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EXACT EXPONENTS FOR INCLUSION OF DISCRETE MUCKENHOUPT CLASSES INTO GEHRING CLASSES AND REVERSE

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Abstract. In this paper, we establish some embedding relationships between Muckenhoupt and Gehring classes $\mathcal{A}_p$ and $\mathcal{G}_r$ by proving transition and inclusion relations. We also identify the exact range of $\tau > 1$ for which $w^\tau \in \mathcal{A}_p$. Additionally, we show that the weights that satisfy the $\mathcal{A}_p$-condition also meet the $\mathcal{A}_\infty$-condition. Next, we prove the Jones factorization property of $\mathcal{A}_p$ weights in terms of two $\mathcal{A}_1$ weights by employing the discrete Rubio De Francia iterated algorithm. Finally, we determine the specific ranges of indices (sharp exponents) for which w belongs to $\mathcal{G}_r\,(\mathcal{A}_p)$ if w belongs to $\mathcal{A}_p\,(\mathcal{G}_r)$ and the precise range of $q < p$ for which w belongs to $\mathcal{A}_q$ given that it belongs to $\mathcal{A}_p$.

1. Introduction

Our aim in this paper is to study the transition and inclusion relations between the discrete Muckenhoupt and Gehring weights and some applications of these results in terms of the boundedness of Hardy-littlewood maximal operator, discrete Rubio De Francia iterated algorithm and discrete Jones factorization Theorem to find the sharp range of embedding and transition exponents. Throughout this paper, $\mathbb{Z}_+$ stands for a set of positive integers i.e. $\mathbb{Z}_+ = \{1,2,3,\ldots\}$. By interval J, we mean a finite subset of $\mathbb{Z}_+$ consisting of consecutive integers, i.e. $J = \{1,2,\ldots,n\}$ for $n \in \mathbb{Z}_+$ and $|J|$ stands for its cardinality. A discrete weight on $\mathbb{Z}_+$ is a sequence $w = \{w(n)\}_{n=1}^{\infty}$ of nonnegative real numbers. The usual discrete weighted Lebesgue space $\ell_u^p(\mathbb{Z}_+)$ is defined for nonnegative sequences f by

$$\ell_u^p(\mathbb{Z}_+) := \left\{ f : \|f\|_{\ell_u^p(\mathbb{Z}_+)} = \left(\sum_{n=1}^{\infty} f^p(n) u(n) \right)^{1/p} < \infty \right\}.$$

Following the usual terminology [3] and [23, 29], a discrete weight u belongs to the discrete $\mathcal{A}_1(A)$ Muckenhoupt class for $A > 1$, if the inequality

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \leqslant A \inf_{J \subset \mathbb{Z}_+} u, \text{ for all } J \subset \mathbb{Z}_+. \tag{1.1}$$

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holds for every subinterval $J \subset \mathbb{Z}_+$. That is a sequence $u \in \mathcal{A}_1$ iff its norm

$$\mathcal{A}_1(u) := \sup_{J \subset \mathbb{Z}_+} \frac{1}{|J|} \left(\sum_{J \subset \mathbb{Z}_+} \frac{u}{ess \inf_{J \subset \mathbb{Z}_+} u} \right) < \infty.$$

A discrete weight u belongs to the discrete $\mathcal{A}_2(A)$ Muckenhoupt class for $A > 1$, if the inequality

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right) \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{-1} \right) \leqslant A, \tag{1.2}$$

holds for every subinterval $J \subset \mathbb{Z}_+$. A discrete nonnegative sequence u belongs to the discrete Muckenhoupt class $\mathcal{A}_p(\mathcal{C})$ for $p > 1$ and $\mathcal{C} > 1$ if the inequality

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right) \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{1-p}} \right)^{p-1} \leqslant \mathcal{C}, \tag{1.3}$$

holds for every subinterval $J \subset \mathbb{Z}_+$. For a given exponent $p > 1$, we define the $\mathcal{A}_p(u)$-norm of the discrete weight u by the following quantity

$$\mathcal{A}_p(u) := \sup_{J \subset \mathbb{Z}_+} \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{1-p}} \right)^{p-1}.$$

The class $\mathcal{A}_\infty$ is naturally defined as the union of all the $\mathcal{A}_p$ classes, that is $\mathcal{A}_\infty \equiv \cup_{p>1} \mathcal{A}_p$.

A discrete nonnegative weight u belongs to the discrete Gehring class $\mathcal{G}_q(\mathcal{K})$ for a given exponent $q > 1$ and a constant $\mathcal{K} > 1$, (or satisfies the reverse Hölder inequality) if

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^q \right)^{\frac{1}{q}} \leqslant \mathcal{K} \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u,$$

holds for every subinterval $J \subset \mathbb{Z}_+$. For given exponent $q > 1$, we define the $\mathcal{G}_q(u)$-norm of the discrete weight u by the following quantity

$$\mathcal{G}_q(u) := \sup_{J \subset \mathbb{Z}_+} \left[\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{-1} \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^q \right)^{1/q} \right]^{\frac{q}{q-1}},$$

where the supremum is taken over all intervals $J \subset \mathbb{Z}_+$. A discrete nonnegative weight u belongs to the discrete Gehring class $\mathcal{G}_\infty(\mathcal{C})$, if

$$\sup_{J \subset \mathbb{Z}_+} u \leqslant \mathcal{C} \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u.$$

The discrete weight u belongs to the discrete Gehring class $\mathcal{G}_1(\mathcal{K})$, if

$$\exp \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} \frac{u}{\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u} \log \frac{u}{\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u} \right) \leqslant \mathcal{K}, \text{ for } \mathcal{K} > 1,$$

for every subinterval $J \subset \mathbb{Z}_+$. We define $\mathcal{G}_1(u)$-norm by the following quantity

$$\mathcal{G}_1(u) := \sup_{J \subset \mathbb{Z}_+} \left[\exp\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} \frac{u}{\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u} \log \frac{u}{\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u} \right) \right].$$

In [1], Ariño and Muckenhoupt introduced the characterizations of $\mathcal{A}_1$-class and proved that if u is nonincreasing and satisfies (1.1), then the space $d(u^{-q^*/q}, q^*)$ is the dual space of the discrete classical Lorentz space

$$d(u,q) = \left\{ x : \|x\|_{u,q} = \left(\sum_{n=1}^{\infty} |x^*(n)|^q u(n) \right)^{1/q} < \infty \right\},$$

where $x^*(n)$ is the nonincreasing rearrangement of $|x(n)|$ and q^* is the conjugate of q. Pavlov [19] and Lyubarskii and Seip [9] introduced the characterizations of $\mathcal{A}_2$-class and gave a full description of all complete interpolating sequences on the real line by using the condition (1.2). A sequence $\lambda(n)$ of complex numbers is called interpolating sequence if for all complex sequence $a(n)$ with

$$\sum_{n \in \mathbb{Z}_+} |a(n)|^2 e^{-2\pi |\Im \lambda(n)|} (1 + |\Im \lambda(n)|) < \infty,$$

the interpolation problem $f(\lambda(n)) = a(n)$ has a unique solution $f \in PW_\pi^2$, where PW_π^2 is the Paly-Wiener space of all entire functions of exponential type at most π which belong to L^2 on the real line. The authors in [9] and [19] proved that the sequence $\lambda(n)$ of real numbers is a complete interpolating sequence if and only if:

(i). the sequence $\lambda(n)$ is a separated sequence,

(ii). the limit

$$F(z) = \lim_{R \to \infty} \prod_{|\lambda(n)| < R} \left(1 - \frac{z}{\lambda(n)} \right),$$

exists uniformly on compact subset of the complex space and defines an entire function F of exponential type π, the generating function,

(iii). there is a relatively dense subsequence $\lambda(n_k)$ such that the numbers $d(k) = \left| F'(\lambda(n_k)) \right|^2$ satisfies the discrete Muckenhoupt condition

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} d(k) \right) \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} d^{-1}(k) \right) \leqslant A, \tag{1.4}$$

for some $A > 0$ and all subintervals $J \subset \mathbb{Z}_+$ of consecutive integers containing $|J|$ elements.

