0^{\sigma_0}(H^{-1}(v))^m\big(r^Ng(r)\big)'dr\\ =-\int_0^{\sigma_0}(H^{-1}(v))^m\big(r^Ng(r)\big)'dr. \tag{3.10}$$

Thanks to the hypothesis $(\mathbf{A_3})$, it follows from (3.7)–(3.10) and Lemma 2.1-(4) that

$$\begin{aligned}0=&\frac{p-1}{p}\sigma_0^N(-v'(\sigma_0))^p\\ &+\int_0^{\sigma_0}\Big[\frac{N-p}{p}\cdot\frac{r^{N-1}f(r)(H^{-1}(v))^{q-1}v}{h(H^{-1}(v))}-\frac{(H^{-1}(v))^q\big(r^Nf(r)\big)'}{q}\Big]dr\\ &+\int_0^{\sigma_0}\Big[\frac{N-p}{p}\cdot\frac{r^{N-1}g(r)(H^{-1}(v))^{m-1}v}{h(H^{-1}(v))}-\frac{(H^{-1}(v))^m\big(r^Ng(r)\big)'}{m}\Big]dr\\ \geqslant&\frac{p-1}{p}\sigma_0^N(-v'(\sigma_0))^p\\ &+\int_0^{\sigma_0}\Big[\frac{N-p}{p(\mu+1)}\cdot r^{N-1}f(r)-\frac{\big(r^Nf(r)\big)'}{q}\Big](H^{-1}(v))^qdr\\ &+\int_0^{\sigma_0}\Big[\frac{N-p}{p(\mu+1)}\cdot r^{N-1}g(r)-\frac{\big(r^Ng(r)\big)'}{m}\Big](H^{-1}(v))^mdr>0.\end{aligned}$$

This is impossible. Hence $U=[0,\infty)$ and obviously $v'(r)<0$ provided $v(r)>0$. To sum up, there exists a positive radial solution $v(r)$ of (1.13).

Step 2. In this step, we will show that $v(r) \to 0$ as $r \to \infty$, i.e., $v(r)$ is a positive ground state solution of (1.13).

Integrating (3.2) from 0 to s, it yields

$$\begin{aligned}\int_0^s (t^{N-1}(-v')^{p-1})' dt &= s^{N-1}(-v'(s))^{p-1} \\ &= \int_0^s \frac{t^{N-1}\big(f(t)(H^{-1}(v))^{q-1} + g(t)(H^{-1}(v))^{m-1}\big)}{h(H^{-1}(v))} dt.\end{aligned} \tag{3.11}$$

Notice that $0 < v(r) \leqslant v_0$, we can choose $0 < v_0 \leqslant 1$, then it follows from Lemma 2.1-(4),(5) and (3.11) that there exists a constant $C > 0$ such that

$$\begin{aligned} s^{N-1}(-v'(s))^{p-1} &\geqslant C\int_0^s t^{N-1}\big(f(t)v^{q-1} + g(t)v^{m-1}\big)dt \\ &\geqslant Cv^{q-1}(s)\int_0^s t^{N-1}f(t)dt + Cv^{m-1}(s)\int_0^s t^{N-1}g(t)dt \\ &\geqslant Cv^{l-1}(s)\int_0^s t^{N-1}\big(f(t)+g(t)\big)dt, \end{aligned} \tag{3.12}$$

where $l = \max\{q,m\}$. Consequently,

$$\frac{-v'(s)}{v^{\frac{l-1}{p-1}}(s)} \geqslant C\Big(s^{1-N}\int_0^s t^{N-1}\big(f(t)+g(t)\big)dt\Big)^{\frac{1}{p-1}}. \tag{3.13}$$

Recalling $1 < p < \max\{q,m\} = l$. Integrating (3.2) from 0 to r, then we get

$$\begin{aligned} \frac{p-1}{l-p}v^{\frac{p-l}{p-1}}(r) &\geqslant \frac{p-1}{l-p}\big(v^{\frac{p-l}{p-1}}(r) - v_0^{\frac{p-l}{p-1}}\big) \\ &= -\int_0^r \frac{v'(s)}{v^{\frac{l-1}{p-1}}(s)}ds \\ &\geqslant C\int_0^r \Big(s^{1-N}\int_0^s t^{N-1}\big(f(t)+g(t)\big)dt\Big)^{\frac{1}{p-1}}ds, \end{aligned}$$

which implies

$$v(r) \leqslant C\left\{\int_0^r \Big(s^{1-N}\int_0^s t^{N-1}\big(f(t)+g(t)\big)dt\Big)^{\frac{1}{p-1}}ds\right\}^{\frac{p-1}{p-l}}. \tag{3.14}$$

Using the hypothesis $(\mathbf{A_2})$, we immediately obtain $v(r) \to 0$ by letting $r \to \infty$ in (3.14).

Finally, from the above derivation process, we point out that for each $0 < v_0 \leqslant 1$, there exists a solution $v(r)$ of (1.13) with $v(0) = v_0$. So, the existence of infinitely many positive ground state solution for (1.13) is established, which indicates that (1.1) possesses infinitely many positive radial ground state solutions. The proof is finished. □

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