Journal of
Mathematical
Inequalities
Volume 17, Number 3 (2023), 1007–1016 doi:10.7153/jmi-2023-17-65

ON GENERALIZED SINGULAR NUMBER OF POSITIVE MATRIX OF τ MEASURABLE OPERATORS

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(*Communicated by L. Mihoković*)

Abstract. Let $(\mathscr{M},\tau)$ be a semi-finite von Neumann algebra, $L_0(\mathscr{M})$ be the set of all τ-measurable operators. We studied generalized singular numbers of 2×2 positive matrices with entries in $L_0(\mathscr{M})$. We proved the equivalence of several inequalities associated with these generalized singular numbers and gave symmetric norm's version of this results, i.e., we extend the related inequalities of 2×2 positive semi-definite block matrices in [1, 5] to the 2×2 positive matrices of τ-measurable operators case.

1. Introduction

We denote the space of all compact linear operators on a complex separable Hilbert space H by $K(H)$. In [4], Bhatia and Kittaneh proved that if $x,y\in K(H)$ are self-adjoint and $\pm y\leqslant x$, then

$$s_j(y)\leqslant s_j(x\oplus x),\qquad j=1,2,\cdots, \tag{1}$$

where $s_j(z)$ $(j=1,2,\cdots)$ is singular value of $z\in K(H)$ and $x\oplus x$ for the block-diagonal operator $\begin{pmatrix} x & 0\\ 0 & x\end{pmatrix}$ defined on $H\oplus H$. They also proved the following arithmetic-geometric mean inequality for singular values (see [3]): if $x,y\in K(H)$, then

$$2s_j(xy^*)\leqslant s_j(x^*x+y^*y),\qquad j=1,2,\cdots. \tag{2}$$

Zhan [13] has proved that if $x,y\in K(H)$ are positive, then

$$s_j(x-y)\leqslant s_j(x\oplus y),\qquad j=1,2,\cdots. \tag{3}$$

Tao has proved in [11] that if $x,y,z\in K(H)$ such that $\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\geqslant 0$, then

$$2s_j(z)\leqslant s_j\left(\begin{pmatrix} x & z\\ z^* & y\end{pmatrix}\right),\qquad j=1,2,\cdots. \tag{4}$$

Mathematics subject classification (2020): 46L52, 47L05.
Keywords and phrases: Generalized singular number, τ-measurable operator, positive matrix of τ-measurable operators, semi-finite von Neumann algebra, symmetric space, noncommutative symmetric space.

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Paper JMI-17-65

Tao has showed that (2)–(4) are equivalent.

Audeh and Kittaneh proved in [1] that if $x,y,z \in K(H)$ such that $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then

$$s_j(z) \leqslant s_j(x \oplus y), \qquad j = 1,2,\cdots. \tag{5}$$

They obtained the following generalization of (1): if $x,y \in K(H)$ are self-adjoint and $\pm y \leqslant x$, then

$$2s_j(y) \leqslant s_j((x+y) \oplus (x-y)), \qquad j = 1,2,\cdots. \tag{6}$$

They have proved that (1) and (5) are equivalent, and (4) and (6) are equivalent. Burqan and Kittaneh [5] have proved that if $x,y,z \in K(H)$ such that $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then

$$s_j(z+z^*) \leqslant s_j((x+y) \oplus (x+y)), \qquad j = 1,2,\cdots \tag{7}$$

and this inequality is equivalent with (1). We recall that while the inequalities in [4, 11, 13] are formulated for matrices, they can be extended in a natural way to compact operators on a complex separable Hilbert space (see [1]).

