Journal of Mathematical Inequalities
Volume 17, Number 3 (2023), 931–942 doi:10.7153/jmi-2023-17-60

THE THIRD–ORDER HERMITIAN TOEPLITZ DETERMINANT FOR SOME CLASSES OF ANALYTIC FUNCTIONS

ADAM LECKO* AND BARBARA ŚMIAROWSKA

(*Communicated by V. R. Allu*)

Abstract. Sharp lower and upper bounds are found of the second and third-order Hermitian Toeplitz determinants for some classes of analytic functions.

1. Introduction

Let $\mathcal{H}$ be the class of analytic functions in $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and $\mathcal{A}$ be its subclass of functions f of the form

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \quad a_1 = 1,\ z \in \mathbb{D}. \tag{1}$$

Let $\mathcal{S}$ be the subclass of $\mathcal{A}$ of all univalent functions. Given $\alpha \in [0,1)$, let $\mathcal{P}'(\alpha)$ and $\mathcal{T}(\alpha)$ denote the subclass of $\mathcal{A}$ of all f satisfying

$$\operatorname{Re} f'(z) > \alpha, \quad z \in \mathbb{D}, \tag{2}$$

and

$$\operatorname{Re} \frac{f(z)}{z} > \alpha, \quad z \in \mathbb{D}. \tag{3}$$

Functions f in $\mathcal{P}'(\alpha)$ are called of bounded turning of order α. Particularly, elements of $\mathcal{P}'(0) =: \mathcal{P}'$ are called of bounded turning (cf. [9, Vol. I, p. 101]). On the other hand, the condition (2) with $\alpha = 0$ is known as a famous criterium of univalence due to Alexander [1] (cf. [9, Vol. I, p. 88]) which means that $\mathcal{P}' \subset \mathcal{S}$. Since $\mathcal{P}'(\alpha) \subset \mathcal{P}'$ for $\alpha \in [0,1)$, it follows that $\mathcal{P}'(\alpha) \subset \mathcal{S}$ for all $\alpha \in [0,1)$. The class $\mathcal{P}'$ is one of the fundamental subfamily of univalent functions and has been extensively studied by many authors e.g., [14], [13].

The family $\mathcal{T}(\alpha)$, particularly $\mathcal{T}(0) =: \mathcal{T}$, plays an important role in the geometric function theory although their elements are functions which are not necessarily

Mathematics subject classification (2020): 30C45, 30C50.
Keywords and phrases: Hermitian Toeplitz determinants, function of bounded turning, analytic functions, Carathéodory class.
* Corresponding author.

Paper JMI-17-60

univalent. One of the significant results belongs to Marx [16] and Strohhäcker [22]. They proved that

$$\mathscr{S}^c \subset \mathscr{S}^*(1/2) \subset \mathscr{T}(1/2) \tag{4}$$

(see also [17, Theorem 2.6a, p. 57]), where $\mathscr{S}^c$ stands for the class of convex functions introduced by Study [23], i.e., the subfamily of $\mathscr{S}$ of all univalent functions which map $\mathbb{D}$ onto convex domains, and $\mathscr{S}^*(1/2)$ means the class of starlike functions of order $1/2$. Functions starlike of order α ($\alpha \in [0,1)$) were introduced by Robertson [20]. By the well known result due to Study ([23], see also [7, p. 42]) a function $f \in \mathscr{A}$ belongs to $\mathscr{S}^c$ if and only if

$$\operatorname{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} > 0, \quad z \in \mathbb{D}.$$

A function $f \in \mathscr{A}$ is in $\mathscr{S}^*(\alpha)$ ($\alpha \in [0,1)$ ([20], see also [9, Vol. I, p. 138] if and only if

$$\operatorname{Re}\frac{zf'(z)}{f(z)} > \alpha, \quad z \in \mathbb{D}.$$

The class $\mathscr{T}$ plays a fundamental role in the theory of semigroups of analytic functions as a generator of one-parameter continues semigroups studied by Berkson, Porta, Shoikhet, Elin and others (e.g., [21], [8]). For other classical results concerning the classes $\mathscr{T}$ and $\mathscr{T}(1/2)$ see e.g., [15], [19].