In [28, Theorem 3.2], the authors characterized the boundedness of the discrete Hardy-Littlewood maximal operator

$$\mathfrak{M}f(n) = \sup_{n \in J} \frac{1}{n} \sum_{k=1}^{n} f(k),$$

on the usual weighted Lebesgue space $\ell_u^p(\mathbb{Z}^+)$ in terms of $\mathcal{A}_p$-condition where f is nonnegative sequence. Precisely, they proved that the operator $\mathfrak{M}f$ is bounded on $\ell_u^p(\mathbb{Z}^+)$ if and only if $u \in \mathcal{A}^p$ and the inequality

$$\sum_{k=1}^{\infty}(\mathfrak{M}f(k))^p u(k) \leqslant C\sum_{k=1}^{\infty}(f(k))^p u(k), \tag{1.5}$$

holds with $C > 0$ independent of p and depending only on the norm $\mathcal{A}_p(u)$. In [25] the authors proved that if u is a nonincreasing sequence and satisfies (1.1) for $A > 1$, then for $p \in [1,\ A/(A-1))$ the inequality

$$\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^p \leqslant A_1\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^p, \tag{1.6}$$

holds for every subinterval interval $J \subset \mathbb{Z}_+$. This result proves that any $\mathcal{A}_1$-Muckenhoupt weight belongs to some Gehring classes $\mathcal{G}_p$ (will be defined later) of weights satisfy reverse Hölder's inequality (*a transition property).* The self-improving properties of discrete Muckenhoupt and Gehring classes have been studied recently by some authors. For example in [27] the authors proved that if $q > 1$, $A > 1$, u is a nondecreasing weight and belongs to $\mathcal{A}_q(A)$, then $u \in \mathcal{A}_p(A_1)$ for $p \in (p_0, q]$ where $p_0 > 1$ is the unique solution of the equation

$$(Ax)^{\frac{1}{q-1}}\left(\frac{q-x}{q-1}\right) = 1. \tag{1.7}$$

This result proves that if $u \in \mathcal{A}_q(A)$ then there exist an $\varepsilon > 0$ and a constant $A_1 = A_1(p,A)$ such that $u \in \mathcal{A}_{q-\varepsilon}(A_1)$, (self-improving property) and thus

$$\mathcal{A}_q(A) \subset \mathcal{A}_{q-\varepsilon}(A_1), \tag{1.8}$$

where $\varepsilon = q-p$ for $p \in (p_0,q]$ and $p_0 > 1$ is the unique solution of the equation (1.7). In [25] the authors proved that if $q > 1$, $\mathcal{K} > 1$ and v is a nonincreasing sequence belongs to $\mathcal{G}_q(\mathcal{K})$, then there exist an $\varepsilon > 0$, and a constant $\mathcal{K}_1 = \mathcal{K}_1(q,\mathcal{K})$ such that $v \in \mathcal{G}_{p+\varepsilon}(\mathcal{K}_1)$, (self-improving property) and thus

$$\mathcal{G}_q(\mathcal{K}) \subset \mathcal{G}_{q+\varepsilon}(\mathcal{K}_1). \tag{1.9}$$

There is an alternative way involves exploiting the correspondence between a weighted Muckenhoupt class and the reverse Hölder class. This in fact provides a simple proof of the self-improving property of $\mathcal{G}_q(\mathcal{K})$ as follows: Assume that $v \in \mathcal{G}_q(\mathcal{K})$ i.e., the condition

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v^q\right)^{1/q} \leqslant \mathcal{K}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v\right), \quad \text{for all } J \subset \mathbb{Z}_+, \tag{1.10}$$

holds. This condition can be rewritten in the form

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v^{q-\frac{1}{p-1}}\left(\frac{1}{v}\right)^{\frac{-1}{p-1}}\right)^{p-1} \leqslant \mathcal{K}^{q(p-1)}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v\right)^{q(p-1)}. \tag{1.11}$$

By taking $p = q/(q-1)$, we have from (1.11) that (here $\Lambda(J) = \sum_J v(n)$)

$$\left(\frac{1}{\Lambda(J)}\sum_{J\subset\mathbb{Z}_+} v\left(\frac{1}{v}\right)\right)\left(\frac{1}{\Lambda(J)}\sum_{J\subset\mathbb{Z}_+} v\left(\frac{1}{v}\right)^{\frac{-1}{p-1}}\right)^{p-1} \tag{1.12}$$

$$\leqslant \mathcal{K}^{q(p-1)}\frac{|J|^p}{\Lambda^p(J)}\left(\frac{\Lambda(J)}{|J|}\right)^p = \mathcal{K}^p, \tag{1.13}$$

for all $J \subset \mathbb{Z}_+$, which is a weighted $\mathcal{A}_{p,v}(\mathcal{K}^p)$ condition for v^{-1} with respect to the weight v and $q = p/(p-1)$. This shows that if $v \in \mathcal{G}_q(\mathcal{K})$ then $v^{-1} \in \mathcal{A}_{p,v}(\mathcal{C})$ with $\mathcal{C} = \mathcal{K}^p$ where $q = p/(p-1)$.

Making use of this discrete Gehring result, the authors in [26] proved that the so-called reverse Hölder inequality for discrete Muckenhoupt weights is also satisfied. Precisely, their result reads as follows: If $1 < p < \infty$ and $u \in \mathcal{A}_p$, then there are constants $q > 1$ and $0 < C < \infty$ such that

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^q(k)\right)^{\frac{1}{q}} \leqslant C\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u(k), \text{ for all } J \subset \mathbb{Z}_+.$$

This equivalence gives the transition property between the discrete Muckenhoupt and Gehring classes in the form: If $u \in \mathcal{A}_p$ for some p then $u \in \mathcal{G}_q$ for some q. In this paper, we will prove the inverse and prove that if $u \in \mathcal{G}_s$ for some s then $u \in \mathcal{A}_q$ for some q. For more details and additional results, we refer the reader to the paper [22].

The interesting question arises now is: *What are the relations between p and q for which the inclusions $\mathcal{A}_p \subset \mathcal{G}_q$ and $\mathcal{G}_q \subset \mathcal{A}_p$ hold?*

In this paper, we give a solution of this question by employing a new approach depends upon the appropriate factorizations of w. Precisely, the aims of this paper are the following:

1). For $w \in \mathcal{A}_p$, we find the precise range of r's such that $w \in \mathcal{G}_r$, and the precise range of $q < p$ for which $w \in \mathcal{A}_q$.

2). For $w \in \mathcal{G}_r$, we find the precise range of p's such that $w \in \mathcal{A}_p$, and the precise range of $q > r$ for which $w \in \mathcal{G}_q$.

3). Find the precise range of $\tau > 1$, such that $w^\tau \in \mathcal{A}_p$.

For classical results of integral forms, we refer the reader to the papers [2, 11, 12, 13, 18, 20, 21, 31] and the references cited therein.

The paper is organized as follows:

In Section 2, we state some basic results that are needed in the proofs of the main results, prove some results of the structure of the discrete Muckenhoupt and Gehring weights (Lemma 2.1 and Theorem 2.1 and Theorem 2.2). Next, we prove that if $u \in \mathcal{A}_p$ then $u \in \mathcal{G}_{1+\delta}$ for some positive $\delta > 0$ that is if $u \in \mathcal{A}_p$ then it satisfies a reverse Hölder inequality (Theorem 2.3). The the self-improving property of the Muckenhoupt weights will be proved in Lemma 2.2. In addition, we prove that the weights satisfy $\mathcal{A}_p$-condition also satisfy $\mathcal{A}_\infty$-condition and if $u \in \mathcal{A}_p$ then $u^\tau \in \mathcal{A}_p$ for some $\tau > 1$. We also show that if $u \in \mathcal{A}_p$, then $u^\delta \in \mathcal{A}_q$ and establish the explicit value of q. Next, we establish the transition and inclusion relations between the two classes which give embedding relations between $\mathcal{A}_p$ and $\mathcal{G}_r$ and prove that if the weights satisfy $\mathcal{G}_\infty$-condition then the weights also satisfy $\mathcal{A}_p$-condition.

In Section 3, we give some applications and start by proving the properties of the discrete Rubio De Francia iterated algorithm and pass to prove the discrete version of Jones's Factorization Theorem of $\mathcal{A}_p$ weights in terms of two $\mathcal{A}_1$ weights and investigate the exact range on $q < p$ such that $w \in \mathcal{A}_p$ implies $w \in \mathcal{A}_q$ and investigate the exact range on $p > r$ such that $w \in \mathcal{G}_r$ implies $w \in \mathcal{A}_p$ and investigate the exact range on r such that $w \in \mathcal{A}_p$ implies $w \in \mathcal{G}_r$. The ranges are optimal.

2. Main results

We begin this section by recalling some properties of Muckenhoupt $\mathcal{A}_p$-classes, which are adapted from [26] and will be used later in the proof of the main results. Recall that two positive quantities A, B are said to be *equivalent*, written $A \simeq B$, if there exists two constants c and C such that the inequality $cB \leqslant A \leqslant CB$ holds. Furthermore, $A \lesssim B$ is satisfied if there exists a constant C such that the inequality $A \leqslant CB$ holds. Clearly the relation $\lesssim$ is transitive, that is, if $A \lesssim B$ and $B \lesssim C$ hold, then $A \lesssim C$ also holds. Throughout this section, we assume that the sequences in the statements of theorems are non-negative and assume for the sake of conventions that $0 \cdot \infty = 0$, $0/0 = 0$, $\sum_{s=a}^{b} y(s) = 0$, for $a > b$, and by $\sum_{J \subset \mathbb{Z}_+} u$ we mean that $\sum_{n \in J \subset \mathbb{Z}_+} u(n)$ and $p^{'}$ is the conjugate exponent given by $1/p + 1/p^{'} = 1$.

