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NEW IMPULSIVE–INTEGRAL INEQUALITY FOR STOCHASTIC DIFFERENTIAL EQUATIONS WITH POISSON JUMPS AND CAPUTO FRACTIONAL DERIVATIVE

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Abstract. In this paper, we study the existence and exponential stability in pth moment of mild solutions for a class of impulsive fractional stochastic differential equations driven by Poisson jumps. Firstly, we discuss the existence and uniqueness of mild solutions for the considered equations by the Banach fixed point theorem. Next, we establish a new impulsive-integral inequality that can effectively improve some previous results [4, 17, 5, 3, 6]. Then, we obtain the exponential stability in the pth moment of mild solutions for the considered equations with the aid of the new inequality. Finally, an example is given to illustrate the efficiency of the obtained theoretical results.

1. Introduction

As we know, stochastic differential equations are viewed as an excellent tool for describing real-life phenomena when noises and stochastic perturbation are non-negligible in a wide variety of applications such as economics, finance, engineering and social sciences and so on. Naturally, studies on the existence, uniqueness and stability of solutions for stochastic differential equations or impulsive stochastic differential equations have been heat research topics. For examples, Chen in literature [4] studied the exponential stability in the pth moment of mild solutions for impulsive stochastic partial differential equations with delays by establishing an impulsive-integral inequality; Shu et al. in literature [6, 14] studied the existence and exponential stability of mild solutions for neutral stochastic functional differential equations by using the noncompact measurement strategy, Mönch fixed point theorem and some inequality techniques; Luo in literature [11] discussed exponential stability of mild solutions of stochastic partial differential equations with delays by fixed point theory; Xu et al. in literature [16] investigated the mean square exponential stability of mild solutions for impulsive stochastic partial differential equations with delays by using a delay differential inequality and stochastic analysis technique; Li and Fan in literature [9] concerned exponential stability of mild solutions for impulsive stochastic partial differential equations with delays by employing the formula for the variation of parameters and

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inequality technique; Xu et al. in literature [15] concerned the pth moment globally exponential stability and quasi sure globally exponential stability for impulsive stochastic differential equations driven by G-Brownian motion by using G-Lyapunov function methods and inequality techniques; Guo et al. in literature [7] discussed both the pth moment and almost sure exponential stability of solutions to stochastic functional differential equations with impulsive by using the Razumikhin-type technique; Li et al. in literature [10] studied the existence and Hyers-Ulam stability of random impulsive stochastic functional differential equations with finite delays via fixed point theorem and Gronwall inequality.

On the other hand, stochastic differential equations driven by Poisson random measures arise in many different fields. For example, they have been used to develop models for neuronal activity that account for synaptic impulses occurring randomly, both in time and at different locations of a spatially extended neuron. Other applications arise in chemical reaction-diffusion systems and stochastic turbulence models. To the best of our knowledge, the existing paper on the existence and stability analysis of the mild solutions for stochastic partial differential equations driven by Poisson jump are relatively few. For example, Anguraj et al. in literature [1] investigated the Hyers-Ulam stability results under the Lipschitz condition on a bounded and closed interval by using stochastic analysis and the Gronwall inequality. In paper [8], Hou et al. considered the exponential stability of energy solutions to stochastic partial differential equations with variable delays and Poisson jumps by estimating the coefficients functions in the stochastic energy equality. Also Chen et al. in [5] concerned exponential stability of a class of impulsive neutral stochastic partial differential equations with delays and Poisson jumps via an impulsive-integral inequality. In recent years, with the development of fractional calculus, many scholars devote themselves to the study of the existence and stability of solutions of fractional stochastic differential equations. For example, In paper [3], the authors studied the existence and exponential stability of neutral stochastic fractional differential equations with impulses driven by Poisson jumps via the fixed point theorem and inequality strategy. It should be pointed out that the restrictive conditions of impulsive-integral inequality in [3] are too strict which shows that impulsive-integral inequality has room for improvement. Based on this discussion, in this paper, we concern with the following stochastic fractional differential equations in the Hilbert space $(X;\|\cdot\|_X)$ with the inner product $(\cdot,\cdot)_X$:

$$\begin{cases} {}^c\mathrm{D}_t^q x(t) = Ax(t) + \mathbb{J}_t^{2-q}\Big(f_1(t,x(t-\delta_1(t)))\Big) + \mathbb{J}_t^{1-q}\Big(f_2(t,x(t-\delta_2(t)))\frac{dW(t)}{dt} \\ \qquad\qquad + \int_Z f_3(t,x(t-\delta_3(t)),y)\widetilde{N}(dt,dy)\Big),\ t \neq t_k,\ t \in [0,T] = J, \\ x(t) = \varphi(t) \in PC,\ t \in [-r,0],\ x'(0) = y_1 \in X,\ \text{a.s.}, \\ \Delta x(t_k) = I_k(x(t_k)),\ \Delta x'(t_k) = J_k(x(t_k)),\ t = t_k,\ k = 1,2,\cdots, \end{cases} \tag{1.1}$$

where ${}^c\mathrm{D}_t^q$ denotes the Caputo fractional derivative of order $1 < q < 2$, $\mathbb{J}^q$ denotes the qth order fractional integral, A is a closed and densely linear operator in a Hilbert space X, $Z \in \mathscr{B}_\sigma(U)$, where $\mathscr{B}_\sigma(U)$ is the Borel σ-algebra of separable Hilbert space U. y_1 is a $\mathfrak{F}_0$-measurable X-valued random variable independent of the Wiener process

$W(t)$. The function $f_1 : J \times C([-r,0],X) \to 2^X$ is non-empty, bounded, closed and convex multivalued map and $f_2 : J \times C([-r,0],X) \to L(Y,X)$, $f_3 : J \times X \times Z \to X$. $I_k(\cdot)$, $J_k(\cdot) : X \to X$ are continuous functions and the fixed times t_k satisfy $0 = t_0 < t_1 < \cdots < t_k < \cdots$, $\Delta x(t_k) = x(t_k^+) - x(t_k^-)$ and $x(t_k^-) = x(t_k)$, where $x(t_k^+)$ and $x(t_k^-)$ are represent the right and left limits of $x(t)$ at t_k, respectively. $\delta_i(t) : R^+ \to [0,r]$, $i = 1,2,3$ are continuous functions. PC be the space of all almost surely bounded, $\mathfrak{F}_0$-measure and continuous functions everywhere except for an infinite number of point s at which $\xi(s)$ and left limit $\xi(s)$ exist and $\xi(s^+) = \xi(s)$ from $[-r,0]$ into X and equipped with the supremum norm $\|\varphi\|_0 = \sup_{\theta\in[-r,0]} \|\varphi(\theta)\|$ and let $PC^b_{\mathfrak{F}_0}([-r,0],X)$ denotes the space of all bounded $\mathfrak{F}_0$-measurable $PC([-r,0],X)$-valued random variable u such that $\|u\|^p_{PC} = \sup_{s\in[-r,0]} E\|u(s)\|^p$. Also, we define the measure $\widetilde{N}$ by $\widetilde{N}(dx,dy) = N_p(dt,dy) - \nu(dy)dt$, where ν is the characteristic measure of N_p, which is called the compensated Poisson random measure. For a Borel set $Z \in \mathscr{B}_\sigma(U - \{0\})$, the space $\mathscr{P}^p(J \times Z, X)$, $p \geqslant 2$ denotes the space of all predictable maps $f : J \times Z \times \Omega \to X$ with $\int_0^T \int_Z E\|f(t,u)\|^p dt\vartheta(du) < \infty$. This section ends by highlighting the main contributions of this paper:

- The existence and uniqueness of solution for impulsive fractional stochastic system are proved.
- A new impulsive-integral inequality is established which can effectively improve some previous results [4, 17, 5, 3, 6].
- Exponential stability results are established by applying appropriate hypotheses and new impulsive-integral inequality.