Let $(\mathscr{M},\tau)$ be a semi-finite von Neumann algebra, $L_0(\mathscr{M})$ be the set of all τ-measurable operators, $\mu_t(x)$ be the generalized singular number of $x \in L_0(\mathscr{M})$. In [7], Han and Shao generalized (1)–(4) for τ-measurable operators associated with $\mathscr{M}$ and proved that the generalized singular numbers version of (2)–(4) are equivalent. In this paper, we prove that if $y \in L_0(\mathscr{M})$ is a self-adjoint operator, then

$$\mu_t(y) \leqslant \mu_t(y_+ \oplus y_-), \qquad t > 0,$$

where y_+, y_- are the positive and negative parts of y, respectively. As application, we extend (5)–(7) to the generalized singular number case. We also prove the equivalence of the corresponding inequalities and some symmetric norm inequalities.

2. Preliminaries

Let $L_0(0,\alpha)$ $(0 < \alpha \leqslant \infty)$ the space of all μ-measurable real-valued functions f on $(0,\alpha)$. We define the decreasing rearrangement function $f^* : (0,\alpha) \mapsto (0,\alpha)$ for $f \in L_0(0,\alpha)$ by

$$f^*(t) = \inf\{s > 0 : \ \mu(\{\omega \in (0,\alpha) : \ |f(\omega)| > s\}) \leqslant t\}, \qquad t \geqslant 0.$$

Let E be a Banach subspace of $L_0(0,\alpha)$, simply called a Banach function space on $(0,\alpha)$ in the sequel. E is said to be *symmetric* if, for $f \in E$ and $g \in L_0(0,\alpha)$ such that $g^*(t) \leqslant f^*(t)$ for all $t \geqslant 0$, one has $g \in E$ and $\|g\|_E \leqslant \|f\|_E$ (see [2, 8]).

We denote by $\mathscr{M}$ a semi-finite von Neumann algebra with a faithful normal semi-finite trace τ and by $L_0(\mathscr{M})$ the set of all τ-measurable operators. For $x \in L_0(\mathscr{M})$, we define the distribution function $\lambda(x)$ of x by $\lambda_t(x) = \tau(e_{(t,\infty)}(|x|))$ for $t > 0$, where $e_{(t,\infty)}(|x|)$ is the spectral projection of $|x|$ in the interval (t,∞), and define the generalized singular numbers $\mu(x)$ of x by $\mu_t(x) = \inf\{s > 0 : \ \lambda_s(x) \leqslant t\}$ for $t > 0$. It is clear that $\mu_t(x) = 0$, for all $t \geqslant \tau(1)$. For further information about elementary properties of the generalized singular numbers, we refer the reader to [6].

We recall that if $\mathscr{M} = \mathbb{M}_n$ and τ is the standard trace, then

$$\mu_t(x) = s_j(x), \quad t \in [j-1, j), \quad j = 1, 2, \cdots.$$

Recall that if $x \in L_0(\mathscr{M})$, then for any $t > 0$,

$$\mu_t(x) = \inf\left\{\|xe\| \ : \ e \text{ is projection in } \mathscr{M},\ \tau(e^{\perp}) \leqslant t\right\}. \tag{8}$$

Moreover, the infimum can be restricted to the family of all spectral projections of $|x|$ (see [6, proof of Proposition 2.2]).

We denote by $\mathbb{M}_2(\mathscr{M})$ the semifinite von Neumann algebra

$$\mathbb{M}_2(\mathscr{M}) = \left\{ \begin{pmatrix} x_{1,1} & x_{1,2} \\ x_{2,1} & x_{2,2} \end{pmatrix},\ x_{i,j} \in \mathscr{M},\ i,j = 1,2 \right\}$$

on Hilbert space $\mathscr{H} \oplus \mathscr{H}$ with trace $Tr \otimes \tau$.

Given a symmetric Banach function space E on $(0, \alpha)$ $(\tau(1) = \alpha)$. Let

$$E(\mathscr{M}, \tau) = \{x \in L_0(\mathscr{M}) : \ \|\mu(x)\|_E < \infty\}, \qquad \|x\|_E = \|\mu(x)\|_E.$$

Then $(E(\mathscr{M}, \tau), \|\cdot\|_E)$ is a Banach space. This space is called noncommutative symmetric space, and denoted by $E(\mathscr{M})$ for convenience. If $1 \leqslant p \leqslant \infty$ and $E = L_p(0, \alpha)$, then $E(\mathscr{M}) = L_p(\mathscr{M})$, which are the usual noncommutative L_p-spaces associated with $(\mathscr{M}, \tau)$ (see [10, 12]).