LEMMA 2.1. *Let u be a nonnegative weight and p and q be positive real numbers. The following properties hold:*

(1). *If $u \in \mathcal{A}_p$ then $u^{\alpha} \in \mathcal{A}_p$, for $0 \leqslant \alpha \leqslant 1$, with $\mathcal{A}_p(u^{\alpha}) = \mathcal{A}_p^{\alpha}(u)$,*

(2). *If $u \in \mathcal{A}_p$, then $u^{\tau} \in \mathcal{A}_p$ for some $\tau > 1$,*

(3). *$u \in \mathcal{A}_p$ if and only if u and $u^{\frac{1}{1-p}}$ are in $\mathcal{A}_\infty$.*

(4). *Given $1 < p$ and $s < \infty$. Then $u \in \mathcal{A}_p \cap \mathcal{G}_s$ if and only if $u^s \in \mathcal{A}_q$, where $q = s(p-1)+1$.*

Now, we start with the following properties for $\mathcal{A}_p$ weights, which will have a clear imprint in the proofs of the main results for this paper.

THEOREM 2.1. (1). *If $1 \leqslant p < q < \infty$, then $\mathcal{A}_p \subset \mathcal{A}_q$.*

(2). *If $p > 1$, then $u \in \mathcal{A}_p$ if and only if $u^{1-p'} \in \mathcal{A}_{p'}$.*

(3). *If $u_1,\ u_2 \in \mathcal{A}_1$, then $u_1 u_2^{1-p} \in \mathcal{A}_p$.*

(4). *If $u \in \mathcal{A}_p$, then*

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \leqslant \mathcal{A}_p(u) \exp\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} \log u \right),$$

for every $J \subset \mathbb{Z}_+$.

Proof. (1). Let $p=1$ and $1<q<\infty$ and assume that $u\in\mathcal{A}_1$. Then

$$\begin{aligned}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{1-q'}\right)^{q-1}&=\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{\frac{1}{q-1}}\right)^{q-1}\\&\leqslant \operatorname*{ess\,sup}_{J\subset\mathbb{Z}_+}\frac{1}{u}=\frac{1}{\inf\limits_{J\subset\mathbb{Z}_+}u}\\&\leqslant\frac{\mathcal{A}_1(u)}{\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u},\quad\text{for every}\quad J\subset\mathbb{Z}_+.\end{aligned}$$

This shows that $u\in\mathcal{A}_q$. Now, we consider the case when $1<p<q<\infty$ and $u\in\mathcal{A}_p$. By applying Hölder's inequality we obtain

$$\begin{aligned}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{\frac{1}{1-q}}\right)^{q-1}&=\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{\frac{1}{q-1}}\right)^{q-1}\\&\leqslant\left(\frac{1}{|J|}\right)^{q-1}\left(\sum_{J\subset\mathbb{Z}_+}\left(\left(\frac{1}{u}\right)^{\frac{1}{q-1}}\right)^{\frac{q-1}{p-1}}\right)^{(q-1)\left(\frac{p-1}{q-1}\right)}\\&\quad\times\left(\sum_{J\subset\mathbb{Z}_+}1^{\frac{q-1}{q-p}}\right)^{(q-1)\left(\frac{q-p}{q-1}\right)}\\&=\left(\frac{1}{|J|}\right)^{q-1}\left(\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{\frac{1}{p-1}}\right)^{p-1}\left(\sum_{J\subset\mathbb{Z}_+}1\right)^{q-p}\\&=\left(\frac{1}{|J|}\right)^{q-1}\left(\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{\frac{1}{p-1}}\right)^{p-1}|J|^{q-p}\\&=\left(\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{\frac{1}{p-1}}\right)^{p-1}|J|^{1-p}\\&=\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{\frac{1}{1-p}}\right)^{p-1}.\end{aligned}$$

This implies

$$\begin{aligned}\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u(n)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{\frac{1}{1-q}}\right)^{q-1}&\leqslant\frac{1}{|J|}\sum_{n\in J}u(n)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{\frac{1}{1-p}}\right)^{p-1}\\&\leqslant[u]_{\mathcal{A}_p},\end{aligned}$$

for every $J\subset\mathbb{Z}_+$. In the last inequality we used the fact that $u\in\mathcal{A}_p$. This shows that $u\in\mathcal{A}_q$.

(2). First, assume that $u \in \mathcal{A}_p$ with $1 < p < \infty$. Then

$$\begin{aligned}
&\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(u^{1-p'}\right)^{\frac{1}{1-p'}}\right)^{p'-1}\\
&=\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{\frac{1}{p-1}}\\
&=\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{\frac{1}{1-p}}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{\frac{1}{p-1}}\\
&\leqslant \mathcal{A}_p^{\frac{1}{p-1}}(u),
\end{aligned}$$

for every $J \subset \mathbb{Z}_+$, which shows that $u^{1-p'} \in \mathcal{A}_{p'}$. Now, we consider the reverse, i.e., we assume that $u^{1-p'} \in \mathcal{A}_{p'}$ with $1 < p < \infty$. As above we see that

$$\begin{aligned}
\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{\frac{1}{1-p}}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{\frac{1}{p-1}} &= \frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{\frac{1}{p-1}}\\
&=\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(u^{1-p'}\right)^{\frac{1}{1-p'}}\right)^{p'-1}\\
&\leqslant \mathcal{A}_{p'}(u^{1-p'}),
\end{aligned}$$

for every $J \subset \mathbb{Z}_+$. This shows that $u \in \mathcal{A}_p$.

(3). Assume that $u_1 \in \mathcal{A}_1$ and $u_2 \in \mathcal{A}_1$. The $\mathcal{A}_p$ condition for $u_1 u_2^{1-p}$ is

$$\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1 u_2{}^{1-p}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1{}^{\frac{1}{1-p}} u_2\right)^{p-1} \leqslant c.$$

Since $u_1 \in \mathcal{A}_1$ and $u_2 \in \mathcal{A}_1$, then

$$\frac{1}{u_i} \leqslant \mathcal{A}_1(u_i)\frac{|J|}{u_i(J)},$$

almost for every $J \subset \mathbb{Z}_+$, and $i = 1,2$. This implies

$$\begin{aligned}
\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1 u_2{}^{1-p} &= \frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1\left(\frac{1}{u_2}\right)^{p-1}\\
&\leqslant \mathcal{A}_1^{p-1}(u_2)\left(\frac{|J|}{u_2(J)}\right)^{p-1}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1(n)\right)\\
&= \mathcal{A}_1^{p-1}(u_2)\left(\frac{|J|}{u_2(J)}\right)^{p-1}\frac{u_1(J)}{|J|},
\end{aligned}$$

and

$$\begin{aligned}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1^{\frac{1}{1-p}}u_2\right)^{p-1} &= \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u_1}\right)^{\frac{1}{p-1}}u_2\right)^{p-1}\\ &\leqslant \mathcal{A}_1(u_1)\frac{|J|}{u_1(J)}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u_2\right)^{p-1}\\ &= \mathcal{A}_1(u_1)\frac{|J|}{u_1(J)}\left(\frac{u_2(J)}{|J|}\right)^{p-1}.\end{aligned}$$

Thus

$$\begin{aligned}&\sum_{J\subset\mathbb{Z}_+} u_1 u_2^{1-p}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1^{\frac{1}{1-p}}u_2\right)^{p-1}\\ &\leqslant \mathcal{A}_1(u_1)\mathcal{A}_1^{p-1}(u_2),\ \text{for every } J\subset\mathbb{Z}_+.\end{aligned}$$

(4). To prove the result in this case we assume that $p>1$. Then, for any $J\subset\mathbb{Z}_+$, we have that

$$\begin{aligned}[\mathcal{A}_p(u)] \ :&= \sup_{J\subset\mathbb{Z}_+}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{\frac{-1}{p-1}}\right)^{p-1}\\ &\geqslant \lim_{q\to\infty}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{\frac{-1}{q-1}}\right)^{q-1}\\ &= \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u\right)\exp\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}-\log u\right),\end{aligned}$$

which proves the desired inequality. The proof is complete. □

REMARK 2.1. From the above results, we see that the $\mathcal{A}_p$ classes are nested and $\mathcal{A}_1$ is the strongest condition. Also, the results show the interpretation of duality and also give a method for constructing $\mathcal{A}_p$ weights from $\mathcal{A}_1$ weights.

In the following lemma, we will present and prove some interesting properties for the Gehring $\mathcal{G}_q$ classes.