The arrangement of the rest paper is as follows. In Section 2, some preliminaries and results which are applied in the later paper are presented. Section 3 is devoted to studying the exponential stability in the pth moment of the mild solutions of (1.1).

2. Preliminaries

Let X and Y be two real, separable Hilbert spaces and $L(Y,X)$ be the space of a bounded linear operator from Y to X. $C(J,X)$ stands for the Banach space of continuous functions from J to X with supremum norm, i.e., $\|x\|_J = \sup_{t\in J} \|x(t)\|$, $\forall x \in C(J,X)$ and $L^1(J,X)$ represents the Banach space of functions from J to X which are Bochner integrable normed by $\|x\|_{L^1} = \int_0^T \|x\| dt$, $\forall x \in L^1(J,X)$. Let $(\Omega, \mathfrak{F}, P)$ be a complete filtered probability space with a filtration $\{\mathfrak{F}_t\}_{t\geqslant 0}$ satisfying the usual conditions (i.e. right continuous and $\mathfrak{F}_0$ containing all P_0-null sets). Let $\beta_n(t)$ $(n = 1,2,\cdots)$ be a sequence of real-valued one-dimensional standard Brownian motions mutually independent over $(\Omega, \mathfrak{F}, P)$. Set $W(t) = \sum_{n=1}^{+\infty} \sqrt{\lambda_n}\beta_n(t)e_n$ $(t \geqslant 0)$, where $\lambda_n \geqslant 0 (n = 1,2,\cdots)$ are nonnegative real numbers and $\{e_n\}$ $(n = 1,2,\cdots)$ is a complete orthonormal basis in Y. Let $Q \in L(Y,X)$ be an operator defined by $Qe_n = \lambda_n e_n$ with a finite trace $trQ = \sum_{n=1}^{+\infty} \lambda_n < +\infty$. Then, the above Y-valued stochastic process $W(t)$ is called a Q-Wiener process.

DEFINITION 1. Let $\phi \in L(Y,X)$ and define

$$\|\phi\|_{L_2^0}^2 := tr(\phi Q \phi^*) = \Big\{ \sum_{n=1}^{+\infty} \| \sqrt{\lambda_n} \phi e_n \Big\}.$$

If $\|\phi\|_{L_2^0}^2 < +\infty$, then ϕ is a Q-Hilbert-Schmidt operator and define $L_2^0(Y,X)$ the space of all Q-Hilbert-Schmidt operators $\phi : Y \to X$.

For more details about the X-valued stochastic integral of an $L_2^0(Y,X)$-valued, $\mathfrak{F}_t$-adapted predictable process $h(t)$ with respect to the Q-Wiener process $W(t)$, we can see reference [12].

DEFINITION 2. The Riemann-Liouville fractional integral operator $\mathbb{J}$ of order $q > 0$ is defined by

$$\mathbb{J}_t^q f(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} f(s) ds,$$

where $f \in L^1(J,X)$ and $0 \leqslant T < \infty$.

DEFINITION 3. The Caputo fractional derivative is defined by

$${}^c\mathrm{D}_t^q f(t) = \frac{1}{\Gamma(n-q)} \int_0^t (t-s)^{n-q-1} f^n(s) ds,\ n-1 < q < n,$$

where $f \in C^{n-1}((0,T),X) \bigcap L^1(J,X)$.

DEFINITION 4. Let $A : D(A) \subset X \to X$ be a closed linear operator. The operator A is said to be a sectorial operator of type $(M, \theta, \alpha, \omega)$ if there exist constants $\omega \in R$, $0 < \theta < \frac{\pi}{2}$, $M > 0$ such that

(i) The $\alpha-$resolvent of A exists outside the sector $\omega + S_\theta = \{\omega + \lambda^\alpha : \lambda \in \mathbb{C}, |Arg(-\lambda^\alpha)| < \theta\}$;

(ii) $\|R(\lambda^\alpha, A)\| = \|(\lambda^\alpha - A)^{-1}\| \leqslant \frac{M}{|\lambda^\alpha - \omega}|$, $\lambda \notin \omega + S_\theta$.

In addition, if A is a sectorial operator of type $(M, \theta, \alpha, \omega)$, then it is obvious to know A generates an α-resolvent family $\{R_\alpha(t) : t \geqslant 0\}$ in a Banach space, where $R_\alpha(t) = \frac{1}{2\pi i} \int_C e^{\lambda t} R(\lambda^q, A) d\lambda$. For more details, we refer readers to see [13].

DEFINITION 5. ([13]) An X-valued stochastic process $\{x(t), t \geqslant 0\}$ is called a mild solution of (1.1) if

(1) $x(t)$ is an $\mathfrak{F}_t$ $(t \geqslant 0)$ adapted process;

(2) $x(t) \in X$ has a càdlàg path on $t \in [0, +\infty)$ almost surely;

(3) for each $t \in [0,+\infty)$, we have

$$
\begin{aligned}
x(t) = {} & \mathscr{I}_q(t)\varphi(0) + \mathscr{K}_q(t)y_1 + \int_0^t \mathscr{K}_q(t-s)f_1(s,x(s-\delta_1(s)))ds \\
& + \int_0^t \mathscr{I}_q(t-s)f_2(s,x(s-\delta_2(s)))dW(s) \\
& + \int_0^t \int_Z \mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\widetilde{N}(ds,dy) \\
& + \sum_{0<t_k<t} \mathscr{I}_q(t-t_k)I_k(x(t_k)) \\
& + \sum_{0<t_k<t} \mathscr{K}_q(t-t_k)J_k(x(t_k)),\ t \geqslant 0,
\end{aligned}
$$

and $x(t) = \varphi(t)$ for $t \in [-r,0]$. Where the operators $\mathscr{I}_q(t)$ and $\mathscr{K}_q(t)$ are defined as

$$
\mathscr{I}_q(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{q-1}R(\lambda^q,A)d\lambda
$$

and

$$
\mathscr{K}_q(t) = \frac{1}{2\pi i}\int_C e^{\lambda t}\lambda^{q-2}R(\lambda^q,A)d\lambda,
$$

where C is a suitable path such that $\lambda^q \notin \omega + S_\theta$, $\lambda \in \mathbb{C}$.

LEMMA 1. ([2]) *For any $p \geqslant 2$ and for an arbitrary $L_2^0(Y,X)$-valued predictable process $\psi(s)$ such that*

$$
\sup_{s\in[0,t]} E\Big\|\int_0^s \psi(u)d\omega(u)\Big\|^p \leqslant C_p\Big(\int_0^t (E\|\psi(s)\|^p_{L_2^0})^{\frac{2}{p}}ds\Big)^{\frac{p}{2}},\ t \in [0,+\infty), \tag{2.1}
$$

where $C_p = (\frac{p(p-1)}{2})^{\frac{p}{2}}$.

