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SHARP BOUNDS ON THE FOURTH–ORDER HERMITIAN TOEPLITZ DETERMINANT FOR STARLIKE FUNCTIONS OF ORDER 1/2

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Abstract. In this paper, we prove the sharp bounds on the fourth-order Hermitian Toeplitz determinant for starlike functions of order $1/2$, which solves a conjecture posed by Cudna *et al.* [10] for the case $q = 4$.

1. Introduction

Let $\mathcal{A}$ denote the class of functions *analytic* in the unit disk $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ and of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n. \tag{1.1}$$

We denote $\mathcal{S}$ by the subclass of $\mathcal{A}$ consisting of univalent functions.

Given $\alpha \in [0,1)$, a function $f \in \mathcal{A}$ is called starlike function of order α, if

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

We denote this class by $\mathcal{S}^*(\alpha)$.

Similarly, for $\alpha \in [0,1)$, a function $f \in \mathcal{A}$ is called convex function of order α, if

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \quad (z \in \mathbb{D}).$$

This class is denoted by $\mathcal{S}^c(\alpha)$.

Both $\mathcal{S}^*(\alpha)$ and $\mathcal{S}^c(\alpha)$ were introduced by Robertson [36], they are subclasses of $\mathcal{S}$. In particular, $\mathcal{S}^* := \mathcal{S}^*(0)$ and $\mathcal{S}^c := \mathcal{S}^c(0)$ are the classes of starlike and convex functions (see [11]), respectively. The class $\mathcal{S}^*(1/2)$ plays important roles in geometry function theory, one of significant results was given by Marx [29] and Strohhäcker [37], they proved that

$$\mathcal{S}^c \subset \mathcal{S}^*(1/2). \tag{1.2}$$

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The function

$$f(z)=\frac{z}{1-z} \quad (z\in\mathbb{D}) \tag{1.3}$$

plays as an extremal function for both $\mathcal{S}^c$ and $\mathcal{S}^*(1/2)$.

Recently, Cunda *et al.* [10] (see also [21]) introduced the notion of Hermitian Toeplitz determinants for the class $\mathcal{A}$, and some of its subclasses. Hermitian Toeplitz matrices play an important role in functional analysis, applied mathematics as well as in physics and technical sciences, e.g., in the Szegö theory, the stochastic filtering, the signal processing, the biological information processing and other engineering problems.

Given $q,n\in\mathbb{N}$, the Hermitian Toeplitz matrix $T_{q,n}(f)$ of a function $f\in\mathcal{A}$ of the form (1.1) is defined by

$$T_{q,n}(f)=\begin{pmatrix} a_n & a_{n+1} & \cdots & a_{n+q-1} \\ \overline{a}_{n+1} & a_n & \cdots & a_{n+q-2} \\ \vdots & \vdots & \vdots & \vdots \\ \overline{a}_{n+q-1} & \overline{a}_{n+q-2} & \cdots & a_n \end{pmatrix},$$

where $\overline{a}_k:=\overline{a_k}$. For convenience, we let $\det(T_{q,n})(f)$ denote the determinant of $T_{q,n}(f)$.

By the definition, $\det(T_{4,1})(f)$ is given by

$$\det(T_{4,1})(f)=\begin{vmatrix} a_1 & a_2 & a_3 & a_4 \\ \overline{a}_2 & a_1 & a_2 & a_3 \\ \overline{a}_3 & \overline{a}_2 & a_1 & a_2 \\ \overline{a}_4 & \overline{a}_3 & \overline{a}_2 & a_1 \end{vmatrix}.$$

Note that for $f\in\mathcal{A}$, we get $a_1=1$, and $\det(T_{4,1})(f)$ reduces to

$$\begin{aligned}\det(T_{4,1})(f)=&1-3|a_2|^2+|a_2|^4-2|a_2|^2|a_3|^2-2|a_3|^2+|a_3|^4+|a_2|^2|a_4|^2-|a_4|^2\\ &+4\Re\left(a_2^2\overline{a}_3\right)+4\Re\left(a_2a_3\overline{a}_4\right)-2\Re\left(a_2^3\overline{a}_4\right)-2\Re\left(a_2\overline{a}_3^2a_4\right).\end{aligned} \tag{1.4}$$

In recent years, many scholars devoted to finding bounds of determinants, whose elements are coefficients of functions in $\mathcal{A}$, or its subclasses. Hankel matrices, i.e., square matrices which have constant entries along the reverse diagonal, and the symmetric Toeplitz determinant (see [1]) are of particular interests.