THEOREM 2.2. *Let u be a nonnegative weight and p and q be real positive numbers such that $p,\ q>1$. The following properties hold:*

(1). $\mathcal{G}_\infty\subset\mathcal{G}_q\subset\mathcal{G}_1$ *for all* $1<q\leqslant\infty$,

(2). $\mathcal{G}_1=\bigcup\limits_{1<q\leqslant\infty}\mathcal{G}_q$ *with* $\mathcal{G}_1(u)=\lim_{q\to1}\mathcal{G}_q$,

(3). $\mathcal{G}_1=\mathcal{A}_\infty=\bigcup\limits_{1<p\leqslant\infty}\mathcal{A}_p=\bigcup\limits_{1<q\leqslant\infty}\mathcal{G}_q$.

Proof. We give the proof of property (1). The proofs of the rest of the properties are similar and will be omitted. If $u \in \mathcal{G}_\infty$, then by the definition of $\mathcal{G}_\infty$, there exists $0 < C < \infty$, we have should be such that

$$u \leqslant C\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right), \text{ for all } J \subset \mathbb{Z}_+. \tag{2.1}$$

For all $1 < q < \infty$, by applying (2.1), we have that

$$\begin{aligned}
&\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^q\right)^{1/q}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{-1}\\
&\leqslant \left[\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(C\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\right)^q\right]^{1/q}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{-1}\\
&= C\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{-1} = C.
\end{aligned}$$

That is $u \in \mathcal{G}_q$ and hence $\mathcal{G}_\infty \subset \mathcal{G}_q$. Now, if $u \in \mathcal{G}_q$, then there exists $G > 1$ such that

$$\left[\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^q\right)^{1/q}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{-1}\right]^{q/(q-1)} \leqslant G, \quad \text{for all } J \subset \mathbb{Z}_+. \tag{2.2}$$

Taking the limit in (2.2) as q tends to 1, we obtain that

$$\begin{aligned}
G &\geqslant \lim_{q\to 1}\left[\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^q\right)^{1/q}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^{-1}\right]^{q/(q-1)}\\
&= \lim_{q\to 1}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{u}{\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u}\right)^q\right)^{1/(q-1)}\\
&= \exp\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\frac{u}{\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u}\log\left(\frac{u}{\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u}\right)\right).
\end{aligned}$$

That is from the definition of $\mathcal{G}_1$ we see that $u \in \mathcal{G}_1$ and hence, $\mathcal{G}_q \subset \mathcal{G}_1$, which is the desired result. The proof is complete. $\square$

The following theorem proves that if $u \in \mathcal{A}_p$ then it satisfies the reverse Hölder inequality.

THEOREM 2.3. *Let $1 < p < \infty$ and assume that $u \in \mathcal{A}_p$. Then there are $\delta > 0$ and $0 < c < \infty$ such that*

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1+\delta}\right)^{\frac{1}{1+\delta}} \leqslant c\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u, \tag{2.3}$$

for every $J \subset \mathbb{Z}_+$.

Proof. By property (1) of Theorem 2.1, we have $\mathcal{A}_p \subset \mathcal{A}_q$, if $1 \leqslant p < q$. Thus we may assume that $p > 2$. From Hölder's inequality, for every positive sequence u, we see that

$$1 \leqslant \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} \left(\frac{1}{u} \right)^{\frac{1}{p-1}} \right) \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{p-1}} \right). \tag{2.4}$$

By applying $\mathcal{A}_p$-condition for u, we get that

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{1-p}} \right)^{p-1} \leqslant c = c \cdot 1^{p-1}.$$

Recalling (2.4), we get that

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{1-p}} \right)^{p-1} \leqslant c \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{1-p}} \right)^{p-1} \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{p-1}} \right)^{p-1}. \tag{2.5}$$

Dividing both sides of (2.5) by

$$0 < \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{1-p}} \right)^{p-1} < \infty,$$

this leads directly to

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \leqslant c \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{p-1}} \right)^{p-1},$$

which gives

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} \left(u^{\frac{1}{p-1}} \right)^{p-1} \right)^{\frac{1}{p-1}} \leqslant c \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{p-1}}. \tag{2.6}$$

But since $p > 2$, then $p - 1 > 1$ and (2.6) shows that $u^{\frac{1}{p-1}}$ satisfies the reverse Hölder inequality. By (1.9) there exist $q > p-1$ and $c' < \infty$, such that

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{q}{p-1}} \right)^{\frac{1}{q}} \leqslant c' \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{p-1}} \leqslant c' \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{\frac{1}{p-1}},$$

for $p - 1 < q < p - 1 + \delta$, $\delta > 0$. The last inequality follows directly from Hölder's inequality. Finally, we get that

$$\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{q}{p-1}} \right)^{\frac{p-1}{q}} \leqslant c \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u.$$

Taking $1 + \delta = q/(p-1)$, we obtain the assertion (2.3). The proof is complete. □

REMARK 2.2. For different approaches to this important result in the continuous case, we refer the interested reader to the books [4, Theorem 4.22], [6, page 397], [7, Corollary 6.10], [8, Theorem 9.2.2], [10, Lemma 4.33], [12] and [17].

As a first application of Theorem 2.3, we present the so-called self improving property for Muckenhoupt weights which is an important in its own seek.

LEMMA 2.2. *Suppose that* $u \in \mathcal{A}_p$*. Then, there is* $\varepsilon > 0$ *such that* $u \in \mathcal{A}_{p-\varepsilon}$*.*

Proof. We apply Theorem 2.3 to $(1/u)^{1/(p-1)}$, which is a weight sequence satisfying condition $\mathcal{A}_q$-condition with $q = p/(p-1)$. This gives us that

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{(1+\delta)/(p-1)}\right)^{(p-1)/(1+\delta)} \leqslant \left(\frac{c}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{1/(p-1)}\right)^{p-1}.$$

Setting $\varepsilon = (\delta/(1+\delta))(p-1) > 0$, so that $(p-\varepsilon)-1 = (p-1)/(1+\delta)$ and multiplying both sides of the above inequality by $(1/|J|)\sum_{J\subset\mathbb{Z}_+} u$, this yields

$$\begin{aligned}&\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{1/(p-\varepsilon-1)}\right)^{p-\varepsilon-1}\\ &\leqslant c\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(\frac{1}{u}\right)^{1/(p-1)}\right)^{p-1},\end{aligned}$$

and then $u \in (\mathcal{A}_{p-\varepsilon})$. The proof is complete. □

Before, we continue to the next result, we need the following definition.

DEFINITION 2.1. A weight sequence u is said to satisfy $\mathcal{A}_\infty$-condition, if

$$\frac{u(E)}{u(J)} \leqslant C\left(\frac{|E|}{|J|}\right)^{\alpha}, \tag{2.7}$$

where $u(E) = \sum_E u$ and $u(J) = \sum_{J\subset\mathbb{Z}_+} u$. The constants $C > 0$ and $\alpha > 0$ in (2.7) are supposed to be independent of E and J.

LEMMA 2.3. *If* u *satisfies* $\mathcal{A}_p$*-condition for some* $p < \infty$*, then* $u \in \mathcal{A}_\infty$*.*

Proof. For $E \subset \mathbb{Z}_+$, Hölder's inequality and Theorem 2.3 give

$$\begin{aligned}\frac{1}{|J|}\sum_E u &\leqslant \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1+\delta}\right)^{1/(1+\delta)}\left(\frac{|E|}{|J|}\right)^{\delta/(1+\delta)}\\ &\leqslant \left(\frac{c}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{|E|}{|J|}\right)^{\delta/(1+\delta)} = c\frac{\sum_{J\subset\mathbb{Z}_+} u}{|J|}\left(\frac{|E|}{|J|}\right)^{\delta/(1+\delta)},\end{aligned}$$

which is the $\mathcal{A}_\infty$-condition (2.7) with $\alpha = \delta/(1+\delta)$. The proof is complete. $\square$

The next result improves the result in [30] by removing the monotonicity condition of the weight sequence u.