LEMMA 2. ([2]) *For any $p \geqslant 2$ and there exists $C_p > 0$ such that*

$$
\begin{aligned}
\sup_{\tau\in[0,t]} E\Big[\Big\|\int_0^t\int_Z H(s,x)\widetilde{N}(d\tau,dx)\Big\|\Big]^p \leqslant {} & C_p\Big\{E\Big[\Big(\int_0^t\int_Z |H(s,x)|^2\lambda dxds\Big)^{\frac{p}{2}}\Big] \\
& + E\Big[\int_0^t\int_Z |H(s,x)|^p\lambda dxds\Big]\Big\}.
\end{aligned} \tag{2.2}
$$

DEFINITION 6. The mild solution of the system (1.1) is said to be exponentially stable in the pth moment if there exist a pair of positive constants $\gamma > 0$ and $M_0 > 0$, for any initial value $\varphi \in PC$, a.s., such that

$$
E\|x(t)\|^p \leqslant M_0 e^{-\gamma t},\ t \geqslant 0,\ p \geqslant 2. \tag{2.3}
$$

3. Existence and uniqueness

Moreover, to obtain our main results, we give the following assumptions:

(H1) The operator families $\mathscr{I}_q$, $\mathscr{K}_q$, $t \geqslant 0$ generated by A are compact in $\overline{D(A)}$ and there exist positive constants γ_1, γ_2 such that

$$\sup_{t\geqslant 0}\|\mathscr{I}_q(t)\| \leqslant M^* e^{-\gamma_1 t}, \quad \sup_{t\geqslant 0}\|\mathscr{K}_q(t)\| \leqslant M^* e^{-\gamma_2 t}, \tag{3.1}$$

where constant $M^* \geqslant 1$.

(H2) There exist positive constants β_1, β_2, β_3 and $\overline{\beta_3} > 0$ such that

$$E\|f_1(t,x_1)-f_1(t,x_2)\| \leqslant \beta_1 E\|x_1-x_2\|, \;\; f_1(t,0)=0, \tag{3.2}$$

$$E\|f_2(t,x_1)-f_2(t,x_2)\| \leqslant \beta_2 E\|x_1-x_2\|, \;\; f_2(t,0)=0, \tag{3.3}$$

$$\int_Z E\|f_3(t,x_1,y)-f_3(t,x_2,y)\|^2\theta(dy) \leqslant \beta_3\|x_1-x_2\|^2, \;\; f_3(t,0,y)=0, \tag{3.4}$$

and

$$\int_Z E\|f_3(t,x_1,y)-f_3(t,x_2,y)\|^p\theta(dy) \leqslant \overline{\beta_3}\|x_1-x_2\|^p, \tag{3.5}$$

where x_1, $x_2 \in X$, $y \in Z$, $t \geqslant 0$.

(H3) There exist positive constants c_k, d_k, $k = 1,2,\cdots$, such that

$$E\|I_k(x_1)-I_k(x_2)\| \leqslant c_k\|x_1-x_2\|, \;\; \|I_k(0)\|=0, \tag{3.6}$$

$$E\|J_k(x_1)-J_k(x_2)\| \leqslant d_k\|x_1-x_2\|, \;\; \|J_k(0)\|=0, \tag{3.7}$$

where x_1, $x_2 \in X$, $\sum_{k=1}^{+\infty} c_k < +\infty$ and $\sum_{k=1}^{+\infty} d_k < +\infty$.

THEOREM 1. *Assume that conditions* (H1)–(H3) *hold, then the system* (1.1) *has a unique mild solution on* $[-r,T]$, $0 < T < \infty$ *provided that*

$$5^{p-1}M^{*p}\Big[\beta_1^p T^2 + C_p(\beta_2^p T^{\frac{p}{2}} + \beta_3^{\frac{p}{2}} T^{\frac{p}{2}} + \overline{\beta_3}T) + \Big((\sum_{k=1}^{\infty} c_k)^p + (\sum_{k=1}^{\infty} d_k)^p\Big)\Big] < 1. \tag{3.8}$$

Proof. Let PC_T be the space of all $\mathfrak{F}$-adapted processes $x(t,\omega) : [-r,T]\times\Omega \to X$ which is almost surely continuous in $t \neq t_k$ $(k = 1,2,\cdots,)$ for fixed $\omega \in \Omega$. $\lim_{t\to t_k^-} x(t)$ and $\lim_{t\to t_k^+} x(t)$ all exist and $x(t_k^-) = x(t_k^+)$. When we define the norm as $\|x\|_{PC_T}^p = \sup_{s\in[-r,T]} E\|x(s)\|^p$, then PC_T is a Banach space with the norm $\|\cdot\|_{PC_T}$. Therefore, we call the $\overline{PC_T}$ be the closed subset of PC_T defined as $\overline{PC_T} = \{x \in PC_T : x(t) = \varphi(t)\ for\ t \in [-r,0]\}$ with norm $\|\cdot\|_{PC_T}$. Next, we transform the system (1.1) into a fixed point problem. Consider the operator $\Psi : \overline{PC_T} \to \overline{PC_T}$ defined as

$$\begin{aligned}(\Psi x)(t) = {} & \mathscr{I}_q(t)\varphi(0) + \mathscr{K}_q(t)y_1 + \int_0^t \mathscr{K}_q(t-s)f_1(s,x(s-\delta_1(s)))ds \\ & + \int_0^t \mathscr{I}_q(t-s)f_2(s,x(s-\delta_2(s)))dW(s)\end{aligned}$$

$$
\begin{aligned}
&+\int_0^t\int_Z \mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\widetilde{N}(ds,dy)\\
&+\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)I_k(x(t_k))+\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)J_k(x(t_k)),\ t\in[0,T],
\end{aligned}
$$

and $(\Psi x)(t)=\varphi(t)$, $t\in[-r,0]$. Then, we prove that the operator Ψ has a fixed point which is exactly the mild solution of the system (1.1). Firstly, we verify that $t\to(\Psi x)(t)$ is continuous on $[0,T]$. Let $x\in\overline{PC_T}$, $t\in(0,T)$ and $|\varepsilon|$ be sufficiently small, we obtain

$$
E\|(\Psi x)(t+\varepsilon)-(\Psi x)(t)\|^p\leqslant 7^{p-1}\sum_{i=1}^{7}E\|N_i(t+\varepsilon)-N_i(t)\|^p. \tag{3.9}
$$

In view of (H1) and the strong continuity of operator $\mathscr{I}_q(t)$, we obtain

$$
\begin{aligned}
E\|N_1(t+\varepsilon)-N_1(t)\|^p &= \|[\mathscr{I}_q(t+\varepsilon)-\mathscr{I}_q(t)]\varphi(0)\|^p\\
&= \|[\mathscr{I}_q(t+\varepsilon)-\mathscr{I}_q(t)]\|^p\|\varphi(0)\|^p\\
&\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Also, by the strong continuity of operator $\mathscr{K}_q(t)$ and (H1), we have

$$
\begin{aligned}
E\|N_2(t+\varepsilon)-N_2(t)\|^p &= \|[\mathscr{K}_q(t+\varepsilon)-\mathscr{K}_q(t)]y_1\|^p\\
&= \|[\mathscr{K}_q(t+\varepsilon)-\mathscr{K}_q(t)]\|^p\|y_1\|^p\\
&\to 0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