The sharp upper bounds on the second Hankel determinants were obtained by [5, 8, 12, 13, 26, 30], for various classes of analytic functions. We refer to [3, 6, 7, 15, 33, 35, 38, 40, 41] for discussions on the upper bounds of the third Hankel determinants for various classes of univalent functions. However, these results are far from sharpness. In a recent paper, Kwon *et al.* [17] found such a formula of expressing c_4 for Carathéodory functions. The sharp results of the third Hankel determinants are found for some classes of univalent functions (see e.g., [4, 16, 18, 20, 23]). Moreover, Wang *et al.* [39] determined the fourth-order Hankel determinant of a subclass of analytic functions.

Recently, Lecko *et al.* [23] and Kowalczyk *et al.* [16] obtained sharp bounds of the third Hankel determinants for the classes $\mathcal{S}^*(1/2)$ and $\mathcal{S}^c$, respectively. We note that Rath *et al.* [34] pointed out there was an error in the proof of Theorem A, they also gave a new corrected proof. Moreover, Nunokawa and Sokół [31] derived several criteria for $\mathcal{S}^*(1/2)$.

THEOREM A. *If $f \in \mathcal{S}^*(1/2)$ be of the form* (1.1)*, then the third Hankel determinant*

$$|\det(H_{3,1})(f)| = \left|a_3(a_2a_4 - a_3^2) - a_4(a_4 - a_2a_3) + a_5(a_3 - a_2^2)\right| \leqslant \frac{1}{9}.$$

The result is sharp, with the extremal function given by

$$f(z) = \frac{z}{\sqrt[3]{1-z^3}} = z + \frac{1}{3}z^4 + \frac{2}{9}z^7 + \cdots. \tag{1.5}$$

THEOREM B. *If $f \in \mathcal{S}^c$ be of the form* (1.1)*, then the third Hankel determinant*

$$|\det(H_{3,1})(f)| \leqslant \frac{4}{135}.$$

The result is sharp, with the extremal function given by

$$f(z) = \arctan z = \frac{1}{2i}\log\frac{1+iz}{1-iz} = z - \frac{1}{3}z^3 + \frac{1}{5}z^5 + \cdots. \tag{1.6}$$

The Hermitian Toeplitz determinants related to normalized analytic functions is a natural concept to study. We refer to [2, 9, 10, 14, 19, 21, 22, 25, 32] for discussions on the sharp bounds of the Hermitian Toeplitz determinants for various classes of univalent functions. Furthermore, in 2020, Cudna *et al.* [10] obtained sharp bounds of the second and third-order Hermitian Toeplitz determinants for starlike and convex functions of order α, and proposed a conjecture about the sharp estimates of $\det(T_{q,1})(f)$ $(q \in \mathbb{N}\setminus\{1\}; \mathbb{N} := \{1,2,3,\cdots\})$.

CONJECTURE 1.1. *If $f \in \mathcal{S}^*(1/2)$ (or $f \in \mathcal{S}^c$) be of the form* (1.1)*, then*

$$0 \leqslant \det(T_{q,1})(f) \leqslant 1 \quad (q \in \mathbb{N}\setminus\{1\}).$$

All of inequalities are sharp.

The cases for $q = 2, 3$ of Conjecture 1.1 were proved by Cudna *et al.* [10]. However, the problem of finding sharp estimates of the Hermitian Toeplitz determinants $\det(T_{q,1})(f)$ for $q \geqslant 4$ is technically much more difficult, and few sharp bounds have been obtained. Recently, Lecko *et al.* [24] proved the above conjecture for $q = 4$ in the class $\mathcal{S}^c$. The purpose of this paper is to prove this conjecture for $q = 4$ in the class $\mathcal{S}^*(1/2)$.