3. Main results

LEMMA 1. *Let $y \in L_0(\mathscr{M})$ be self-adjoint operator. Then*

$$\mu_t(y) \leqslant \mu_t(y_+ \oplus y_-), \qquad t > 0,$$

where y_+, y_- are the positive and negative parts of y, respectively.

Proof. It is clear that $|y| = y_+ + y_-$, $y = y_+ - y_-$, $y_+ y_- = 0$, $y_- y_+ = 0$. Let $y_+ = \int_0^\infty \lambda de_\lambda(y_+)$ (respectively, $y_- = \int_0^\infty \lambda de_\lambda(y_-)$) be the spectral decomposition of y_+ (respectively, y_-). Then

$$y_+ \oplus y_- = \int_0^\infty \lambda de_\lambda(y_+) \oplus \int_0^\infty \lambda de_\lambda(y_-) = \int_0^\infty \lambda d(e_\lambda(y_+) \oplus e_\lambda(y_-)). \tag{9}$$

By (8) and (9), we get that for any $\varepsilon > 0$, there is a spectral projection e of $y_+ \oplus y_-$ such that $\tau(e^{\perp}) \leqslant t$, $\mu_t(y_+ \oplus y_-) + \varepsilon > \|(y_+ \oplus y_-)e\|$ and $e = e_1 \oplus e_2$, where e_1 is a spectral projection of y_+ and e_2 is a spectral projection of y_-. Hence, $e_1 \perp e_2$, and so $f = e_1 + e_2$ is a projection. Since

$$f^{\perp} = 1 - f \leqslant 1 - e_1 + 1 - e_2 = e_1^{\perp} + e_2^{\perp},$$

we deduce that $\tau(f^{\perp}) \leqslant t$. Applying [6, Lemma 2.5]), we obtain that

$$\begin{aligned}\mu_s(y) = \mu_t(|y|) &\leqslant \||y|f\| = \|(y_+ + y_-)f\| = \|y_+e_1 + y_-e_2\| \\ &= \|e_1y_+e_1 + e_2y_-e_2\| = \max\{\|e_1y_+e_1\|, \|e_2y_-e_2\|\} \\ &= \|e_1y_+e_1 \oplus e_2y_-e_2\| = \|y_+e_1 \oplus y_-e_2\| \\ &= \|(y_+ \oplus y_-)(e_1 \oplus e_2)\| \\ &= \|(y_+ \oplus y_-)e\| < \mu_t(y_+ \oplus y_-) + \varepsilon.\end{aligned}$$

The proof is complete since ε is arbitrary. $\square$

LEMMA 2. *Let $x, y \in L_0(\mathscr{M})$ be self-adjoint operators such that $\pm y \leqslant x$. Then*

$$\mu_t(y) \leqslant \mu_t(x \oplus x), \qquad t > 0.$$

Proof. Let y_+, y_- be the positive and negative parts of y, respectively. Then

$$\begin{aligned}y_+ &= e_{[0,\infty)}(y)ye_{[0,\infty)}(y) \leqslant e_{[0,\infty)}(y)xe_{[0,\infty)}(y) \\ y_- &= e_{(-\infty,0)}(y)ye_{(-\infty,0)}(y) \leqslant e_{(-\infty,0)}(y)xe_{(-\infty,0)}(y).\end{aligned}$$