Then, from conditions (H1)–(H3), Lemma 2.1 and Lemma 2.2, we get

$$
\begin{aligned}
&E\|N_3(t+\varepsilon)-N_3(t)\|^p\\
&= E\Big\|\int_0^{t+\varepsilon}\mathscr{K}_q(t+\varepsilon-s)f_1(s,x(s-\delta_1(s)))ds-\int_0^t\mathscr{K}_q(t-s)f_1(s,x(s-\delta_1(s)))ds\Big\|^p\\
&\leqslant 2^{p-1}E\Big\|\int_0^t[\mathscr{K}_q(t+\varepsilon-s)-\mathscr{K}_q(t-s)]f_1(s,x(s-\delta_1(s)))ds\Big\|^p\\
&\quad+2^{p-1}E\Big\|\int_t^{t+\varepsilon}\mathscr{K}_q(t+\varepsilon-s)f_1(s,x(s-\delta_1(s)))ds\Big\|^p\\
&\leqslant 2^{p-1}E\Big[\int_0^t\Big\|[\mathscr{K}_q(t+\varepsilon-s)-\mathscr{K}_q(t-s)]f_1(s,x(s-\delta_1(s)))\Big\|_X ds\Big]^p\\
&\quad+2^{p-1}E\Big[\int_t^{t+\varepsilon}\Big\|\mathscr{K}_q(t+\varepsilon-s)f_1(s,x(s-\delta_1(s)))\Big\|_X ds\Big]^p\\
&\leqslant 2^{p-1}\Big[\int_0^t\Big\|\mathscr{K}_q(t+\varepsilon-s)-\mathscr{K}_q(t-s)\Big\|_{L(X)}^{\frac{p}{p-1}}ds\Big]^{p-1}\times\int_0^t E\Big\|f_1(s,x(s-\delta_1(s)))\Big\|_X^p ds\\
&\quad+2^{p-1}\Big[\int_t^{t+\varepsilon}\Big\|\mathscr{K}_q(t+\varepsilon-s)\Big\|_{L(X)}^{\frac{p}{p-1}}ds\Big]^{p-1}\times\int_t^{t+\varepsilon}E\Big\|f_1(s,x(s-\delta_1(s)))\Big\|_X^p ds\\
&\to 0 \text{ as } \varepsilon\to 0;
\end{aligned}
$$

$$
\begin{aligned}
& E\|N_4(t+\varepsilon)-N_4(t)\|^p \\
= {} & E\Big\|\int_0^{t+\varepsilon}\mathscr{I}_q(t+\varepsilon-s)f_2(s,x(s-\delta_2(s)))dW(s) \\
& -\int_0^t \mathscr{I}_q(t-s)f_2(s,x(s-\delta_2(s)))dW(s)\Big\|^p \\
\leqslant {} & 2^{p-1}E\Big\|\int_0^t[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]f_2(s,x(s-\delta_2(s)))dW(s)\Big\|^p \\
& +2^{p-1}E\Big\|\int_t^{t+\varepsilon}\mathscr{I}_q(t+\varepsilon-s)f_2(s,x(s-\delta_2(s)))dW(s)\Big\|^p \\
\leqslant {} & 2^{p-1}C_p\Big[\int_0^t\Big[E\Big\|[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]f_2(s,x(s-\delta_2(s)))\Big\|^p_{L_2^0}\Big]^{\frac{2}{p}}ds\Big]^{\frac{p}{2}} \\
& +2^{p-1}C_p\Big[\int_t^{t+\varepsilon}\Big(E\Big\|\mathscr{I}_q(t+\varepsilon-s)f_2(s,x(s-\delta_2(s)))\Big\|^p_{L_2^0}\Big)^{\frac{2}{p}}ds\Big]^{\frac{p}{2}} \\
\to {} & 0 \text{ as } \varepsilon\to 0;
\end{aligned}
$$

$$
\begin{aligned}
& E\|N_5(t+\varepsilon)-N_5(t)\|^p \\
= {} & E\Big\|\int_0^{t+\varepsilon}\int_Z\mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\widetilde{N}(ds,dy) \\
& -\int_0^t\int_Z\mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\widetilde{N}(ds,dy)\Big\|^p \\
\leqslant {} & 2^{p-1}E\Big\|\int_0^t\int_Z[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]f_3(s,x(s-r),y)\widetilde{N}(ds,dy)\Big\|^p \\
& +2^{p-1}E\Big\|\int_t^{t+\varepsilon}\int_Z\mathscr{I}_q(t+\varepsilon-s)f_3(s,x(s-r),y)\widetilde{N}(ds,dy)\Big\|^p \\
\leqslant {} & 2^{p-1}C_pE\Big[\int_0^t\int_Z\|[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]f_3(s,x(s-r),y)\|^2 ds\lambda dy\Big]^{\frac{p}{2}} \\
& +2^{p-1}C_p\int_0^t\int_Z E\|[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]f_3(s,x(s-r),y)\|^p ds\lambda dy \\
& +2^{p-1}C_pE\Big[\int_t^{t+\varepsilon}\int_Z\|\mathscr{I}_q(t+\varepsilon-s)f_3(s,x(s-r),y)\|^2 ds\lambda dy\Big]^{\frac{p}{2}} \\
& +2^{p-1}C_p\int_t^{t+\varepsilon}\int_Z E\|\mathscr{I}_q(t+\varepsilon-s)f_3(s,x(s-r),y)\|^p ds\lambda dy \\
\leqslant {} & 2^{p-1}C_pE\Big[\int_0^t\|[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]\|^2\int_Z\|f_3(s,x(s-r),y)\|^2\lambda(dy)ds\Big]^{\frac{p}{2}} \\
& +2^{p-1}C_p\int_0^t\|[\mathscr{I}_q(t+\varepsilon-s)-\mathscr{I}_q(t-s)]\|^p\int_Z E\|f_3(s,x(s-r),y)\|^p\lambda(dy)ds \\
& +2^{p-1}C_p\Big[\int_t^{t+\varepsilon}\|\mathscr{I}_q(t+\varepsilon-s)\|^2\int_Z E\|f_3(s,x(s-r),y)\|^2\lambda(dy)ds\Big]^{\frac{p}{2}} \\
& +2^{p-1}C_p\int_t^{t+\varepsilon}\|\mathscr{I}_q(t+\varepsilon-s)\|^p\int_Z E\|f_3(s,x(s-r),y)\|^p\lambda(dy)ds \\
\to {} & 0 \text{ as } \varepsilon\to 0;
\end{aligned}
$$

$$
\begin{aligned}
&E\|N_6(t+\varepsilon)-N_6(t)\|^p\\
=\ &E\Big\|\Big[\sum_{0<t_k<t+\varepsilon}\mathscr{I}_q(t+\varepsilon-t_k)I_k(x(t_k))-\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)I_k(x(t_k))\Big]\Big\|^p\\
\leqslant\ &2^{p-1}E\Big\|\sum_{0<t_k<t}[\mathscr{I}_q(t+\varepsilon-t_k)-\mathscr{I}_q(t-t_k)]I_k(x(t_k))\Big\|^p\\
&+2^{p-1}E\Big\|\sum_{t<t_k<t+\varepsilon}\mathscr{I}_q(t+\varepsilon-t_k)I_k(x(t_k))\Big\|^p\\
\to\ &0 \text{ as } \varepsilon\to 0;
\end{aligned}
$$

$$
\begin{aligned}
&E\|N_7(t+\varepsilon)-N_7(t)\|^p\\
=\ &E\Big\|\Big[\sum_{0<t_k<t+\varepsilon}\mathscr{K}_q(t+\varepsilon-t_k)J_k(x(t_k))-\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)J_k(x(t_k))\Big]\Big\|^p\\
\leqslant\ &2^{p-1}E\Big\|\sum_{0<t_k<t}[\mathscr{K}_q(t+\varepsilon-t_k)-\mathscr{K}_q(t-t_k)]J_k(x(t_k))\Big\|^p\\
&+2^{p-1}E\Big\|\sum_{t<t_k<t+\varepsilon}\mathscr{K}_q(t+\varepsilon-t_k)J_k(x(t_k))\Big\|^p\\
\to\ &0 \text{ as } \varepsilon\to 0.
\end{aligned}
$$

In view of above estimations, we obtain

$$
\lim_{\varepsilon\to 0}E\|(\Psi x)(t+\varepsilon)-(\Psi x)(t)\|^p=0,
$$

which implies that the function $t\to(\Psi x)(t)$ is continuous on $[0,T]$.