Denote $\mathcal{P}$ by the class of *Carathéodory functions* p normalized by

$$p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \quad (z \in \mathbb{D}), \tag{1.7}$$

and satisfy the condition $\Re(p(z)) > 0$.

The following result will be required in the proof of our main result.

LEMMA 1.1. (See [27, 28]) *If $p \in \mathcal{P}$, then*

$$2c_2 = c_1^2 + (4 - c_1^2)\zeta, \tag{1.8}$$

and

$$4c_3 = c_1^3 + (4 - c_1^2)c_1\zeta(2-\zeta) + 2(4 - c_1^2)(1 - |\zeta|^2)\eta \tag{1.9}$$

for some $\zeta, \eta \in \overline{\mathbb{D}} := \{z \in \mathbb{C} : |z| \leqslant 1\}$.

2. Main result

In this section, we will prove the sharp estimates of fourth-order Hermitian Toeplitz determinant $\det(T_{4,1})(f)$ for $f \in \mathcal{S}^*(1/2)$, which gives an affirmative answer to Conjecture 1.1 for the case $q = 4$.

THEOREM 2.1. *If $f \in \mathcal{S}^*(1/2)$, then*

$$0 \leqslant \det(T_{4,1})(f) \leqslant 1. \tag{2.1}$$

The left inequality is sharp for the extremal function given by (1.3)*, and the right inequality is sharp for the identity function $f(z) = z$.*

Proof. For the function $f \in \mathcal{S}^*(1/2)$ of the form (1.1), we know that there exists an analytic function $p \in \mathcal{P}$ in the unit disk $\mathbb{D}$ with $p(0) = 1$ and $\Re(p(z)) > 0$ such that

$$\frac{zf'(z)}{f(z)} = \frac{1}{2}\big(p(z)+1\big) \quad (z \in \mathbb{D}).$$

By elementary calculations, we have

$$z + \sum_{n=2}^{\infty} na_n z^n = \frac{1}{2}\left(2 + \sum_{n=1}^{\infty} c_n z^n\right)\left(z + \sum_{n=2}^{\infty} a_n z^n\right). \tag{2.2}$$

It follows from (2.2) that

$$a_2 = \frac{1}{2}c_1, \ \ a_3 = \frac{1}{8}(2c_2 + c_1^2), \ \ a_4 = \frac{1}{48}(8c_3 + 6c_1c_2 + c_1^3). \tag{2.3}$$

Since the class $\mathcal{S}^*(1/2)$ and $\det(T_{4,1})$ are rotationally invariants, we may assume that $c := c_1 \in [0,2]$. Thus, we find from (1.4) and (2.3) that

$$\begin{aligned}\det(T_{4,1})(f) = \frac{1}{36864}\Big[&36864 - 27648c^2 + 5760c^4 - 304c^6 + c^8 \\ &- 4608|c_2|^2 + 144|c_2|^4 + 72c^2(8-c^2)|c_2|^2 - 256(4-c^2)|c_3|^2 \\ &+ 192c^2(24-5c^2)\cdot\Re(c_2) + 144c^4\cdot\big[\Re(c_2)\big]^2 - 48c^4\cdot\Re(c_2^2) \\ &- 32c^3(8+c^2)\cdot\Re(c_3) + 1536c\cdot\Re(c_2\overline{c_3}) - 384c\cdot\Re(c_2^2\overline{c_3})\Big].\end{aligned}$$

Hence, by using (1.8) and (1.9), we get

$$\begin{aligned}\det(T_{4,1})(f) = \frac{1}{36864}(4-c^2)^3 \cdot \Big[&576 - 4(72+c^2)\cdot|\zeta|^2 + 4c^2\cdot|\zeta|^2\cdot\Re(\zeta) \\ &+ (36-c^2)\cdot|\zeta|^4 - 32c\cdot(1-|\zeta|^2)\cdot\Re(\overline{\zeta}\eta) \\ &+ 16c\cdot(1-|\zeta|^2)\cdot\Re(\overline{\zeta^2}\eta) - 64\cdot(1-|\zeta|^2)^2\cdot|\eta|^2\Big]\end{aligned} \tag{2.4}$$

for some $\zeta,\ \eta \in \overline{\mathbb{D}}$ and $c \in [0,2]$.