Given $q,n \in \mathbb{N}$, define the matrix $T_{q,n}(f)$ of a function $f \in \mathscr{A}$ of the form (1) by

$$T_{q,n}(f) := \begin{bmatrix} a_n & a_{n+1} & \dots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \dots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \dots & a_n \end{bmatrix},$$

where $\overline{a}_k := \overline{a_k}$. In the case when a_n is a real number, $T_{q,n}(f)$ is called the Hermitian Toeplitz matrix. In particular,

$$\begin{aligned} \det T_{3,1}(f) &= \begin{vmatrix} 1 & a_2 & a_3 \\ \overline{a}_2 & 1 & a_2 \\ \overline{a}_3 & \overline{a}_2 & 1 \end{vmatrix} \\ &= 1+2\operatorname{Re}\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2. \end{aligned} \tag{5}$$

In recent years a lot of papers has been devoted to the estimation of determinants whose entries are coefficients of functions in the class $\mathscr{A}$ or its subclasses. Hankel matrices i.e., square matrices which have constant entries along the reverse diagonal (see e.g., [4], [5] and [12], with further references), and the symmetric Toeplitz determinant (see [2]) are of particular interest.

For this reason looking on the interest of specialists in [6], [10] and [11] the study of the Hermitian Toeplitz determinants $T_{q,1}(f)$ on the class $\mathscr{A}$ or its subclasses has begun. Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics as well as in physics and technical sciences.

In this paper, we continue the research study of the Hermitian Toeplitz determinants. We will find sharp lower and upper bounds of the second and third order Hermitian Toeplitz determinants for the classes $\mathscr{P}'(\alpha)$ and $\mathscr{T}(\alpha)$ with $\alpha \in [0,1)$.

Let $\mathscr{P}$ be the class of all $p \in \mathscr{H}$ of the form

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n, \quad z \in \mathbb{D}, \tag{6}$$

which have positive real part.

In the proof of the main result we will use the following lemma which contains the well-known formulas for c_1 and c_2 ([3], [18, p. 166]) with further remarks in [5]).

LEMMA 1.1. *If $p \in \mathscr{P}$ is of the form (6), then*

$$c_1 = 2\zeta_1 \tag{7}$$

and

$$c_2 = 2\zeta_1^2 + 2(1 - |\zeta_1|^2)\zeta_2 \tag{8}$$

for some $\zeta_i \in \overline{\mathbb{D}}$, $i \in \{1,2\}$.

For $\zeta_1 \in \mathbb{T}$, there is a unique function $p \in \mathscr{P}$ with c_1 as in (7), namely,

$$p(z) = \frac{1+\zeta_1 z}{1-\zeta_1 z}, \quad z \in \mathbb{D}.$$

For $\zeta_1 \in \mathbb{D}$ and $\zeta_2 \in \mathbb{T}$, there is a unique function $p \in \mathscr{P}$ with c_1 and c_2 as in (7) and (8), namely,

$$p(z) = \frac{1+(\overline{\zeta_1}\zeta_2 + \zeta_1)z + \zeta_2 z^2}{1+(\overline{\zeta_1}\zeta_2 - \zeta_1)z - \zeta_2 z^2}, \quad z \in \mathbb{D}. \tag{9}$$

Recall now the following observation ([11]). Given a compact subclass $\mathscr{F}$ of $\mathscr{A}$, let $A_2(\mathscr{F}) := \max\{|a_2| : f \in \mathscr{F}\}$. Thus if $f \in \mathscr{A}$, ten

$$\det T_{2,1}(f) = 1 - |a_2|^2,$$

and the result below is clear. Equality for the lower bound is attained by a function in $\mathscr{F}$ which is extremal for $A_2(\mathscr{F})$ and for the upper bound when f is the identity function.

THEOREM 1.2. *Let $\mathscr{F}$ be a compact subclass of $\mathscr{A}$. If the identity is an element of $\mathscr{F}$, then*

$$1 - A_2^2(\mathscr{F}) \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

2. The class $\mathscr{P}'(\alpha)$

We will now find the upper and lower bounds of $\det T_{2,1}(f)$ and $\det T_{3,1}(f)$ in the class of $\mathscr{P}'(\alpha)$.