Then, we will verify that Ψ is a contraction operator in $\overline{PC_T}$. For $x_1(t),\ x_2(t)\in\overline{PC_T}$ and $t\in[0,T]$, together with conditions (H1)–(H3), we have

$$
\begin{aligned}
&E\|(\Psi x_1)(t)-(\Psi x_2)(t)\|^p\\
\leqslant\ &5^{p-1}E\Big\|\int_0^t\mathscr{K}_q(t-s)[f_1(s,x_1(s-\delta_1(s)))-f_1(s,x_2(s-\delta_1(s)))]ds\Big\|^p\\
&+5^{p-1}E\Big\|\int_0^t\mathscr{I}_q(t-s)[f_2(s,x_1(s-\delta_2(s)))-f_2(s,x_2(s-\delta_2(s)))]dW(s)\Big\|^p\\
&+5^{p-1}E\Big\|\int_0^t\int_Z\mathscr{I}_q(t-s)[f_3(s,x_1(s-\delta_3(s)),y)-f_3(s,x_2(s-\delta_3(s)),y)]\widetilde{N}(ds,dy)\Big\|^p\\
&+5^{p-1}E\Big\|\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)[I_k(x_1(t_k))-I_k(x_2(t_k))]\Big\|^p\\
&+5^{p-1}E\Big\|\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)[J_k(x_1(t_k))-J_k(x_2(t_k))]\Big\|^p\\
\leqslant\ &5^{p-1}E\Big[\int_0^t\Big\|\mathscr{K}_q(t-s)\Big\|_{L(X)}^{\frac{p}{p-1}}ds\Big]^{p-1}\\
&\times\int_0^t E\Big\|[f_1(s,x_1(s-\delta_1(s)))-f_1(s,x_2(s-\delta_1(s)))]ds\Big\|_X^p ds
\end{aligned}
$$

$$
\begin{aligned}
&+5^{p-1}C_p\Big[\int_0^t\Big[E\Big\|[\mathscr{I}_q(t-s)][f_2(s,x_1(s-\delta_2(s)))-f_2(s,x_2(s-\delta_2(s)))]\Big\|_{L_2^0}^p\Big]^{\frac{2}{p}}ds\Big]^{\frac{p}{2}}\\
\leqslant\ & 5^{p-1}C_pE\Big[\int_0^t\|\mathscr{I}_q(t-s)\|^2\\
&\times\int_Z\|[f_3(s,x_1(s-\delta_3(s)),y)-f_3(s,x_2(s-\delta_3(s)),y)]\|^2\lambda(dy)ds\Big]^{\frac{p}{2}}\\
&+5^{p-1}C_p\int_0^t\|\mathscr{I}_q(t-s)\|^p\\
&\times\int_Z E\|[f_3(s,x_1(s-\delta_3(s)),y)-f_3(s,x_2(s-\delta_3(s)),y)]\|^p\lambda(dy)ds\\
&+5^{p-1}E\Big\|\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)[I_k(x_1(t_k))-I_k(x_2(t_k))]\Big\|^p\\
&+5^{p-1}E\Big\|\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)[J_k(x_1(t_k))-J_k(x_2(t_k))]\Big\|^p\\
\leqslant\ & 5^{p-1}M^{*p}\Big[\beta_1^pT^2+C_p(\beta_2^pT^{\frac{p}{2}}+\beta_3^{\frac{p}{2}}T^{\frac{p}{2}}+\overline{\beta_3}T)+\Big((\sum_{k=1}^{\infty}c_k)^p+(\sum_{k=1}^{\infty}d_k)^p\Big)\Big]\\
&\times\sup_{\theta\in[-r,0]}\|x_1(t+\theta)-x_2(t+\theta)\|^p,
\end{aligned}
$$

which implies that

$$
\sup_{s\in[-r,t]}E\|(\Psi x_1)(t)-(\Psi x_2)(t)\|^p<\Omega\sup_{s\in[-r,t]}\|x_1(t+\theta)-x_2(t+\theta)\|^p. \tag{3.10}
$$

Since $\Omega<1$ from inequality (3.8), we obtain that Ψ is a contraction map. Therefore, Ψ has a unique fixed point $x(t)$ in $\overline{PC_T}$ which is the solution of the system (1.1) by the Banach fixed point theorem. □

4. Exponential stability in the pth moment

To obtain the exponential stability in the pth moment of mild solutions of (1.1), we need to establish an improved impulsive-integral inequality as follows firstly.

LEMMA 3. *For $\gamma_1,\ \gamma_2>0$, and there exist positive constants: ξ, ω, ξ^*, ω^*, ξ_k, ω_k $(k=1,2,\cdots)$ and a function $u:[-r,+\infty)\to[0,+\infty)$. If the following inequality:*

$$
u(t)\leqslant\begin{cases}
\xi e^{-\gamma_1 t}+\omega e^{-\gamma_2 t}+\xi^*\int_0^t e^{-\gamma_1(t-s)}\sup_{\theta\in[-r,0]}u(s+\theta)ds\\
+\omega^*\int_0^t e^{-\gamma_2(t-s)}\sup_{\theta\in[-r,0]}u(s+\theta)ds+\sum_{t_k<t}\xi_k e^{-\gamma_1(t-t_k)}u(t_k^-)\\
+\sum_{t_k<t}\omega_k e^{-\gamma_2(t-t_k)}u(t_k^-),\ t\geqslant 0,\\
\xi e^{-\gamma_1 t}+\omega e^{-\gamma_2 t},\ t\in[-r,0]
\end{cases}
$$

holds. Then, we have $u(t)\leqslant(\xi+\omega)e^{-\mu t}$ $(t\geqslant -r)$, where μ is a positive constant defined by $\mu=\min\{\gamma_1,\ \gamma_2\}-(\xi^*+\omega^*)e^{\min\{\gamma_1,\ \gamma_2\}r}-\overline{\xi}$, where $\overline{\xi}$ satisfies $\prod_{0<t_k<t}\lambda_k<e^{\overline{\xi}t}$ and $\overline{\xi}<\min\{\gamma_1,\ \gamma_2\}-(\xi^*+\omega^*)e^{\min\{\gamma_1,\ \gamma_2\}r}$, $\lambda_k=\max\{1+\xi_k+\omega_k,1\}$.

Proof. It is easy to see that $u(t) \leqslant (\xi+\omega)e^{-\mu t}$ for $t \in [-r,0]$. Then for any $t \geqslant 0$, we will also verify $u(t) \leqslant (\xi+\omega)e^{-\mu t}$.