We now consider the lower and upper bounds for the class $\mathcal{S}^*(1/2)$ for various cases.

Case 1. Suppose that $\zeta = 0$. Then

$$0 \leqslant \det(T_{4,1})(f) = \frac{1}{576}(4-c^2)^3\cdot\left(9-|\eta|^2\right) \leqslant 1 \tag{2.5}$$

for all $c \in [0,2]$ and $\eta \in \overline{\mathbb{D}}$.

Case 2. Suppose that $\eta = 0$. Then

$$\begin{aligned}\det(T_{4,1})(f) = \frac{1}{36864}(4-c^2)^3 \cdot \Big[&576 - 4(72+c^2)\cdot|\zeta|^2 + 4c^2\cdot|\zeta|^2\cdot\Re(\zeta) \\ &+ (36-c^2)\cdot|\zeta|^4\Big].\end{aligned} \tag{2.6}$$

We observe that

$$\begin{aligned}&576 - 4(72+c^2)\cdot|\zeta|^2 + 4c^2\cdot|\zeta|^2\cdot\Re(\zeta) + (36-c^2)\cdot|\zeta|^4 \\ \geqslant\ & 576 - 4(72+c^2)\cdot|\zeta|^2 - 4c^2\cdot|\zeta|^3 + (36-c^2)\cdot|\zeta|^4 \\ =\ & 576 - 288|\zeta|^2 + 36|\zeta|^4 - c^2\cdot|\zeta|^2\cdot(2+|\zeta|)^2 \\ \geqslant\ & 576 - 288|\zeta|^2 + 36|\zeta|^4 - 4|\zeta|^2\cdot(2+|\zeta|)^2 \\ =\ & 576 - 304|\zeta|^2 - 16|\zeta|^3 + 32|\zeta|^4 \\ \geqslant\ & 576 - 304|\zeta|^2 - 16|\zeta|^3 \geqslant 256 \quad (0 \leqslant |\zeta| \leqslant 1).\end{aligned}$$

It follows from (2.6) that $\det(T_{4,1})(f) \geqslant 0$.

Similarly, we know that

$$\begin{aligned}&576 - 4(72+c^2)\cdot|\zeta|^2 + 4c^2\cdot|\zeta|^2\cdot\Re(\zeta) + (36-c^2)\cdot|\zeta|^4 \\ \leqslant\ & 576 - 4(72+c^2)\cdot|\zeta|^2 + 4c^2\cdot|\zeta|^3 + (36-c^2)\cdot|\zeta|^4 \\ =\ & 576 - 288|\zeta|^2 + 36|\zeta|^4 - c^2\cdot|\zeta|^2\cdot(2-|\zeta|)^2 \\ \leqslant\ & 576 - 288|\zeta|^2 + 36|\zeta|^4 \leqslant 576 \quad (0 \leqslant |\zeta| \leqslant 1).\end{aligned}$$

Hence, we find from (2.6) that $\det(T_{4,1})(f) \leqslant 1$.