Let $\alpha \in [0,1)$. Since by (14), $A_2(\mathscr{P}'(\alpha)) = 1-\alpha$ with the extremal function $f \in \mathscr{A}$ satisfying

$$f'(z) = \frac{1+(1-2\alpha)z}{1-z}, \quad z \in \mathbb{D}, \tag{10}$$

and since the identity function belongs to the class $\mathscr{P}'(\alpha))$, by Theorem 1.2 we have

THEOREM 2.1. *Let $\alpha \in [0,1)$. If $f \in \mathscr{P}'(\alpha)$, then*

$$\alpha(2-\alpha) \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

Particularly, for $\alpha = 0$ i.e., for the class $\mathscr{P}'$ we have

COROLLARY 2.2. *If $f \in \mathscr{P}'$, then*

$$0 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

Now we will compute the upper and lower bounds of $\det T_{3,1}(f)$.

THEOREM 2.3. *Let $\alpha \in [0,1)$. If $f \in \mathscr{P}'(\alpha)$, then*

$$\det T_{3,1}(f) \leqslant 1, \tag{11}$$

and

$$\det T_{3,1}(f) \geqslant \begin{cases} \dfrac{(2\alpha^2-7\alpha+1)^2}{8(3\alpha-1)}, & 0 \leqslant \alpha < \dfrac{1}{18}, \\ -\dfrac{1}{9}(2\alpha+1)(6\alpha^2-10\alpha+1), & \dfrac{1}{18} \leqslant \alpha < 1, \end{cases} \tag{12}$$

All inequalities are sharp.

Proof. Fix $\alpha \in [0,1)$ and let $f \in \mathscr{P}'(\alpha)$ be of the form (1). Then by (2),

$$f'(z) = (1-\alpha)p(z) + \alpha, \quad z \in \mathbb{D}, \tag{13}$$

for a certain $p \in \mathscr{P}$ of the form (6). Putting the series (1) and (6) into (13), by equating the coefficients we get

$$a_2 = \frac{1}{2}(1-\alpha)c_1, \quad a_3 = \frac{1}{3}(1-\alpha)c_2. \tag{14}$$

Since the class $\mathscr{P}'(\alpha)$ and $\det T_{3,1}$ are rotation invariant, we may assume that $a_2 \geqslant 0$, i.e., by (14) that $c_1 \geqslant 0$, so by (7) that $\zeta_1 \in [0,1]$.

A. Since by (14),

$$|a_3| \leqslant \frac{2}{3}(1-\alpha) < 1,$$

by (5) we have

$$\begin{aligned}\det T_{3,1}(f) &= 1 + 2\operatorname{Re}\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2\\ &= 1 - 2a_2^2(1-\operatorname{Re}(\overline{a}_3)) - |a_3|^2\\ &\leqslant 1,\end{aligned}$$

which shows (11).

B. Now we show the inequality (12). From (5) with (14), (7) and (8) we have

$$\begin{aligned}9\det T_{3,1}(f) =& -4(1-\alpha)^2(3\alpha-2)\zeta_1^4 - 18(1-\alpha)^2\zeta_1^2 + 9\\ & -4(1-\alpha)^2(3\alpha-1)(1-\zeta_1^2)\operatorname{Re}\left(\zeta_1^2\overline{\zeta_2}\right)\\ & -4(1-\alpha)^2(1-\zeta_1^2)^2|\zeta_2|^2\end{aligned} \tag{15}$$

for some $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$.

Define

$$\begin{aligned}F(x,y,t) :=& -4(1-\alpha)^2(3\alpha-2)x^2 - 18(1-\alpha)^2x + 9\\ & -4(1-\alpha)^2(3\alpha-1)(1-x)xy\cos t - 4(1-\alpha)^2(1-x)^2y^2\end{aligned}$$

for $x,y \in [0,1]$ and $t \in [0,2\pi]$.

When $\zeta_2 \neq 0$, then $\zeta_2 = |\zeta_2|e^{i\theta}$ for a unique $\theta \in [0,2\pi)$. Thus by (15),

$$9\det T_{3,1}(f) = F(\zeta_1^2, |\zeta_2|, \theta).$$

When $\zeta_2 = 0$, then by (15),

$$9\det T_{3,1}(f) = F(\zeta_1^2, 0, \theta) = F(\zeta_1^2, 0, 0).$$

We now find the minimum value of F.