Case 1. When $\gamma_1 \leqslant \gamma_2$, multiplying $e^{\gamma_1 t}$ on both sides of the first inequality of the lemma 4.1, we get

$$\begin{aligned} u(t)e^{\gamma_1 t} &\leqslant \xi + \omega e^{-(\gamma_2-\gamma_1)t} + \sum_{t_k<t} \xi_k e^{\gamma_1 t_k} u(t_k^-) + \sum_{t_k<t} \omega_k e^{\gamma_1 t} e^{-\gamma_2 (t-t_k)} u(t_k^-) \\ &\quad + \xi^* \int_0^t e^{\gamma_1 s} \sup_{\theta\in[-r,0]} u(s+\theta)ds + \omega^* \int_0^t e^{\gamma_1 t} e^{-\gamma_2 (t-s)} \sup_{\theta\in[-r,0]} u(s+\theta)ds \\ &\leqslant (\xi+\omega) + (\xi^*+\omega^*) \int_0^t e^{-\gamma_1 \theta} e^{\gamma_1 (s+\theta)} \sup_{\theta\in[-r,0]} u(s+\theta)ds + \sum_{t_k<t} (\xi_k+\omega_k) e^{\gamma_1 t_k} u(t_k^-) \\ &\leqslant (\xi+\omega) + (\xi^*+\omega^*)e^{\gamma_1 r} \cdot \int_0^t e^{\gamma_1 (s+\theta)} \sup_{\theta\in[-r,0]} u(s+\theta)ds + \sum_{t_k<t} (\xi_k+\omega_k) e^{\gamma_1 t_k} u(t_k^-). \end{aligned}$$

Let $v(t) = u(t)e^{\gamma_1 t}$, the above inequality is changed as

$$v(t) \leqslant (\xi+\omega) + (\xi^*+\omega^*)e^{\gamma_1 r} \cdot \int_0^t \sup_{\theta\in[-r,0]} v(s+\theta)ds + \sum_{t_k<t} (\xi_k+\omega_k) v(t_k).$$

Also, let

$$z(t) = (\xi+\omega) + (\xi^*+\omega^*)e^{\gamma_1 r} \cdot \int_0^t \sup_{\theta\in[-r,0]} v(s+\theta)ds + \sum_{t_k<t} (\xi_k+\omega_k) v(t_k). \tag{4.1}$$

Then, for $t \neq t_k$, we obtain

$$z'(t) = (\xi^*+\omega^*)e^{\gamma_1 r} \sup_{\theta\in[-r,0]} v(t+\theta) \leqslant (\xi^*+\omega^*)e^{\gamma_1 r} \sup_{\theta\in[-r,0]} z(t+\theta), \tag{4.2}$$

and

$$z(t_k^+) \leqslant \lambda_k z(t_k), \tag{4.3}$$

where $\lambda_k = \max\{1+\xi_k+\omega_k, 1\}$.

Next, consider the following equation

$$z'(t) = (\xi^*+\omega^*)e^{\gamma_1 r} \sup_{\theta\in[-r,0]} z(t+\theta), \tag{4.4}$$

it is easy to find that the solution of (4.4) is $z(t) = \zeta e^{[(\xi^*+\omega^*)e^{\gamma_1 r}]t}$, where $\zeta = \|z_0\|$. From the comparison principle, we know

$$v(t) \leqslant z(t) = \zeta e^{[(\xi^*+\omega^*)e^{\gamma_1 r}]t},\ t \in [-r, t_1] \tag{4.5}$$

and

$$z(t_k^+) \leqslant \zeta \lambda_1 e^{[(\xi^*+\omega^*)e^{\gamma_1 r}]t_1}. \tag{4.6}$$

For $t \in (t_1, t_2]$, from (4.5) and (4.6), we have

$$z(t) \leqslant \|z_{t_1}\| e^{[(\xi^*+\omega^*)e^{\gamma_1 r}](t-t_1)} \leqslant \|z_0\| \lambda_1 e^{[(\xi^*+\omega^*)e^{\gamma_1 r}]t}. \tag{4.7}$$

By the mathematical induction, for $t \in (t_k, t_{k+1}]$, we have

$$z(t) \leqslant \|z_0\| \prod_{t_k<t} \lambda_k e^{[(\xi^*+\omega^*)e^{\gamma_1 r}]t}. \tag{4.8}$$

Thus, we have

$$u(t) \leqslant \|z_0\| \prod_{t_k<t} \lambda_k e^{-[\gamma_1-(\xi^*+\omega^*)e^{\gamma_1 r}]t} = (\xi+\omega) \prod_{t_k<t} \lambda_k e^{-[\gamma_1-(\xi^*+\omega^*)e^{\gamma_1 r}]t} = (\xi+\omega) e^{-\mu_1 t}, \tag{4.9}$$

where $\mu_1 = \gamma_1 - (\xi^*+\omega^*)e^{\gamma_1 r} - \overline{\xi}$, $\overline{\xi}$ satisfies $\prod_{t_k<t} \lambda_k < e^{\overline{\xi} t}$ and $\overline{\xi} < \gamma_1 - (\xi^* + \omega^*)e^{\gamma_1 r}$.

Case 2. When $\gamma_2 \leqslant \gamma_1$, by the similar methods, we obtain

$$u(t) \leqslant (\xi+\omega) e^{-\mu_2 t}, \tag{4.10}$$

where $\mu_2 = \gamma_2 - (\xi^*+\omega^*)e^{\gamma_2 r} - \overline{\xi}$, $\overline{\xi}$ satisfies $\prod_{t_k<t} \lambda_k < e^{\overline{\xi} t}$ and $\overline{\xi} < \gamma_2 - (\xi^* + \omega^*)e^{\gamma_2 r}$.

Therefore, we can always obtain that

$$u(t) \leqslant (\xi+\omega) e^{-\mu t}, \tag{4.11}$$

where $\mu = \min\{\gamma_1, \gamma_2\} - (\xi^*+\omega^*)e^{\min\{\gamma_1, \gamma_2\} r} - \overline{\xi}$, $\overline{\xi}$ satisfies $\prod_{t_k<t} \lambda_k < e^{\overline{\xi} t}$ and $\overline{\xi} < \min\{\gamma_1, \gamma_2\} - (\xi^*+\omega^*)e^{\min\{\gamma_1, \gamma_2\} r}$. $\square$

REMARK 1. Compared with previous results [3], it is easy to see that the ξ_k and ω_k in our results is more simple than that in [3], which is required to satisfy $\frac{\xi^*}{\gamma_1} + \frac{\omega^*}{\gamma_2} + \sum_{k=1}^{+\infty} \xi_k + \sum_{k=1}^{+\infty} \omega_k < 1$. If $\sum_{k=1}^{+\infty} \xi_k \geqslant 1$ or $\sum_{k=1}^{+\infty} \omega_k \geqslant 1$, the corresponding Lemma in [3] will not hold. But in our result, ξ_k and ω_k can be greater than or equal to 1. When $\omega = \omega^* = \omega_k = 0$, our results can also improve the previous results [4, 17, 5, 6].

THEOREM 2. *Assume that conditions* (H1)–(H3) *hold, then the mild solution of the system* (1.1) *is exponentially stable in the* p*th moment.*

Proof. Taking the mathematical expectation for the mild solution of (1.1), we have

$$\begin{aligned} E\|x(t)\|^p = E\Big\| &\mathscr{I}_q(t)\varphi(0) + \mathscr{K}_q(t)y_1 + \int_0^t \mathscr{K}_q(t-s) f_1(s, x(s-\delta_1(s)))ds \\ &+ \int_0^t \mathscr{I}_q(t-s) f_2(s, x(s-\delta_2(s)))dW(s) \\ &+ \int_0^t \int_Z \mathscr{I}_q(t-s) f_3(s, x(s-\delta_3(s)), y)\widetilde{N}(ds, dy) \\ &+ \sum_{0<t_k<t} \mathscr{I}_q(t-t_k) I_k(x(t_k)) + \sum_{0<t_k<t} \mathscr{K}_q(t-t_k) J_k(x(t_k)) \Big\|^p \end{aligned}$$

$$
\begin{aligned}
&\leqslant 5^{p-1}E\Big\|\mathscr{I}_q(t)\varphi(0)+\mathscr{K}_q(t)y_1\Big\|^p\\
&\quad+5^{p-1}E\Big\|\int_0^t \mathscr{K}_q(t-s)f_1(s,x(s-\delta_1(s)))ds\Big\|^p\\
&\quad+5^{p-1}E\Big\|\int_0^t \mathscr{I}_q(t-s)f_2(s,x(s-\delta_2(s)))dW(s)\Big\|^p\\
&\quad+5^{p-1}E\Big\|\int_0^t\int_Z \mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\widetilde{N}(ds,dy)\Big\|^p\\
&\quad+5^{p-1}E\Big\|\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)I_k(x(t_k))+\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)J_k(x(t_k))\|^p\\
&=5^{p-1}\sum_{i=1}^{5}\Phi_i.
\end{aligned}
$$