Case 3. Suppose that $\zeta,\ \eta \in \overline{\mathbb{D}} \setminus \{0\}$. Then, there exist unique θ and φ in $[0,2\pi)$ such that $\zeta = xe^{i\theta}$ and $\eta = ye^{i\varphi}$, where $x := |\zeta| \in (0,1]$ and $y := |\eta| \in (0,1]$. Thus, from (2.4), we get

$$\det(T_{4,1})(f) = \frac{1}{36864}(4-c^2)^3 \cdot F(c,x,y,\theta,\varphi), \tag{2.7}$$

where

$$\begin{aligned} F(c,x,y,\theta,\varphi) = 576 &- 4(72+c^2)x^2 + 4c^2x^3\cos\theta + (36-c^2)x^4 \\ &- 32c(1-x^2)xy\cos(\theta-\varphi) + 16c(1-x^2)x^2y\cos(2\theta-\varphi) \\ &- 64(1-x^2)^2y^2. \end{aligned}$$

For $c \in [0,2]$ and $x,\ y \in (0,1]$, we have

$$G(c,x,y) \leqslant F(c,x,y,\theta,\varphi) \leqslant H(c,x,y), \tag{2.8}$$

where

$$\begin{aligned} G(c,x,y) := 576 - 4(72+c^2)x^2 - 4c^2x^3 + (36-c^2)x^4 \\ - 16c(1-x^2)(2+x)xy - 64(1-x^2)^2y^2, \end{aligned} \tag{2.9}$$

and

$$\begin{aligned} H(c,x,y) := 576 - 4(72+c^2)x^2 + 4c^2x^3 + (36-c^2)x^4 \\ + 16c(1-x^2)(2+x)xy - 64(1-x^2)^2y^2. \end{aligned} \tag{2.10}$$

Case 3.1. We discuss the lower bound of $\det(T_{4,1})(f)$. By observing that

$$\begin{aligned} G(c,x,y) &= 576 - 4(72+c^2)x^2 - 4c^2x^3 + (36-c^2)x^4 \\ &\quad - 16c(1-x^2)(2+x)xy - 64(1-x^2)^2y^2 \\ &\geqslant 576 - 304x^2 - 16x^3 + 32x^4 - 32(1-x^2)(2+x)x - 64(1-x^2)^2 \\ &= 512 - 64x - 208x^2 + 48x^3 \\ &\geqslant 512 - 64x - 208x^2 \geqslant 240 \quad (c \in [0,2];\, x,y \in (0,1]), \end{aligned}$$

it follows from (2.7) and (2.8) that

$$\begin{aligned} \det(T_{4,1})(f) &= \frac{1}{36864}(4-c^2)^3 \cdot F(c,x,y,\theta,\varphi) \\ &\geqslant \frac{1}{36864}(4-c^2)^3 \cdot G(c,x,y) \geqslant \frac{5}{768}(4-c^2)^3 \geqslant 0 \quad (c \in [0,2]), \end{aligned}$$

which together with Cases 1 and 2 lead to the lower bound in (2.1).

Case 3.2. We discuss the upper bound of $\det(T_{4,1})(f)$.

Let $x = 1$. Then

$$H(c,1,y) = 324 - c^2 \leqslant 324 \quad (c \in [0,2];\, y \in (0,1]).$$

Let $x \in (0,1)$. Then

$$y_0 = \frac{cx(2+x)}{8(1-x^2)} \geqslant 0, \;\; -64(1-x^2)^2 < 0.$$

Therefore, we consider the following two cases.

Case 3.2.1. If $y_0 < 1$, that is $x \in (0, x_0(c))$, where

$$x_0(c) = \frac{-c+\sqrt{c^2+8c+64}}{c+8} \quad (c \in [0,2]).$$

Note that $x_0(c) \leqslant 1$ for all $c \in [0,2]$ and $x_0(0) = 1$. Let

$$\triangle_1 := \Big\{(c,x):\; 0 \leqslant c \leqslant 2;\; 0 \leqslant x \leqslant x_0(c)\Big\}.$$

We find that

$$H(c,x,y) \leqslant H(c,x,y_0) =: h(c,x) \quad ((c,x) \in \triangle_1;\, y \in (0,1]),$$

where

$$\begin{aligned} h(c,x) &= 576 - 288x^2 + 8c^2x^3 + 36x^4 \\ &= 36(4-x^2)^2 + 8c^2x^3 \quad ((c,x) \in \triangle_1). \end{aligned}$$