B1. Suppose first that $\alpha \in [0,1/3)$. Then

$$\begin{aligned}F(x,y,t) \geqslant& F(x,y,\pi)\\ =& -4(1-\alpha)^2(3\alpha-2)x^2 - 18(1-\alpha)^2x + 9\\ & +4(1-\alpha)^2(3\alpha-1)(1-x)xy - 4(1-\alpha)^2(1-x)^2y^2\\ =&: J(x,y), \quad x,y \in [0,1],\ t \in [0,2\pi].\end{aligned}$$

(a) For $x = 1$,

$$\begin{aligned}J(1,y) &= -4(1-\alpha)^2(3\alpha-2) - 18(1-\alpha)^2 + 9\\ &= -(2\alpha+1)(6\alpha^2-10\alpha+1), \quad y \in [0,1].\end{aligned}$$

(b) Let $x \in [0,1)$. Because

$$y_w := \frac{(3\alpha-1)x}{2(1-x)} < 0,$$

it follows that

$$J(x,y) \geqslant J(x,1) = 8(1-3\alpha)(\alpha-1)^2x^2 + 2(6\alpha-7)(\alpha-1)^2x - (2\alpha+1)(2\alpha-5), \quad x \in [0,1).$$

Set

$$x_w := \frac{6\alpha-7}{8(3\alpha-1)}.$$

(c) Since $x_w < 1$ holds for $\alpha \in [0,1/18)$, then

$$J(x,1) \geqslant J(x_w,1) = \frac{9(2\alpha^2-7\alpha+1)^2}{8(3\alpha-1)}, \quad x \in [0,1).$$

Note additionally here that $J(1,1) \geqslant J(x_w,1)$ for $x \in [0,1)$, i.e., that for $\alpha \in [0,1/18)$,

$$-(2\alpha+1)(6\alpha^2-10\alpha+1) \geqslant \frac{9(2\alpha^2-7\alpha+1)^2}{8(3\alpha-1)}$$

equivalent to

$$(3\alpha-1)(18\alpha-1)^2(\alpha-1)^2 \leqslant 0,$$

which is true for $\alpha \in [0,1/18)$.

(d) It remains to consider the case $x_w \geqslant 1$ which holds only when $1/18 \leqslant \alpha < 1/3$. Then

$$J(x,1) \geqslant J(1,1) = -(2\alpha+1)(6\alpha^2-10\alpha+1), \quad x \in [0,1).$$

B2. Suppose now that $\alpha \in [1/3,1)$. Then

$$\begin{aligned} F(x,y,t) \geqslant & F(x,y,0) \\ &= -4(1-\alpha)^2(3\alpha-2)x^2 - 18(1-\alpha)^2x + 9 \\ &\quad - 4(1-\alpha)^2(3\alpha-1)(1-x)xy - 4(1-\alpha)^2(1-x)^2y^2 \\ &=: K(x,y), \quad x,y \in [0,1],\ t \in [0,2\pi]. \end{aligned}$$

(a) For $x=1$,

$$\begin{aligned} K(1,y) &= -4(1-\alpha)^2(3\alpha-2) - 18(1-\alpha)^2 + 9 \\ &= -(2\alpha+1)(6\alpha^2-10\alpha+1), \quad y \in [0,1]. \end{aligned}$$

(b) Let $x \in [0,1)$. Because

$$y_w := -\frac{(3\alpha-1)x}{2(1-x)} \leqslant 0,$$

we see that

$$\begin{aligned}K(x,y) &\geqslant K(x,1)\\ &= -6(2\alpha+1)(\alpha-1)^2x-(2\alpha+1)(2\alpha-5)\\ &\geqslant -(2\alpha+1)(6\alpha^2-10\alpha+1)=K(1,1), \quad x\in[0,1).\end{aligned}$$

C. Note that for $\zeta_2=0$, by (2),

$$9\det T_{3,1}(f)=F(\zeta_1^2,0,0)=G(\zeta_1^2,0)=H(\zeta_1^2,0).$$

D. Summarizing, from Parts A-C we get inequalities (11) and (12), respectively.

It remains to show sharpness of all inequalities. The identity function is extremal for the first inequality in (11). The function f given by (10) for which $a_2=1-\alpha$ and $a_3=2(1-\alpha)/3$ is extremal for the second inequality in (12).