THEOREM 2.4. *If $u \in \mathcal{A}_p$ with $1 \leqslant p < \infty$, then there exists $\varepsilon = \varepsilon(p, \mathcal{A}_p(u)) > 0$ such that $u^{1+\varepsilon} \in \mathcal{A}_p$.*

Proof. If $u \in \mathcal{A}_p$ with $1 \leqslant p < \infty$, then by Theorem 2.3 there exists $\varepsilon = \varepsilon(p, \mathcal{A}_p(u)) > 0$ and $c = c(p, \mathcal{A}_p(u))$ such that

$$\sum_{J \subset \mathbb{Z}_+} u^{1+\varepsilon} \leqslant c \left(\frac{1}{J} \sum_{J \subset \mathbb{Z}_+} u \right)^{1+\varepsilon}, \quad \text{for every} \quad J \subset \mathbb{Z}_+. \tag{2.8}$$

For $p = 1$, assume that $u \in \mathcal{A}_1$. Then (2.8) and the $\mathcal{A}_1$-condition (1.1) imply that

$$\begin{aligned} \sum_{J \subset \mathbb{Z}_+} u^{1+\varepsilon} &\leqslant c \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{1+\varepsilon} \\ &\leqslant c \left(\operatorname*{ess\,inf}_{J \subset \mathbb{Z}_+} u \right)^{1+\varepsilon} \\ &= c \operatorname*{ess\,inf}_{J \subset \mathbb{Z}_+} u^{1+\varepsilon}. \end{aligned}$$

This shows that $u^{1+\varepsilon} \in \mathcal{A}_1$. Assume that $u \in \mathcal{A}_p$ with $1 < p < \infty$. Property (2) in Theorem 2.1 implies that $u^{1-p'} \in \mathcal{A}_{p'}$. Applying Theorem 2.3, we may choose $\varepsilon > 0$, so that both u and $u^{1-p'}$ satisfy a revere Hölder inequality with the same exponent $1+\varepsilon$, that is, (2.8) holds and

$$\sum_{J \subset \mathbb{Z}_+} u^{(1-p')(1+\varepsilon)} \leqslant c \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{1-p'} \right)^{1+\varepsilon}, \quad \text{for every} \quad J \subset \mathbb{Z}_+.$$

Together with $\mathcal{A}_p$ condition (1.3), this implies that

$$\begin{aligned} &\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{1+\varepsilon} \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{(1+\varepsilon)(1-p')} \right)^{p-1} \\ &\leqslant c \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{1+\varepsilon} \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{1-p'} \right)^{(1+\varepsilon)(p-1)} \\ &= c \left[\left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right) \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{1-p'} \right)^{(p-1)} \right]^{(1+\varepsilon)} \leqslant c_1 < \infty. \end{aligned}$$

This shows that $u^{1+\varepsilon} \in \mathcal{A}_p$. The proof is complete $\square$

Before proceeding to the next application, we recall the following property for the Muckenhoupt weights from Lemma 2.1. Precisely, if $u \in \mathcal{A}_p$, then $u^\delta \in \mathcal{A}_p$. Here, we give here an improvement for this result by presenting the explicit value for q such that $u^\delta \in \mathcal{A}_q$ as follows.

LEMMA 2.4. *Suppose that $u \in \mathcal{A}_p$ for some $1 < p < \infty$ and $0 < \delta < 1$. Then, $u^\delta \in \mathcal{A}_q$ for $q = \delta p + 1 - \delta$.*

Proof. For $0 \leqslant \delta \leqslant 1$, and $u \in \mathcal{A}_p$, we have $1/(p-1) \geqslant \delta/(p-1) > 0$, and hence for all $J \subset \mathbb{Z}_+$, we have

$$\begin{aligned}
&\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^\delta\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(u^\delta\right)^{\frac{1}{1-q}}\right)^{q-1} \\
&= \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^\delta\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(u^\delta\right)^{\frac{1}{\delta-\delta p}}\right)^{\delta p-\delta} \\
&\leqslant \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^\delta\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{\frac{\delta}{\delta-\delta p}}\right)^{\delta p-\delta} \\
&\leqslant \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)^\delta\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{\frac{1}{1-p}}\right)^{\delta(p-1)} \\
&= \left[\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{\frac{1}{1-p}}\right)^{(p-1)}\right]^\delta \\
&\leqslant \mathcal{A}_p^\delta(u),
\end{aligned}$$

that is $u^\delta \in \mathcal{A}_q$, with $\mathcal{A}_q\left(u^\delta\right) \leqslant \mathcal{A}_p(u)^\delta$, which is the desired result. The proof is complete. $\square$

COROLLARY 2.1. *For any $1 < p < \infty$ and for every $u \in \mathcal{A}_p$ there is q with $q < p$ such that $u \in \mathcal{A}_q$. In other words, we have*

$$\mathcal{A}_p = \bigcup_{q\in(1,p)} \mathcal{A}_q.$$

Proof. Given $u \in \mathcal{A}_p$, by property (2) of Theorem 2.1, we have $u^{-1/(p-1)} \in \mathcal{A}_{p'}$. On the other hand, using Theorem 2.3 for the new weight $u^{-1/(p-1)}$, we know that there exist $\delta > 0$, $C > 0$ such that for every $J \subset \mathbb{Z}_+$

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{-(1+\delta)/(p-1)}\right)^{\frac{1}{1+\delta}} \leqslant \frac{C}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{-1/(p-1)}.$$

But $\frac{1+\delta}{p-1} > \frac{1}{p-1}$ implies that $\frac{1+\delta}{p-1} = \frac{1}{q-1}$ for some $1 < q < p$. Then

$$\begin{aligned}
&\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{-1/(q-1)}\right)^{q-1}\\
&= \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{-(1+\delta)/(p-1)}\right)^{\frac{p-1}{1+\delta}}\\
&\leqslant C^{p-1}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{-1/(p-1)}\right)^{p-1} \leqslant C,
\end{aligned}$$

since u itself satisfies Theorem 2.3. The proof is complete. $\square$

It has been proved (see [26]) that the norms of the classes $\mathcal{A}_\infty$ and $\mathcal{G}_p$ are related quantitatively in the following theorem.

THEOREM 2.5. *Let u be a nonnegative weight and $1 < p < \infty$. Then*

$$\frac{\mathcal{A}_\infty^{1/p}(u^p)}{\mathcal{A}_\infty(u)} \leqslant \mathcal{G}_p^{1/p'}(u) \leqslant \mathcal{A}_\infty^{1/p}(u^p). \tag{2.9}$$

In the continuous versions, the classes $\mathcal{A}_1$ and $\mathcal{G}_\infty$ generate $\mathcal{A}_\infty$ in the sense that $\mathcal{A}_\infty = \mathcal{A}_1 \cdot \mathcal{G}_\infty$, and $\mathcal{G}_\infty$ plays the same role relative to $\mathcal{G}_r$ as $\mathcal{A}_1$ does to $\mathcal{A}_p$, (see [5] for more details). The next theorem will play a crucial role in the proof of our main results for the next section. Specifically, it gives the embedding relation between the two discrete classes $\mathcal{A}_1(\mathcal{C})$ and $\mathcal{G}_p(A)$ which adapted from [25].

THEOREM 2.6. *Let u be a nonincreasing weight. If*

$$\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u \leqslant \mathcal{C} \inf_{J\subset\mathbb{Z}_+} u, \text{for some} \quad \mathcal{C} > 1, \tag{2.10}$$

then, for $r \in [1, \ \mathcal{C}/(\mathcal{C}-1))$, we have that

$$\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^r\right)^{1/r} \leqslant A^{1/r}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right), \tag{2.11}$$

where A is given by

$$A := \frac{\mathcal{C}^{1-r}}{r-(r-1)\mathcal{C}}. \tag{2.12}$$

REMARK 2.3. Theorem 2.6 proves that if $u \in \mathcal{A}_1(\mathcal{C})$, then $u \in \mathcal{G}_p(A)$ for $p \in [1, \mathcal{C}/(\mathcal{C}-1))$.

In the following, we will show that the so-called reverse Hölder inequality for discrete Muckenhoupt weights is also satisfied. The proof requires a weighted versions of the $\mathcal{A}_p$ and $\mathcal{G}_s$ conditions. Given a non-negative weight sequence w, we say that a positive sequence u is in $\mathcal{A}_{p,w}(\mathcal{C})$ for some p, $1<p<\infty$, if

$$\left(\frac{1}{w(J)}\sum_{J\subset\mathbb{Z}_+} uw\right)\left(\frac{1}{w(J)}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}w\right)^{p-1} \leqslant \mathcal{C},$$

where $w(J)=\sum_{J\subset\mathbb{Z}_+} w$. Similarly, we write $u\in\mathcal{G}_{s,w}(\mathcal{K})$ if it satisfies the reverse Hölder's inequality

$$\left(\frac{1}{w(J)}\sum_{J\subset\mathbb{Z}_+} u^s w\right)^{1/s} \leqslant \mathcal{K}\frac{1}{w(J)}\sum_{J\subset\mathbb{Z}_+} uw,$$

for $s>1$. The following theorem proves that the converse of Theorem 2.3 is also true.