Using Lemma 1 and [6, Lemma 2.5], we get that

$$\begin{aligned}\mu_t(y) &\leqslant \mu_t(y_+ \oplus y_-) \leqslant \mu_t(e_{[0,\infty)}(y)xe_{[0,\infty)}(y) \oplus e_{(-\infty,0)}(y)xe_{(-\infty,0)}(y)) \\ &= \mu_t((e_{[0,\infty)}(y) \oplus e_{(-\infty,0)}(y))(x \oplus x)(e_{[0,\infty)}(y) \oplus e_{(-\infty,0)}(y))) \\ &\leqslant \|(e_{[0,\infty)}(y) \oplus e_{(-\infty,0)}(y)\|^2 \mu_t(x \oplus x) \\ &\leqslant \mu_t(x \oplus x). \quad \square\end{aligned}$$

We will use the following result (see [9, Proposition 3]), for easy reference give its proof.

LEMMA 3. *Let $x \in L_0(\mathscr{M})$. Then*

$$\mu_t(x \oplus x^*) = \mu_{\frac{t}{2}}(x), \qquad t > 0.$$

Proof. It is clear that $|x \oplus x^*| = |x| \oplus |x^*|$. Similar to the proof of Lemma 1, we have that

$$e_{(t,\infty)}(|x| \oplus |x^*|) = e_{(t,\infty)}(|x|) \oplus e_{(t,\infty)}(|x^*|), \qquad t > 0.$$

Hence, $\lambda_t(x \oplus x^*) = \lambda_t(x) + \lambda_t(x^*)$ for any $t > 0$. As the map: $s \to \lambda_s(x \oplus x^*)$ is continuous from the right, it is obvious that $\lambda_{\mu_t(x \oplus x^*)}(x \oplus x^*) \leqslant t$ for any $t > 0$. Therefore, $\lambda_{\mu_t(x \oplus x^*)}(x) + \lambda_{\mu_t(x \oplus x^*)}(x^*) \leqslant t$. It follows that $\lambda_{\mu_t(x \oplus x^*)}(x) \leqslant \frac{t}{2}$ or $\lambda_{\mu_t(x \oplus x^*)}(x^*) \leqslant \frac{t}{2}$. It implies that $\mu_{\frac{t}{2}}(x) \leqslant \mu_t(x \oplus x^*)$ or $\mu_{\frac{t}{2}}(x^*) \leqslant \mu_t(x \oplus x^*)$. Since $\mu_{\frac{t}{2}}(x) = \mu_{\frac{t}{2}}(x^*)$ (see [6, Lemma 2.5]), we have that $\mu_{\frac{t}{2}}(x) \leqslant \mu_t(x \oplus x^*)$. Conversely, for $\varepsilon > 0$, we choose projections e_1 and e_2 such that $\tau(e_1^{\perp}) \leqslant \frac{t}{2}$, $\tau(e_2^{\perp}) \leqslant \frac{t}{2}$, $\|xe_1\| < \mu_{\frac{t}{2}}(x)+\varepsilon$ and $\|x^*e_2\| < \mu_{\frac{t}{2}}(x)+\varepsilon$. Set $e = e_1 \oplus e_2$. Then $\tau(e^{\perp}) \leqslant t$ and

$$\mu_t(x\oplus x^*) \leqslant \|(x\oplus x^*)e\| = \|xe_1 \oplus x^*e_2\| = \max\{\|xe_1\|, \|x^*e_2\|\} < \mu_{\frac{t}{2}}(x)+\varepsilon.$$

Letting $\varepsilon \to 0$, we obtain the desired result. □

THEOREM 1. *Let $x,y,\in L_0(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$. Then*

$$\mu_t(z) \leqslant \mu_t(x\oplus y), \qquad t>0.$$

Proof. Since $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, we have that

$$0 \leqslant \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} x & -z \\ -z^* & y \end{pmatrix}.$$

It follows that $\pm\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix} \leqslant x\oplus y$. By Lemma 2, we get

$$\mu_t(\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}) \leqslant \mu_t((x\oplus y)\oplus(x\oplus y)), \qquad t>0.$$

On the other hand, $\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} z & 0 \\ 0 & z^* \end{pmatrix}$. By [6, Lemma 2.5], it follows that $\mu_t(\begin{pmatrix} 0 & z \\ z^* & 0 \end{pmatrix}) = \mu_t(z\oplus z^*)$. Hence, $\mu_t(z\oplus z^*) \leqslant \mu_t((x\oplus y)\oplus(x\oplus y))$ for any $t>0$. Using Lemma 3, we obtain desired result. □

We use a similar method in the proof of [1, Theorem 2.4] to obtain the following result.