From the assumption (H1) and Hölder inequality, we get

$$
\begin{aligned}
\Phi_1&=E\Big\|\mathscr{I}_q(t)\varphi(0)+\mathscr{K}_q(t)y_1\Big\|^p\\
&\leqslant 2^{p-1}M^{*p}E\|\varphi\|_0^p e^{-\gamma_1 t}+2^{p-1}M^{*p}E\|y_1\|^p e^{-\gamma_2 t}.
\end{aligned}
$$

Then, from the assumption (H1), (H2) and Hölder inequality, we have

$$
\begin{aligned}
\Phi_2&=E\Big\|\int_0^t \mathscr{K}_q(t-s)f_1(s,x(s-\delta_1(s)))ds\Big\|^p\\
&\leqslant E\Big(\int_0^t\|\mathscr{K}_q(t-s)\|\|f_1(s,x(s-\delta_1(s)))\|ds\Big)^p\\
&\leqslant M^{*p}E\Big(\int_0^t e^{-[\frac{\gamma_2(p-1)}{p}](t-s)}e^{-(\frac{\gamma_2}{p})(t-s)}\|f_1(s,x(s-\delta_1(s)))\|ds\Big)^p\\
&\leqslant M^{*p}\beta_1^p\Big(\int_0^t e^{-\gamma_2(t-s)}ds\Big)^{p-1}\int_0^t e^{-\gamma_2(t-s)}E\|x(s-\delta(s))\|^p ds\\
&\leqslant M^{*p}\beta_1^p\gamma_2^{1-p}\int_0^t e^{-\gamma_2(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds.
\end{aligned}
$$

Furthermore, by the assumption (H1), (H2), Hölder inequality, Lemma 2.1 and Lemma 2.2, we obtain

$$
\begin{aligned}
\Phi_3&=E\Big\|\int_0^t \mathscr{I}_q(t-s)f_2(s,x(s-\delta_2(s)))dW(s)\Big\|^p\\
&\leqslant C_pM^{*p}\Big(\int_0^t[e^{-\gamma_1 p(t-s)}E\|f_2(s,x(s-\delta_2(s)))ds\|_{L_2^0}^p]^{\frac{2}{p}}ds\Big)^{\frac{p}{2}}\\
&\leqslant C_pM^{*p}\beta_2^p\Big(\int_0^t[e^{-\gamma_1(p-1)(t-s)}e^{-\gamma_1(t-s)}E\|x(s-\delta_2(s))\|^p]^{\frac{2}{p}}ds\Big)^{\frac{p}{2}}\\
&\leqslant C_pM^{*p}\beta_2^p\Big(\int_0^t e^{-[\frac{2(p-1)}{p-2}]\gamma_1(t-s)}ds\Big)^{p-1}\int_0^t e^{-\gamma_1(t-s)}E\|x(s-\delta_2(s))\|^p ds\\
&\leqslant C_pM^{*p}\beta_2^p\Big(\frac{2\gamma_1(p-1)}{p-2}\Big)^{1-\frac{p}{2}}\int_0^t e^{-\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds
\end{aligned}
$$

and

$$
\begin{aligned}
\Phi_4 &= E\Big\|\int_0^t\int_Z \mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\widetilde{N}(ds,dy)\Big\|^p \\
&\leqslant C_p\Big\{E\Big(\int_0^t\int_Z\Big\|\mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\Big\|^2 ds\lambda dy\Big)^{\frac{p}{2}} \\
&\quad+\int_0^t\int_Z\Big\|\mathscr{I}_q(t-s)f_3(s,x(s-\delta_3(s)),y)\Big\|^p ds\lambda dy\Big\} \\
&\leqslant C_pM^{*p}\Big\{\Big(\int_0^t e^{-2\gamma_1(t-s)}\int_Z E\Big\|f_3(s,x(s-\delta_3(s)),y)\Big\|^2\lambda dsdy\Big)^{\frac{p}{2}} \\
&\quad+\int_0^t e^{-p\gamma_1(t-s)}\int_Z E\Big\|f_3(s,x(s-\delta_3(s)),y)\Big\|^p\lambda dsdy \\
&\leqslant C_pM^{*p}\beta_3^{\frac{p}{2}}\Big(\int_0^t e^{-\frac{2(p-1)}{p}\gamma_1(t-s)}e^{-\frac{2}{p}\gamma_1(t-s)}E\|x(s-\delta_3(s))\|^2 ds\Big)^{\frac{p}{2}} \\
&\quad+C_pM^{*p}\overline{\beta_3}\int_0^t e^{-p\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds \\
&\leqslant C_pM^{*p}\beta_3^{\frac{p}{2}}\Big(\int_0^t e^{-\frac{2(p-1)}{p}\cdot\frac{p}{p-2}\gamma_1(t-s)}ds\Big)^{\frac{p-2}{2}}\int_0^t e^{-\gamma_1(t-s)}E\|x(s-\delta_3(s))\|^p ds \\
&\quad+C_pM^{*p}\overline{\beta_3}\int_0^t e^{-p\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds \\
&\leqslant C_pM^{*p}(\beta_3^{\frac{p}{2}}+\overline{\beta_3})\Big(\frac{p-2}{2(p-1)\gamma_1}\Big)^{\frac{p-2}{2}}\int_0^t e^{-\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds.
\end{aligned}
$$

Next, for $p\geqslant 2$ and $1<m\leqslant 2$ with $\frac{1}{p}+\frac{1}{m}=1$ and the assumption (H3), we obtain

$$
\begin{aligned}
\Phi_5 &= E\Big\|\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)I_k(x(t_k))+\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)J_k(x(t_k))\Big\|^p \\
&\leqslant 2^{p-1}\Big[E\Big\|\sum_{0<t_k<t}\mathscr{I}_q(t-t_k)I_k(x(t_k))\Big\|^p+E\Big\|\sum_{0<t_k<t}\mathscr{K}_q(t-t_k)J_k(x(t_k))\Big\|^p\Big] \\
&\leqslant 2^{p-1}M^{*p}E\Big[\Big\|\sum_{0<t_k<t}c_ke^{-\gamma_1(t-t_k)}\|x(t_k)\Big\|^p+\Big\|\sum_{0<t_k<t}d_ke^{-\gamma_2(t-t_k)}\|x(t_k)\Big\|^p\Big] \\
&\leqslant 2^{p-1}M^{*p}\Big[(\sum_{k=1}^{\infty}c_k)^{\frac{p}{m}}\sum_{0<t_k<t}c_ke^{-\gamma_1(t-t_k)}+(\sum_{k=1}^{\infty}d_k)^{\frac{p}{m}}\sum_{0<t_k<t}d_ke^{-\gamma_2(t-t_k)}\Big]\Big\|x(t_k)\Big\|^p.
\end{aligned}
$$