THEOREM 2.7. *If $u\in\mathcal{G}_s$ for $1<s<\infty$, then there exists $1<p<\infty$, such that $u\in\mathcal{A}_p$. The value of p depends only on s and $\mathcal{G}_s(u)$.*

Proof. We know that if $u\in\mathcal{A}_p$, then the $\mathcal{A}_p$ condition can be rewritten in the form

$$\left(\sum_{J\subset\mathbb{Z}_+} u\right)^{p'-1}\sum_{J\subset\mathbb{Z}_+}\left((u)^{-1}\right)^{p'} u \leqslant (\mathcal{A}_p(u))^{p'-1}\left(\sum_{J\subset\mathbb{Z}_+}(u)^{-1}u\right)^{p'},$$

which in turn is equivalent to the weighted Muckenhoupt condition

$$\frac{1}{w(J)}\sum_{J\subset\mathbb{Z}_+}\left((u)^{-1}\right)^{p'} w \leqslant (\mathcal{A}_p(u))^{p'-1}\left(\frac{1}{w(J)}\sum_{J\subset\mathbb{Z}_+}(u)^{-1}w\right)^{p'},$$

for the sequence $1/u$ with a weight $w=u$. It follows immediately that $u\in\mathcal{A}_p$ if and only if $u^{-1}\in\mathcal{G}_{p',w}$. A similar argument shows that $u\in\mathcal{G}_s$ if and only if $u^{-1}\in\mathcal{A}_{s',w}$. Therefore, given $u\in\mathcal{G}_s$ there exists some $p>1$ such that $u^{-1}\in\mathcal{G}_{p',w}$, which in turn is equivalent to $u\in\mathcal{A}_p$. This completes our proof. □

Now, we are ready to state and prove one of the important results in this section which explicitly gives the relation between the two classes $\mathcal{G}_\infty$ and $\mathcal{A}_p$ with the best range for the exponent p, for related results see also [16]. To provide more clarity on the previous concepts, the authors in [24] investigated following estimates for power low discrete weights by making use of some bounds of functions in [15, Lemma 2.2] and [14, Lemma 2.2], respectively. These estimates will play a crucial role for the precise range of constants of the classes.

LEMMA 2.5. *If $p>1$ and $-1<\lambda<p-1$, then the norm $\mathcal{A}_p(n^\lambda)\simeq\Phi(p,\lambda)$, where*

$$\Phi(p,\lambda)=\frac{1}{(1+\lambda)}\left(\frac{p-1}{p-\lambda-1}\right)^{p-1}. \tag{2.13}$$

Taking limit as $p \to 1$, we get the following corollary.

COROLLARY 2.2. *If* $-1 < \lambda < 0$, *then the norm* $\mathcal{A}_1(n^\lambda) \simeq \Phi(1,\lambda)$, *where*

$$\Phi(1,\lambda) = \frac{1}{(1+\lambda)}. \tag{2.14}$$

LEMMA 2.6. *If* $p > 1$ *and* $\alpha > -1/p$, *then the norm* $(\mathcal{G}_p(n^\alpha))^{\frac{p-1}{p}} \simeq \Psi(p,\alpha)$, *where*

$$\Psi(p,\alpha) = \frac{(1+\alpha)}{(1+p\alpha)^{1/p}}. \tag{2.15}$$

Taking limit as $p \to \infty$, we get the following corollary.

COROLLARY 2.3. *The norm* $(\mathcal{G}_\infty(n^\alpha))^{\frac{p-1}{p}} \simeq \Psi(\infty,\alpha)$, *where*

$$\Psi(\infty,\alpha) = (1+\alpha); \;\; \textit{if } \alpha > 0 \textit{ or } \alpha < 0. \tag{2.16}$$

THEOREM 2.8. *Let* $w \in \mathcal{G}_\infty$ *and assume that* $\beta = \mathcal{G}_\infty(w)$. *Then, there exists* $p > \beta$ *such that* $w \in \mathcal{A}_p$.

Proof. Suppose that $w \in \mathcal{G}_\infty$, it follows from Property (1) in Lemma 2.2 that w belongs to $\mathcal{G}_q$ for some $q > 1$. To finish the proof, it remains only to recall the result in Theorem 2.7, which claims the assertion. To prove the that the range for the exponent p is optimal, we use the weight sequence $w(n) = n^{\beta-1}$ for $\beta > 1$. Applying Lemma 2.3 for this sequence with $\alpha = \beta - 1$, we get that $[\mathcal{G}_\infty(w)] \simeq (1+\alpha) = \beta$. Lemma 2.5 asserts also that this sequence belongs to $\mathcal{A}_p$ for $p > \beta$, but does not belong to $\mathcal{A}_\beta$. This completes the proof. $\square$

3. Applications

In this section, we will prove some applications of the results in Section 2. We start with the properties of the discrete Rubio De Francia iterated algorithm which is interesting in itself and of potential future use in different contexts. To do this, we will consider the discrete Hardy-Littlewood maximal operator $\mathfrak{M}f$ which is defined by

$$\mathfrak{M}f(n) = \sup_{n \in J} \frac{1}{n} \sum_{k=1}^{n} f(k), \quad \text{for} \quad J \subset \mathbb{Z}_+. \tag{3.1}$$

THEOREM 3.1. *Fix* $1 < p < \infty$ *and* $u \in \mathcal{A}_p$. *For any non-negative sequence* $f \in \ell^p(u)$, *define*

$$\Psi f(n) := \sum_{i=0}^{\infty} \frac{\mathfrak{M}^i f(n)}{2^i \|\mathfrak{M}\|^i_{\ell^p(u)}}, \tag{3.2}$$

where for $i > 0$, $\mathfrak{M}^i f := \mathfrak{M} \circ \cdots \circ \mathfrak{M} f$ *denotes* i *iterations of the maximal operator and* $\mathfrak{M}^0 f = f$. *Then:*

(1). $f(n) \leqslant \Psi f(n)$,
(2). $\|\Psi f\|_{\ell^p(u)} \leqslant 2\|f\|_{\ell^p(u)}$,
(3). $\Psi f \in \mathcal{A}_1$ *and* $\mathcal{A}_1(\Psi f) \leqslant 2\|\mathfrak{M}\|_{\ell^p(u)}$.

Proof. (1). It is sufficient to consider the first term in (3.2) for $i=0$.

(2). By applying Minkowski's inequality to the norm of (3.2), we obtain that

$$\|\Psi f\|_{\ell^p(u)} \leqslant \sum_{i=0}^{\infty} \frac{\|\mathfrak{M}^i f\|_{\ell^p(u)}}{2^i\|\mathfrak{M}\|^i_{\ell^p(u)}} \leqslant \sum_{i=0}^{\infty} 2^{-i}\|f\|_{\ell^p(u)} = 2\|f\|_{\ell^p(u)},$$

where we have used the boundedness property of the discrete maximal operator $\mathfrak{M}f$ on the weighted discrete Lebesgue space $\ell^p(u)$.

(3). Finally, since the maximal operator is subadditive, we get that

$$\begin{aligned} \mathfrak{M}(\Psi f)(n) = \mathfrak{M}\left(\sum_{i=0}^{\infty} \frac{\mathfrak{M}^i f(n)}{2^i\|\mathfrak{M}\|^i_{\ell^p(u)}}\right) &\leqslant \sum_{i=0}^{\infty} \frac{\mathfrak{M}^{i+1} f(n)}{2^i\|\mathfrak{M}\|^i_{\ell^p(u)}} \\ &\leqslant 2\|\mathfrak{M}\|_{\ell^p(u)}\Psi f(n), \end{aligned}$$

that is $\Psi f \in \mathcal{A}_1$ with a constant $\mathcal{A}_1(\Psi f) \leqslant 2\|\mathfrak{M}\|_{\ell^p(u)}$. The proof is complete. $\square$

For applications, we will employ the discrete Rubio De Francia iterated algorithm of two $\mathcal{A}_1$ weights (reverse factorization).

THEOREM 3.2. *For $1 < p < \infty$, a weight u is in $\mathcal{A}_p$ if and only if there exist u_1, $u_2 \in \mathcal{A}_1$ such that $u = u_1 u_2^{1-p}$.*

Proof. ($\Longrightarrow$) Fix p and $u_1,\ u_2 \in \mathcal{A}_1$. Then, for any interval $J \subset \mathbb{Z}^+$, we can write that

$$\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_m \leqslant \mathcal{A}_1(u_m)u_m(k), \quad \text{for} \quad m = 1,2.$$

Let $u = u_1 u_2^{1-p}$; then we have that

$$\begin{aligned} &\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\right)^{p-1} \\ &= \frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1 u_2^{1-p}\left(\frac{1}{J}\sum_{J\subset\mathbb{Z}_+}\left[u_1 u_2^{1-p}\right]^{1-p'}\right)^{p-1} \\ &\leqslant \mathcal{A}_1(u_1)\mathcal{A}_1^{p-1}(u_2)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_2\right)^{1-p} \\ &\quad\times\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_2\right)^{p-1}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u_1\right)^{-1} = \mathcal{A}_1(u_1)\mathcal{A}_1^{p-1}(u_2). \end{aligned}$$

($\Longleftarrow$) Fix $u \in \mathcal{A}_p, 1 < p < \infty$, and let $q = pp' > 1$. Define the operator

$$S_1 f = (u)^{\frac{1}{q}} \mathfrak{M}\left(f^{p'} u^{-\frac{1}{p}}\right)^{\frac{1}{p'}}.$$