Let now $0\leqslant\alpha<1/18$. Set

$$\tau:=\sqrt{\frac{6\alpha-7}{8(3\alpha-1)}}=\sqrt{x_w}.$$

Since $\tau\leqslant 1$, by (9) the function

$$\tilde{p}(z):=\frac{1-z^2}{1-2\tau z+z^2}=1+2\tau z+(4\tau^2-2)z^2+\cdots, \quad z\in\mathbb{D},$$

belongs to $\mathscr{P}$. Thus the function f given by (13), where p is replaced by $\tilde{p}$, being of the form (1) with

$$a_2=(1-\alpha)\tau, \quad a_3=\frac{2}{3}(1-\alpha)(2\tau^2-1),$$

belongs to $\mathscr{P}'(\alpha)$ and is extremal for the first inequality in (12). This ends the proof of the theorem. $\square$

Particularly, for $\alpha=0$ we get the followning result.

COROLLARY 2.4. *If $f\in\mathscr{P}'$, then*

$$-\frac{1}{8}\leqslant\det T_{3,1}(f)\leqslant 1.$$

Both inequalities are sharp.

3. The class $\mathscr{T}(\alpha)$

Now we will compute the sharp upper and lower bounds of $\det T_{2,1}(f)$ and $\det T_{3,1}(f)$ in the class $\mathscr{T}(\alpha)$.

Let $\alpha\in[0,1)$. Since by (20) below, $A_2(\mathscr{F}_2(\alpha))=2(1-\alpha)$ with the extremal function

$$f(z)=z\frac{1+(1-2\alpha)z}{1-z}, \quad z\in\mathbb{D}, \tag{16}$$

and since the identity function belongs to the class $\mathscr{T}(\alpha)$, by Theorem 1.2 we have

THEOREM 3.1. *Let* $\alpha \in [0,1)$. *If* $f \in \mathscr{T}(\alpha)$, *then*

$$(2\alpha-1)(3-2\alpha) \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

Particularly, for $\alpha = 0$, we have the following result.

COROLLARY 3.2. *If* $f \in \mathscr{T}$, *then*

$$-3 \leqslant \det T_{2,1}(f) \leqslant 1.$$

Both inequalities are sharp.

Now we will estimate $\det T_{3,1}(f)$.

THEOREM 3.3. *Let* $\alpha \in [0,1)$. *If* $f \in \mathscr{T}(\alpha)$, *then*

$$\det T_{3,1}(f) \leqslant \begin{cases} (5-4\alpha)(2\alpha-1)^2, & 0 \leqslant \alpha < \frac{1}{4}, \\ 1, & \frac{1}{4} \leqslant \alpha < 1, \end{cases} \tag{17}$$

and

$$\det T_{3,1}(f) \geqslant \begin{cases} (2\alpha-1)(\alpha-2)^2, & 0 \leqslant \alpha < \frac{1}{2}, \\ (5-4\alpha)(2\alpha-1)^2, & \frac{1}{2} \leqslant \alpha < 1, \end{cases} \tag{18}$$

All inequalities are sharp.

Proof. Fix $\alpha \in [0,1)$ and let $f \in \mathscr{T}(\alpha)$ be of the form (1). Then by (3),

$$\frac{f(z)}{z} = (1-\alpha)p(z) + \alpha, \quad z \in \mathbb{D}, \tag{19}$$

for a certain $p \in \mathscr{P}$ of the form (6). Substituting the series (1) and (6) into (19), by equating the coefficients we get

$$a_2 = (1-\alpha)c_1, \quad a_3 = (1-\alpha)c_2. \tag{20}$$

Since the class $\mathscr{T}(\alpha)$ and $\det T_{3,1}$ are rotation invariant, we may assume that $a_2 \geqslant 0$, i.e., by (20) that $c_1 \geqslant 0$, so by (7) that $\zeta_1 \in [0,1]$.