Finally, combining with the estimations of $\Phi_1-\Phi_5$ for $t\geqslant 0$, we have

$$
\begin{aligned}
&E\|x(t)\|^p \\
&\leqslant 10^{p-1}M^{*p}E\|\varphi\|^pe^{-\gamma_1 t}+10^{p-1}M^{*p}E\|y_1\|^pe^{-\gamma_2 t} \\
&\quad+5^{p-1}M^{*p}\beta_1^p\gamma_2^{1-p}\int_0^t e^{-\gamma_2(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds \\
&\quad+5^{p-1}C_pM^{*p}\beta_2^p\Big(\frac{2\gamma_1(p-1)}{p-2}\Big)^{1-\frac{p}{2}}\int_0^t e^{-\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds
\end{aligned}
$$

$$
\begin{aligned}
&+5^{p-1}C_pM^{*p}(\beta_3^{\frac{p}{2}}+\overline{\beta_3})\Big(\frac{p-2}{2(p-1)\gamma_1}\Big)^{\frac{p-2}{2}}\int_0^t e^{-\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^p ds\\
&+10^{p-1}M^{*p}\Big[(\sum_{k=1}^{\infty}c_k)^{\frac{p}{m}}\sum_{0<t_k<t}c_ke^{-\gamma_1(t-t_k)}+(\sum_{k=1}^{\infty}d_k)^{\frac{p}{m}}\sum_{0<t_k<t}d_ke^{-\gamma_2(t-t_k)}\Big]\Big\|x(t_k)\Big\|^p\\
=\;&10^{p-1}M^{*p}E\|\varphi\|^pe^{-\gamma_1 t}+10^{p-1}M^{*p}E\|y_1\|^pe^{-\gamma_2 t}\\
&+5^{p-1}M^{*p}\beta_1^p\gamma_2^{1-p}\int_0^t e^{-\gamma_2(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^pds\\
&+5^{p-1}C_pM^{*p}\Gamma\int_0^t e^{-\gamma_1(t-s)}\sup_{\theta\in[-r,0]}E\|x(s+\theta)\|^pds\\
&+10^{p-1}M^{*p}\Big[(\sum_{k=1}^{\infty}c_k)^{\frac{p}{m}}\sum_{0<t_k<t}c_ke^{-\gamma_1(t-t_k)}+(\sum_{k=1}^{\infty}d_k)^{\frac{p}{m}}\sum_{0<t_k<t}d_ke^{-\gamma_2(t-t_k)}\Big]\Big\|x(t_k)\Big\|^p,
\end{aligned}
$$

where

$$\Gamma=\Big(\beta_2^p+\beta_3^{\frac{p}{2}}+\overline{\beta_3}\Big)\Big(\frac{2\gamma_1(p-1)}{p-2}\Big)^{1-\frac{p}{2}}.$$

And it is obvious to see that for $t\in[-r,0]$, we have

$$E\|x(t)\|^p\leqslant M_0e^{-\gamma t},$$

where $M_0=\max\Big\{10^{p-1}M^{*p}[E\|\varphi\|_0^p+E\|y_1\|^p],\ E\|\varphi\|_0^p\Big\}$. Then, by lemma 4.1, for all $t\geqslant -r$, we have

$$E\|x(t)\|^p\leqslant M_0E\|\varphi\|_0e^{-\gamma t},$$

where $\gamma=\min\{\gamma_1,\ \gamma_2\}-\theta-\overline{\xi}$, $\theta=5^{p-1}M^{*p}(\beta_1^p\gamma_2^{1-p}+C_pM^{*p}\Gamma)e^{\min\{\gamma_1,\ \gamma_2\}r}$, $\overline{\xi}$ satisfies $\prod_{t_k<t}\lambda_k<e^{\overline{\xi}t}$, $\lambda_k=1+10^{p-1}M^{*p}\Big[(\sum_{k=1}^{\infty}c_k)^{\frac{p}{m}}\sum_{0<t_k<t}c_k+(\sum_{k=1}^{\infty}d_k)^{\frac{p}{m}}\sum_{0<t_k<t}d_k\Big]$ and $\overline{\xi}<\min\{\gamma_1,\ \gamma_2\}-5^{p-1}M^{*p}(\beta_1^p\gamma_2^{1-p}+C_pM^{*p}\Gamma)e^{\min\{\gamma_1,\ \gamma_2\}r}$. So, we can obtain that the mild solutions of the system (1.1) is exponentially stable in the pth moment. $\square$

5. An example

In this section, an example is given to illustrate the effectiveness and feasibility of the theoretical results in our paper.

EXAMPLE 5.1. Consider the following system:

$$
\begin{cases}
{}^cD_t^{0.5}u(t,x)=\frac{\partial^2}{\partial x^2}u(t,x)+\mathbb{J}_t^{1.5}\Big(\beta_1\sin u(\frac{t}{4},x))\Big)+\mathbb{J}_t^{1.5}\Big(\beta_2\sin u(\frac{t}{3},x))\frac{dW(t)}{dt}\\
+\int_Z\beta_3\sin u(\frac{t}{2},x),y)\widetilde{N}(dt,dy)\Big),\ t\in[0,\pi],\ t\neq t_k,\ x\in[0,\pi],\\
u(t,0)=u(t,\pi)=0,\ t\in[0,\pi],\\
\Delta u(t_k,x)=c_ku(t_k,x),\ \Delta x'(t_k)=d_ku(t_k,x),\ t=t_k,\ k=1,2,\cdots,\\
u(t,x)=\varphi(t,x),\ t\in(-r,0],\ x\in[0,\pi],
\end{cases}
\tag{5.1}
$$

where $W(t)$ is a standard cylindrical Wiener process in X, $A: D(A) \subset X \to X$, which is defined by $Ay = z''$ with the domain $D(A) = \{y \in X,\ y,\ y'$ are absolutely continuous $y'' \in X,\ y(0) = y(\pi) = 0\}$ then

$$Ay = \sum_{n=1}^{\infty} n^2 (y, y_n) y_n, \quad y \in D(A),$$

where $y_n(x) = \sqrt{\frac{2}{\pi}} \sin(nx),\ n \in N$ is the orthonormal set of eigenvectors of A. Also, A is the infinitesimal generator of an analytic semigroup $(S(t))_{t \geqslant 0}$ in X and $\|S(t)\| \leqslant e^{-\pi^2 t}$.

Then, (5.1) can be transformed in the abstract form of (1.1), where $f_1(t,x) = \beta_1 \sin x,\ f_2(t,x) = \beta_2 \sin x,\ f_3(t,x,y) = \beta_3 y \sin x$, the delay functions are $\delta_1(t) = \frac{t}{4}$, $\delta_2(t) = \frac{t}{3},\ \delta_3(t) = \frac{t}{2}$, the impulsive functions are $I_k(x) = c_k x,\ J_k(x) = d_k x,\ k \in N$. Thus, it is easy to verify the conditions (H1)–(H3) of Theorem 1 all hold, so the system (5.1) has at least a mild solution on $[0, \pi]$.

Next, we will prove that the mild solution of (5.1) is exponentially stable in the 4th moment $(p = 4)$. In fact, we know $M^* = 1$, $\gamma_1 = \gamma_2 = \pi^2$. Let $\beta_1 = \beta_2 = \beta_3 = 0.1$, $\overline{\beta_3} = 0.001$, $r = \pi^{-2}$, $c_1 = c_2 = d_1 = d_2 = 0.1$, $k = 2$, by simple calculation, we have $\theta \approx 1.5366$, $\lambda_1 = 1.2$, $\lambda_2 = 4.2$, then let $\overline{\xi} = 6$, so $\gamma = \min\{\gamma_1, \gamma_2\} - \theta - \overline{\xi} \approx 2.3230 > 0$. Thus, the conditions of Theorem 2 all hold, so the mild solution of the system (5.1) is exponentially stable in the 4th moment.

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