Then S_1 is sublinear and $S_1 : \ell^q \to \ell^q$ since

$$\sum_{J \subset \mathbb{Z}_+} (S_1 f)^q = \sum_{J \subset \mathbb{Z}_+} \mathfrak{M}\left(f^{p'} u^{-\frac{1}{p}}\right)^p u \leqslant \mathcal{C} \sum_{J \subset \mathbb{Z}_+} f^q,$$

where $\mathcal{C}$ depending only on $\mathcal{A}_p(u)$. Similarly, for $\sigma = u^{1-p'} \in \mathcal{A}_{p'}$, consider

$$S_2 f = \sigma^{\frac{1}{q}} \mathfrak{M}\left(f^p \sigma^{-\frac{1}{p}}\right)^{\frac{1}{p}},$$

we know that S_2 is sublinear and $S_2 : \ell^q \to \ell^q$. Define $S = S_1 + S_2$ and form the Rubio de Francia iteration algorithm

$$\Psi f(n) = \sum_{k=0}^{\infty} \frac{S^k f(n)}{2^k \|S\|_{\ell^q}^k},$$

then, by the proof of Theorem 3.1, it follows that $\Psi : \ell^q \to \ell^q$. Fix any non-zero function $f \in \ell^q$ then Ψf is finite almost everywhere. Moreover, $S(\Psi f)(n) \leqslant 2\|S\|_{\ell^q} \Psi f(n)$. In particular, we have that

$$u^{\frac{1}{q}} \mathfrak{M}\left((\Psi f)^{p'} u^{-\frac{1}{p}}\right)^{\frac{1}{p'}} = S_1(\Psi f) \lesssim \Psi f.$$

Hence, if we let $u_2 = (\Psi f)^{p'} u^{-\frac{1}{p}}$, then this inequality becomes $\mathfrak{M} u_2 \lesssim u_2$, so $u_2 \in \mathcal{A}_1$. Similarly, if we repeat this argument with S_2 in place of S_1, we get $u_1 = (\Psi f)^p \sigma^{-\frac{1}{p'}} \in \mathcal{A}_1$. Moreover, it is immediate that $u_1 u_2^{1-p} = u^{\frac{1}{p}} u^{\frac{1}{p'}} = u$. This completes the proof. $\square$

Next, we present the equivalence between the weight u that belongs to the class $\mathcal{A}_1 \cap \mathcal{G}_s$ and the weight u^s that belongs to $\mathcal{A}_1$ class, which extends Lemma 3.3 to $\mathcal{A}_1$ weights. This result will be used later for the proof of Theorem 3.3 below.

LEMMA 3.1. *Given a weight u and $s > 1$, then $u \in \mathcal{A}_1 \cap \mathcal{G}_s$ if and only if $u^s \in \mathcal{A}_1$.*

Proof. Suppose first that $u \in \mathcal{A}_1 \cap \mathcal{G}_s$. Then, we have that

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^s \lesssim \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u\right)^s \lesssim \operatorname*{ess\,inf}_{J \subset \mathbb{Z}^+} u^s.$$

Hence, $u^s \in \mathcal{A}_1$. Conversely, suppose $u^s \in \mathcal{A}_1$. By Hölder's inequality, we can write that

$$\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \leqslant \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^s\right)^{\frac{1}{s}} \lesssim \operatorname*{ess\,inf}_{J \subset \mathbb{Z}^+} u \leqslant \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u.$$

It follows at once that $u \in \mathcal{A}_1 \cap \mathcal{G}_s$. This completes the proof. $\square$

The next two lemmas consider dilations of $\mathcal{A}_1$ and $\mathcal{G}_\infty$ weights.

LEMMA 3.2. *If $u \in \mathcal{A}_1$, then $u^{-r} \in \mathcal{G}_\infty$ for any $r > 0$.*

Proof. By Hölder's inequality with exponent $p = 1 + r$, we can write that

$$1 = \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{\frac{1}{p'}} u^{-\frac{1}{p'}} \leqslant \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{\frac{r}{1+r}} \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{-r} \right)^{\frac{1}{1+r}}.$$

If we combine this with the fact that $u \in \mathcal{A}_1$, we get that

$$u^{-r} \lesssim \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{-r} \leqslant \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{-r}.$$

Hence, $u^{-r} \in \mathcal{G}_\infty$. This completes the proof. $\square$

LEMMA 3.3. *If $u \in \mathcal{G}_\infty$, then $u^r \in \mathcal{G}_\infty$ for any $r > 0$.*

Proof. If $r > 1$, from Hölder's inequality we see that

$$u^r \lesssim \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u \right)^{r} \leqslant \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^{r}.$$

If $r < 1$, then, since $u \in \mathcal{A}_\infty$, by Lemma 3.3, $u^r \in \mathcal{G}^{1/r}$. Hence, we can repeat the above argument using the reverse Hölder inequality to get that $u^r \in \mathcal{G}_\infty$. This completes the proof. $\square$

We conclude this part by the following result, which gives the factorization of some weights u belong to the class $\mathcal{A}_p \cap \mathcal{G}_s$ in terms of other weights belong to $\mathcal{A}_1 \cap \mathcal{G}_s$ and $\mathcal{A}_p \cap \mathcal{G}_\infty$, respectively.

THEOREM 3.3. *For $1 < p,\ s < \infty$, the weight $u \in \mathcal{A}_p \cap \mathcal{G}_s$ if and only if there exist weights $v_1,\ v_2$ such that $u = v_1 v_2$, $v_1 \in \mathcal{A}_1 \cap \mathcal{G}_s$ and $v_2 \in \mathcal{A}_p \cap \mathcal{G}_\infty$.*

Proof. We first fix $v_1 \in \mathcal{A}_1 \cap \mathcal{G}_s$ and $v_2 \in \mathcal{A}_p \cap \mathcal{G}_\infty$. By Lemmas 3.1 and 3.3, $v_1^s \in \mathcal{A}_1$ and $v_2^s \in \mathcal{G}_\infty$. Then, we have that

$$\begin{aligned} \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} u^s &\lesssim \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} v_1^s \right) \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} v_2^s \right) \\ &\lesssim \frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} v_1^s \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} v_2 \right)^s \lesssim \left(\frac{1}{|J|} \sum_{J \subset \mathbb{Z}_+} v_1 v_2 \right)^s. \end{aligned}$$

Thus, $u \in \mathcal{G}_s$. Similarly, by Lemma 3.2, $v_1^{1-p'} \in \mathcal{G}_\infty$ and $v_1,\ v_2 \in \mathcal{A}_p$, and so

$$
\begin{aligned}
&\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v_1 v_2 \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} [v_1 v_2]^{1-p'}\right)^{p-1} \\
&\lesssim \left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v_1\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v_2\right)\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v_1^{1-p'}\right)^{p-1}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v_2^{1-p'}\right)^{p-1} \\
&\leqslant \mathcal{A}_p(v_1)\mathcal{A}_p(v_2).
\end{aligned}
$$

Thus $u \in \mathcal{A}_p$. To prove the converse, fix $u \in \mathcal{A}_p \cap \mathcal{G}_s$. Then by Lemma 3.3, $u^s \in \mathcal{A}_q$ with $q = s(p-1)+1$. But then by Theorem 3.2 there exist $u_1, u_2 \in \mathcal{A}_1$ such that $u^s = u_1 u_2^{1-q}$, or equivalently, $u = u_1^{\frac{1}{s}} u_2^{1-p} = v_1 v_2$. By Lemma 3.1, $v_1 \in \mathcal{A}_1 \cap \mathcal{G}_s$, and again by Theorem 3.2 and Lemma 3.2, $v_2 \in \mathcal{A}_p \cap \mathcal{G}_\infty$. This completes the proof. $\square$

In the following, we use Theorems 2.6, 2.8 for the extension to the classes $\mathcal{A}_p$ and $\mathcal{G}_r$, and, as we shall see, the range of the indices will be governed by factorizations of the weights. For the continuous case, the authors in [5] have shown that $w \in \mathcal{G}_r$ iff $w = w_0 w_1$ where $w_0 \in \mathcal{G}_\infty$ and $w_1 \in \mathcal{G}_r \cap \mathcal{A}_1$. But then $w_1^r \in \mathcal{A}_1$, and thus $w \in \mathcal{G}_r$ iff $w = uv^{1/r}$ with $u \in \mathcal{G}_\infty$ and $v \in \mathcal{A}_1$. We shall also use the Jones Factorization Theorem for $\mathcal{A}_p$, Theorem 3.2, i.e. $w \in \mathcal{A}_p$ iff $w = uv^{1-p}$ with $u,\ v \in \mathcal{A}_1$.