A. By (20),

$$0 \leqslant a_2 \leqslant 2(1-\alpha), \quad |a_3| \leqslant 2(1-\alpha), \tag{21}$$

and by (5),

$$\begin{aligned} \det T_{3,1}(f) &= 1 + 2\,\mathrm{Re}\left(a_2^2\overline{a}_3\right) - 2|a_2|^2 - |a_3|^2 \\ &= 1 + 2a_2^2(\mathrm{Re}(\overline{a_3}) - 1) - |a_3|^2. \end{aligned} \tag{22}$$

A1. Suppose first that $\operatorname{Re}(a_3) \leqslant 1$. Then from (22) we get at once

$$\det T_{3,1}(f) \leqslant 1. \tag{23}$$

A2. Suppose now that $\operatorname{Re}(a_3) > 1$. Then from (22) and (21) we obtain

$$\begin{aligned} \det T_{3,1}(f) &= 1 - |a_3|^2 + 2\left(\operatorname{Re}(\overline{a}_3) - 1\right) a_2^2 \\ &\leqslant 1 - |a_3|^2 + 8\left(|a_3| - 1\right)(1-\alpha)^2 = \gamma(|a_3|), \end{aligned} \tag{24}$$

where

$$\gamma(t) := -t^2 + 8(1-\alpha)^2 t + 1 - 8(1-\alpha)^2, \quad 0 \leqslant t \leqslant 2(1-\alpha).$$

Observe that for $0 \leqslant t \leqslant 2(1-\alpha)$,

$$|\gamma(t)| \leqslant \begin{cases} (5-4\alpha)(2\alpha-1)^2, & 0 \leqslant \alpha \leqslant \dfrac{1}{2}, \\ \left(4(1-\alpha)^2 - 1\right)^2, & \dfrac{1}{2} < \alpha < 1. \end{cases} \tag{25}$$

Since $1 \geqslant \left(4(1-\alpha)^2 - 1\right)^2$ for $1/2 < \alpha < 1$ and $1 \geqslant (5-4\alpha)(2\alpha-1)^2$ for $1/4 \leqslant \alpha \leqslant 1/2$, we deduce from (23), (24) and (25) that the upper bound (17) is true.

B. Now we show the inequality (18). From (5) with (20), (7) and (8) we have

$$\begin{aligned} \det T_{3,1}(f) = &-4(4\alpha-3)(\alpha-1)^2\zeta_1^4 - 8(\alpha-1)^2\zeta_1^2 + 1 \\ &- 8(2\alpha-1)(\alpha-1)^2(1-\zeta_1^2)\zeta_1^2 \operatorname{Re}(\zeta_2) \\ &- 4(\alpha-1)^2(1-\zeta_1^2)^2|\zeta_2|^2 \end{aligned} \tag{26}$$

for some $\zeta_1, \zeta_2 \in \overline{\mathbb{D}}$.

Define

$$\begin{aligned} F(x,y,t) := &-4(4\alpha-3)(\alpha-1)^2x^2 - 8(\alpha-1)^2x + 1 \\ &- 8(2\alpha-1)(\alpha-1)^2(1-x)xy\cos(t) - 4(\alpha-1)^2(1-x)^2y^2 \end{aligned}$$

for $x, y \in [0,1]$ and $t \in [0, 2\pi]$.

When $\zeta_2 \neq 0$, then $\zeta_2 = |\zeta_2|\mathrm{e}^{\mathrm{i}\theta}$ for unique $\theta \in [0, 2\pi)$. Thus by (26),

$$\det T_{3,1}(f) = F(\zeta_1^2, |\zeta_2|, \theta).$$

When $\zeta_2 = 0$, then by (26),

$$\det T_{3,1}(f) = F(\zeta_1^2, 0, \theta) = F(\zeta_1^2, 0, 0).$$

We now find the minimum value of F.

B1. Suppose first that $\alpha \in [0, 1/2)$. Then

$$\begin{aligned} F(x,y,t) \geqslant & F(x,y,\pi) \\ &= -4(4\alpha-3)(\alpha-1)^2x^2 - 8(\alpha-1)^2x + 1 \\ &\quad - 8(1-2\alpha)(\alpha-1)^2(1-x)xy - 4(\alpha-1)^2(1-x)^2y^2 \\ &=: J(x,y), \quad x, y \in [0,1],\ t \in [0, 2\pi]. \end{aligned}$$

(a) For $x=1$,

$$J(1,y)=(5-4\alpha)(2\alpha-1)^2,\quad y\in[0,1].$$

(b) Let $x\in[0,1)$. Since

$$y_w:=\frac{(2\alpha-1)x}{1-x}\leqslant 0,$$

we deduce that

$$\begin{aligned}J(x,y)\geqslant J(x,1)&=-16(2\alpha-1)(\alpha-1)^2x^2+8(2\alpha-1)(\alpha-1)^2x-(2\alpha-1)(2\alpha-3)\\&\geqslant(2\alpha-1)(\alpha-2)^2=J\left(\frac{1}{4},1\right),\quad x\in[0,1).\end{aligned}$$