(i) On the vertices of $\triangle_1$, we have

$$h(0,0) = 576, \;\; h(0,x_0(0)) = h(0,1) = 324, \;\; h(2,0) = 576,$$
$$h(2,x_0(2)) = h\left(2, \frac{\sqrt{21}-1}{5}\right) = \frac{32\big(6619+471\sqrt{21}\big)}{625} \approx 449.403 < 576.$$

(ii) On the side $x = 0$, we get

$$h(c,0) = 576 \quad (c \in (0,2)).$$

(iii) On the side $x = x_0(c)$ for $c \in (0,2)$, we obtain

$$\begin{aligned} h(c,x_0(c)) =& \frac{32}{(c+8)^4}\Big[-c^6 - 14c^5 - 87c^4 - 96c^3 + 4680c^2 + 24192c + 41472 \\ &+ (c^5 + 10c^4 + 41c^3 + 380c^2 + 864c)\sqrt{c^2+8c+64}\Big] =: \gamma(c). \end{aligned}$$

Now, we shall prove that γ is an increasing function. Note that

$$\gamma'(c) = \frac{64\Big[\sigma_1(c) + \sigma_2(c)\sqrt{c^2+8c+64}\Big]}{(c+8)^5\sqrt{c^2+8c+64}} > 0 \quad (c \in (0,2))$$

is equivalent to

$$\sigma_1(c)+\sigma_2(c)\sqrt{c^2+8c+64}>0 \quad (c\in(0,2)),$$

where

$$\sigma_1(c)=c^7+35c^6+428c^5+3104c^4+14936c^3 \\ +35840c^2+153088c+221184 \quad (c\in(0,2))$$

and

$$\sigma_2(c)=-c^6-31c^5-280c^4-1344c^3-5832c^2+1152c+13824 \quad (c\in(0,2)).$$

Since

$$\nabla:=\sqrt{c^2+8c+64}\leqslant 2\sqrt{21}\approx 9.1652 \quad (c\in(0,2)),$$

we have

$$\begin{aligned}\sigma_1(c)+\sigma_2(c)\nabla &= c^7+(35-\nabla)c^6+(428-31\nabla)c^5+(3104-280\nabla)c^4\\ &\quad+(14936-1344\nabla)c^3+(35840-3000\nabla)c^2+(153088-2832c\nabla)c\\ &\quad+1152c\nabla+13824\nabla+221184>0 \quad (c\in(0,2)),\end{aligned}$$

we see that γ is an increasing function for $c\in(0,2)$, thus,

$$h(c,x_0(c))=\gamma(c)\leqslant\gamma(2)=\frac{32\left(6619+471\sqrt{21}\right)}{625}\approx 449.403<576 \quad (c\in(0,2)).$$

(iv) On the side $c=0$, we get

$$h(0,x)=36(4-x^2)^2\leqslant 576 \quad (x\in(0,1]).$$

(v) On the side $c=2$, we have

$$h(2,x)=576-(288-32x-36x^2)x^2\leqslant 576 \quad (x\in(0,x_0(2)]).$$

(vi) It remains to consider the interior of $\triangle_1$. Since the system of equations

$$\begin{cases}\partial h/\partial c=16cx^3=0\\ \partial h/\partial x=-576x+24c^2x^2+144x^3=0\end{cases}$$

has solutions $(0,0)$, $(0,2)$ and $(0,-2)$, we see that h has no critical point in the interior of $\triangle_1$.

Case 3.2.2. If $y_0\geqslant 1$, that is $x\in[x_0(c),1]$ for all $c\in[0,2]$. Let

$$\triangle_2:=\{(c,x):\ 0\leqslant c\leqslant 2;\ x_0(c)\leqslant x\leqslant 1\}.$$

Then

$$H(c,x,y)\leqslant H(c,x,1)=:g(c,x) \quad ((c,x)\in\triangle_2;\, y\in(0,1]),$$

where

$$g(c,x)=576-4(72+c^2)x^2+4c^2x^3+(36-c^2)x^4+16c(1-x^2)(2+x)x-64(1-x^2)^2.$$

(i) On the vertices of $\triangle_2$, we get

$$g(0,x_0(0))=g(0,1)=324,\ \ g(2,1)=320,$$
$$g(2,x_0(2))=g\left(2,\frac{\sqrt{21}-1}{5}\right)=\frac{32\left(6619+471\sqrt{21}\right)}{625}\approx 449.403<576.$$