THEOREM 3.4. *Let $w = uv^{1/r}$ be in $\mathcal{G}_r$ with $u \in \mathcal{G}_\infty$ and $v \in \mathcal{A}_1$, and let $\mathcal{C}_1 = \mathcal{G}_\infty(u)$. Then $w \in \mathcal{A}_p$ for all $p > \mathcal{C}_1$. This range of p is optimal.*

Proof. Let $p > \mathcal{C}_1$. Then

$$
\begin{aligned}
&\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} uv^{1/r}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\left(uv^{1/r}\right)^{1-p'}\right)^{p-1} \\
&\leqslant \sup_{J\subset\mathbb{Z}^+} u\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} v^{1/r}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\right)^{p-1}\sup_{J\subset\mathbb{Z}^+}\frac{1}{v^{1/r}} \\
&\leqslant \frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u\inf_J v^{1/r}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} u^{1-p'}\right)^{p-1}\sup_{J\subset\mathbb{Z}^+}\frac{1}{v^{1/r}} \leqslant \mathcal{C},
\end{aligned}
$$

since $u \in \mathcal{A}_p$ for $p > \mathcal{C}_1$ by Theorem 2.8. From Corollary 2.3, we conclude that the sequence $w(n) = n^{\mathcal{C}_1 - 1}$ belongs to $\mathcal{G}_\infty \subset \mathcal{G}_r$ which shows that the range of p is best possible. This completes the proof. $\square$

The next result will give us the precise range of higher summability for $w \in \mathcal{G}_r$.

THEOREM 3.5. *Let $w = uv^{1/r}$ be in $\mathcal{G}_r$ with $u \in \mathcal{G}_\infty$ and $v \in \mathcal{A}_1$. If $\mathcal{C}_2 = \mathcal{A}_1(v)$ then $w \in \mathcal{G}_p$ for all $r \leqslant p < \mathcal{C}_2 r/(\mathcal{C}_2 - 1)$. This range of p is optimal.*

Proof. Let p satisfies the above inequality, and then choose $q>1$ such that

$$p<\frac{\mathcal{C}_2 r}{q(\mathcal{C}_2-1)}.$$

Since

$$1\leqslant\frac{pq}{r}<\frac{\mathcal{C}_2}{\mathcal{C}_2-1},$$

then by Theorem 2.6, we get that $v\in\mathcal{G}_{qp/r}$. This and Hölder's inequality and the fact $u\in\mathcal{G}_\infty$ give us that

$$\begin{aligned}\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}w^p=\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^pv^{p/r}&\leqslant\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^{q'p}\right)^{1/q'}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}v^{qp/r}\right)^{1/q}\\&\leqslant\sup_J u^p\cdot\mathcal{C}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}v\right)^{p/r}\leqslant\mathcal{C}'\sup_J u^p\cdot\inf_J v^{p/r}\\&\leqslant\mathcal{C}''\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}uv^{1/r}\right)^p.\end{aligned}$$

Let $0<\alpha<1$ and consider the sequence $w(n)=n^{-\alpha}$. Using the estimate in Lemma (2.6), then $w\in\mathcal{G}_r$ for $1<r<1/\alpha$. We fix such an r and write $w=v^{1/r}$, and $v=n^{-\alpha r}$. This is the factorization $w=uv^{1/r}$ with $u\equiv1$. By recalling the estimate in Corollary 2.2, we get that $\mathcal{A}_1(v)=\mathcal{C}_2=1/(1-\alpha r)$. That is $\mathcal{C}_2r/(\mathcal{C}_2-1)=1/\alpha$ which is the precise upper bound of higher summability for this weight. This completes the proof. $\square$

For the next Theorem, we need the fact that $v\in\mathcal{A}_1$ implies that $(1/v)^\gamma\in\mathcal{G}_\infty$ for every $\gamma>0$ (see Lemma 3.2).

THEOREM 3.6. *Let $w=uv^{1-p}$ be in $\mathcal{A}_p$ with $u,\ v\in\mathcal{A}_1$, and let $\mathcal{C}=\mathcal{A}_1(u)$. Then $w\in\mathcal{G}_r$ for all $1<r<\mathcal{C}/(\mathcal{C}-1)$. This range of r is optimal.*

Proof. Using Theorem 2.6, we obtain that

$$\begin{aligned}\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}w^r=\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^r\frac{1}{v^{r(p-1)}}&\leqslant\frac{1}{|J|}\frac{1}{v^{r(p-1)}}\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u^r\\&\leqslant\mathcal{C}\sup_J\frac{1}{v^{r(p-1)}}\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}u\right)^r\\&\leqslant\mathcal{C}\mathcal{A}_1^r(u)\sup_{J\subset\mathbb{Z}^+}\frac{1}{v^{r(p-1)}}\cdot\left(\inf_{J\subset\mathbb{Z}^+}u\right)^r\\&\leqslant\mathcal{C}A_1^r(u)\mathcal{C}^r\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}\frac{1}{v^{p-1}}\cdot\inf_{J\subset\mathbb{Z}^+}u\right)^r\\&\leqslant\mathcal{C}\mathcal{A}_1^r(u)\mathcal{C}^r\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+}w\right)^r.\end{aligned}$$

To prove the optimality of the range for the exponent r, we use the weight sequence $w(n) = n^{\frac{1}{\mathcal{C}}-1}$ for $\mathcal{C} > 1$. Applying Lemma 2.6 for this sequence with $\alpha = \frac{1}{\mathcal{C}} - 1$, we get that $w \in \mathcal{G}_r(w)$ for $\alpha > -1/r$. Which leads to the best possible range $1 < r < \frac{\mathcal{C}}{\mathcal{C}-1}$ after direct substitutions. This completes the proof. □

COROLLARY 3.1. *Let $w = uv^{1-p}$ be in $\mathcal{A}_p$ with $u,\ v \in \mathcal{A}_1$. If $\mathcal{C} = \max\{\mathcal{A}_1(u), \mathcal{A}_1(v)\}$, then $w^\tau \in \mathcal{A}_p$ for $1 \leqslant \tau < \mathcal{C}/(\mathcal{C}-1)$. This range of τ is optimal.*

Proof. From Theorem 3.6 we have that w and $w^{1-p'}$ are in $\mathcal{G}_\tau$ for $1 \leqslant \tau < \mathcal{C}/(\mathcal{C}-1)$ and hence $w^\tau \in \mathcal{A}_p$. Again using Corollary 2.2, we observe that the sequence $w(n) = n^{-\alpha}, 0 < \alpha < 1$ belongs to $\mathcal{A}_1 \subset \mathcal{A}_p$, and thus we can take $u = n^{-\alpha}$, $v \equiv 1$. Then $\mathcal{C} = 1/(1-\alpha)$ and thus $\mathcal{C}/(\mathcal{C}-1) = 1/\alpha$ which is best possible. This completes the proof. □

We will now use Theorem 3.6 to investigate the exact range on $q < p$ such that $w \in \mathcal{A}_p$ implies $w \in \mathcal{A}_q$.

THEOREM 3.7. *Let $w = uv^{1-p}$ be in $\mathcal{A}_p$ with $u,\ v \in \mathcal{A}_1$ and let $\mathcal{C}_* = \mathcal{A}_1(v)$. Then $w \in \mathcal{A}_q$ for all q satisfying*

$$\frac{(p-1)(\mathcal{C}_*-1)}{\mathcal{C}_*} + 1 < q \leqslant p.$$

This range of q is optimal.

Proof. Since $w^{1-p'} = vu^{1-p'}$ is in $\mathcal{A}_{p'}$ we get from Theorem 3.6 that $w^{1-p'} \in \mathcal{G}_r$ for $1 < r < \mathcal{C}_*/(\mathcal{C}_*-1)$. Hence since $w \in \mathcal{A}_p$

$$\begin{aligned}
&\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} w\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} w^{r(1-p')}\right)^{(p-1)/r} \\
&\leqslant \mathcal{C}\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} w\left(\frac{1}{|J|}\sum_{J\subset\mathbb{Z}_+} w^{1-p'}\right)^{p-1} \\
&\leqslant \mathcal{C}' < \infty.
\end{aligned}$$

Thus $w \in \mathcal{A}_q$, for $q = 1 + (p-1)/r$, that is $(p-1)(\mathcal{C}_*-1)/\mathcal{C}_* + 1 < q \leqslant p$. We will now show that this range is best possible by considering the sequence $w(n) = n$ and fix $p_0 > 2$. Then, applying the estimates in Lemma 2.5 and Corollary 2.2 respectively to get that $w \in \mathcal{A}_{p_0}$ and $w = v^{1-p_0}$ with $v = n^{1-p_0'} \in \mathcal{A}_1$. Since $\mathcal{A}_1(v) = 1/(2-p_0') = \mathcal{C}_*$ the lower bound of the range of q given above is 2. This completes the proof. □

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