B2. Suppose $\alpha\in[1/2,1)$. Then

$$\begin{aligned}F(x,y,t)\geqslant&F(x,y,0)\\&=-4(4\alpha-3)(\alpha-1)^2x^2-8(\alpha-1)^2x+1\\&\quad-8(2\alpha-1)(\alpha-1)^2(1-x)xy-4(\alpha-1)^2(1-x)^2y^2\\&=:K(x,y),\quad x,y\in[0,1],\ t\in[0,2\pi].\end{aligned}$$

(a) For $x=1$,

$$K(1,y)=(5-4\alpha)(2\alpha-1)^2,\quad y\in[0,1].$$

(b) Let $x\in[0,1)$. Since

$$y_w:=\frac{(1-2\alpha)x}{1-x}\leqslant 0,$$

we deduce that

$$\begin{aligned}K(x,y)\geqslant&K(x,1)\\&=-8(2\alpha-1)(\alpha-1)^2x-(2\alpha-1)(2\alpha-3)\\&\geqslant K(1,1)=(5-4\alpha)(2\alpha-1)^2,\quad x\in[0,1).\end{aligned}$$

C. Note that for $\zeta_2=0$, by (3),

$$\det T_{3,1}(f)=F(\zeta_1^2,0,0)=J(\zeta_1^2,0)=K(\zeta_1^2,0).$$

D. Summarizing, from Parts A-C we get inequalities (17) and (18), respectively.

It remains to show sharpness of all inequalities. The identity function is extremal for the second inequality in (17). The function f given by (16) for which $a_2=2(1-\alpha)$ and $a_3=2(1-\alpha)$ is extremal for the first inequality in (17) and the second one in (18).

Let now $0\leqslant\alpha<1/2$. Set $\tau:=1/2$. Since $\tau\leqslant 1$, by (9) the function

$$\tilde{p}(z):=\frac{1-z^2}{1-2\tau z+z^2}=1+2\tau z+(4\tau^2-2)z^2+\cdots,\quad z\in\mathbb{D},$$

belongs to $\mathscr{P}$. Thus the function f given by (19), where p is replaced by $\tilde{p}$, being of the form (1) with

$$a_2 = 1 - \alpha, \quad a_3 = -(1 - \alpha),$$

belongs to $\mathscr{T}(\alpha)$ and is extremal for the first inequality in (18). This ends the proof of the theorem. □

Particularly, for $\alpha = 0$ we get the following result.

COROLLARY 3.4. *If $f \in \mathscr{T}$, then*

$$-4 \leqslant \det T_{3,1}(f) \leqslant 5.$$

Both inequalities are sharp.

In view of the inclusions (4) the case $\alpha = 1/2$ is of particular interest. Then

COROLLARY 3.5. *If $f \in \mathscr{T}(1/2)$, then*

$$0 \leqslant \det T_{3,1}(f) \leqslant 1.$$

Both inequalities are sharp.

Note that the above result is the same for the class of convex function $\mathscr{S}^c$ and for the class $\mathscr{S}^*(1/2)$ of starlike functions of order $1/2$ ([6]).

Acknowledgements. We would like to express gratitude to the referee for his constructive comments and suggestions that helped to improve the clarity of this manuscript.