(ii) On the side $x=x_0(c)$, see the Case 3.2.1 (iii).
(iii) On the side $x=1$, we see that

$$g(c,1)=324-c^2\leqslant 324 \quad (c\in(0,2)).$$

(iv) On the side $c=2$, we know that

$$g(2,x)=-64x^4-48x^3-144x^2+64x+512 \qquad \left(x\in\left[\frac{\sqrt{21}-1}{5},\ 1\right)\right).$$

Since

$$-256x^3-144x^2-288x+64=0$$

has a unique real solution $x\approx 0.196247$, we know that the function $g(2,x)$ is decreasing for $x\in[x_0(2),1)$. Therefore, we get

$$g(2,x)\leqslant g(2,x_0(2))=g\left(2,\frac{\sqrt{21}-1}{5}\right)\approx 553.184 \qquad \left(x\in\left[\frac{\sqrt{21}-1}{5},\ 1\right)\right).$$

(v) It remains to consider the interior of $\triangle_2$. Since the system of equations

$$\begin{cases}\partial g/\partial c=-8cx^2+8cx^3-2cx^4+16(1-x^2)(2+x)x=0\\ \partial g/\partial x=-8(72+c^2)x+12c^2x^2+4(36-c^2)x^3\\ \qquad\qquad +16c(2+2x-6x^2-4x^3)+256x(1-x^2)=0\end{cases} \tag{2.11}$$

has the solution $c=x=0$. Let $x\neq 0$. From the first equation, we get

$$c=\frac{8(2+x)(1-x^2)}{x(2-x)^2}, \tag{2.12}$$

which satisfies the inequality $0\leqslant c\leqslant 2$ only for $x\in[x',1)$, where $x'\approx 0.954858$ satisfies the condition

$$-5x^3-4x^2+8=0.$$

Now, by substituting (2.12) into the second equation of (2.11), we obtain

$$-9x^7-42x^6+248x^5-80x^4+48x^3-256x^2+64x=0,$$

which exists a unique solution in $(0,1)$, namely,

$$x'' \approx 0.261557 \notin [x', 1).$$

Thus, g has no critical point in the interior of $\triangle_2$.

Therefore, in view of Case 3.2, it follows that

$$\det(T_{4,1})(f) = \frac{1}{36864}(4-c^2)^3 \cdot F(c,x,y,\theta,\varphi) \leqslant \frac{1}{36864} \times 4^3 \times 576 = 1.$$

It is clear that equality for the lower bound in (2.1) holds for the function

$$f(z) = \frac{z}{1-z} = z + z^2 + z^3 + \cdots + z^n + \cdots,$$

and for the upper bound in (2.1), equality holds for the identity function $f(z) = z$. □

Finally, we give an example to illustrate the inequality (2.1) in Theorem 2.1.

EXAMPLE 2.1. Let f_1 and f_2 be given by (1.5) and (1.6), respectively. In view of Theorem A, Theorem B and (1.2), we see that $f_1, f_2 \in \mathcal{S}^*(1/2)$, and the fourth-order Hermitian Toeplitz determinants of f_1 and f_2 satisfy

$$0 \leqslant \det(T_{4,1})(f_1) = \begin{vmatrix} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1/3 & 0 & 0 & 1 \end{vmatrix} = \frac{8}{9} \leqslant 1,$$

and

$$0 \leqslant \det(T_{4,1})(f_2) = \begin{vmatrix} 1 & 0 & -1/3 & 0 \\ 0 & 1 & 0 & -1/3 \\ -1/3 & 0 & 1 & 0 \\ 0 & -1/3 & 0 & 1 \end{vmatrix} = \frac{80}{81} \leqslant 1,$$

respectively.

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