REFERENCES

[1] J. W. ALEXANDER, *Functions which map the interior of the unit circle upon simple regions*, Ann. of Math. **17** (1915), 12–22.
[2] MD FIROZ ALI, D. K. THOMAS, A. VASUDEVARAO, *Toeplitz determinants whose elements are the coefficients of analytic and univalent functions*, Bull. Austr. Math. Soc. **97** (2018), no. 2, 253–264.
[3] C. CARATHÉODORY, *Über den Variabilitatsbereich der Koeffizienten von Potenzreihen, die gegebene werte nicht annehmen*, Math. Ann. **64** (1907), 95–115.
[4] N. E. CHO, B. KOWALCZYK, O. S. KWON, A. LECKO, Y. J. SIM, *Some coefficient inequalities related to the Hankel determinant for strongly starlike functions of order alpha*, J. Math. Ineq. **11** (2017), no. 2, 429–439.
[5] N. E. CHO, B. KOWALCZYK, A. LECKO, *Sharp bounds of some coefficient functionals over the class of functions convex in the direction of the imaginary axis*, Bull. Aust. Math. Soc. **100** (2019), 86–96.
[6] K. CUDNA, O. S. KWON, A. LECKO, Y. J. SIM, B. ŚMIAROWSKA, *The second and third-order Hermitian Toeplitz determinants for starlike and convex functions of order α*, Boletin Soc. Mat. Mexicana **26** (2020), 361–375.
[7] P. L. DUREN, *Univalent Functions*, Grundlehren der Mathematischen Wissenschaften, **259**, Springer-Verlag, New York, 1983.
[8] M. ELIN, D. SHOIKHET, *Linearization Models for Complex Dynamical Systems*, Operator Theory: Advances and Applications, **208**, Birkhäuser Verlag, Basel, 2010.
[9] A. W. GOODMAN, *Univalent Functions*, Mariner, Tampa, Florida, 1983.

[10] P. JASTRZĘBSKI, B. KOWALCZYK, O. S. KWON, A. LECKO, Y. J. SIM, *Hermitian Toeplitz determinants of the second and third-order for classes of close-to-star functions*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. **114** (2020), Art. 166.

[11] B. KOWALCZYK, O. S. KWON, A. LECKO, Y. J. SIM, B. ŚMIAROWSKA, *The third-order Hermitian Toeplitz determinant for classes of functions convex in one direction*, Bull. Malays. Math. Sci. Soc. **43** (2020), 3143–3158.

[12] B. KOWALCZYK, A. LECKO, Y. J. SIM, *The sharp bound of the Hankel determinant of the third kind for convex functions*, Bull. Aust. Math. Soc. **97** (2018), 435–445.

[13] R. J. LIBERA, E. J. ZLOTKIEWICZ, *Coefficient bounds for the inverse of a function with derivatives in* $\mathscr{P}$, Proc. Amer. Math. Soc. **87** (1983), no. 2, 251–257.

[14] T. H. MACGREGOR, *Functions whose derivative has a positive real part*, Trans. Amer. Math. Soc. **104** (1962), no. 4, 532–537.

[15] T. H. MACGREGOR, *The radius of univalence of certain analytic functions*, Proc. Amer. Math. Soc. **14** (1963), 514–520.

[16] A. MARX, *Untersuchungen über schlichte Abbildungen*, Math. Ann. **107** (1933), no. 1, 40–67.

[17] S. S. MILLER, P. T. MOCANU, *Differential Subordinations, Monographs and Textbooks in Pure and Applied Mathematics*, **225**, Marcel Dekker, Inc., New York, 2000.

[18] C. POMMERENKE, *Univalent Functions*, Vandenhoeck & Ruprecht, Göttingen, 1975.

[19] J. S. RATTI, *The radius of univalence of certain analytic functions*, Math. Z. **107** (1968), 241–248.

[20] M. S. ROBERTSON, *On the theory of univalent functions*, Ann. of Math. **37** (2) (1936), 374–408.

[21] D. SHOIKHET, *Rigidity and parametric embedding of semi-complete vector fields on the unit disk*, Milan J. Math. **84** (2016), no. 1, 159–202.

[22] E. STROHHÄCKER, *Beiträge zur Theorie der schlichten Funktionen*, Math. Z. **37** (1933), no. 1, 356–380.

[23] E. STUDY, *Vorlesungen über ausgewählte Gegenstände der Geometrie, Zweites Heft; Konforme Abbildung Einfach-Zusammenhängender Bereiche*, Druck und Verlag von B.G. Teubner, Leipzig und Berlin, 1913.

(Received July 5, 2021)

Adam Lecko
Department of Complex Analysis
Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn
ul. Słoneczna 54, 10-710 Olsztyn, Poland
e-mail: alecko@matman.uwm.edu.pl

Barbara Śmiarowska
Department of Complex Analysis
Faculty of Mathematics and Computer Science
University of Warmia and Mazury in Olsztyn
ul. Słoneczna 54, 10-710 Olsztyn, Poland
e-mail: b.smiarowska@matman.uwm.edu.pl