THEOREM 2. *Let $x,y \in L_0(\mathscr{M})$ be self-adjoint operators such that $\pm y \leqslant x$. Then*

$$2\mu_t(y) \leqslant \mu_t((x+y)\oplus(x-y)) \leqslant 2\mu_t(x\oplus x), \qquad t>0.$$

Proof. Since $\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$ is a unitary operator in $\mathbb{M}_2(\mathscr{M})$ and $\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix} \geqslant 0$, we deduce that

$$0 \leqslant \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix}\frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} = \begin{pmatrix} x & y \\ y & x \end{pmatrix}.$$

Hence, $\mu_t(\begin{pmatrix} x & y \\ y & x \end{pmatrix}) = \mu_t(\begin{pmatrix} x+y & 0 \\ 0 & x-y \end{pmatrix})$. By [7, Lemma 3.3], we have that

$$2\mu_t(y) \leqslant \mu_t(\begin{pmatrix} x & y \\ y & x \end{pmatrix}) = \mu_t((x+y)\oplus(x-y)), \qquad t>0.$$

On the other hand, $0 \leqslant (x+y)\oplus(x-y) = x\oplus x + y\oplus(-y) \leqslant 2(x\oplus x)$, and so the second inequality holds. $\square$

COROLLARY 1. *Let $x,y \in L_0(\mathscr{M})$ be self-adjoint operators. Then*

$$\mu_t(x+y) \leqslant \mu_t((x_+ + y_+)\oplus(x_- + y_-)), \qquad t>0.$$

Proof. It is clear that $\pm x \leqslant |x|$, $\pm y \leqslant |y|$ and $\pm(x+y) \leqslant |x|+|y|$. By Theorem 2, we get that

$$\begin{aligned} \mu_t(x+y) &\leqslant \frac{1}{2}\mu_t(((|x|+|y|)+(x+y))\oplus((|x|+|y|)-(x+y))) \\ &= \frac{1}{2}\mu_t((2(x_+ + y_+))\oplus(2(x_- + y_-))) \\ &= \mu_t((x_+ + y_+)\oplus(x_- + y_-)), \qquad t>0. \quad \square \end{aligned}$$

THEOREM 3. *The following statements are equivalent:*

(i) *If $x,y \in L_0(\mathscr{M})$ are positive operators, then*

$$\mu_t(x-y) \leqslant \mu_t(x\oplus y), \qquad t>0.$$

(ii) *If $x,y \in L_0(\mathscr{M})$, then*

$$2\mu_t(xy^*) \leqslant \mu_t(x^*x + y^*y), \qquad t>0.$$

(iii) *If $x,y,z \in L_0(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$2\mu_t(z) \leqslant \mu_t(\begin{pmatrix} x & z \\ z^* & y \end{pmatrix}), \qquad t>0.$$

(iv) *If $x,y \in L_0(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then*

$$2\mu_t(y) \leqslant \mu_t((x+y)\oplus(x-y)), \qquad t>0.$$

(v) *If $x,y \in L_0(\mathscr{M})$ are self-adjoint operators, then*

$$\mu_t(x+y) \leqslant \mu_t((x_+ + y_+)\oplus(x_- + y_-)), \qquad t>0.$$

Proof. It is known from [7] that (i), (ii) and (iii) are equivalent.

(iii) $\Rightarrow$ (iv) follows from the proof of Theorem 2.

(iv) $\Rightarrow$ (v) follows from the proof of Corollary 1.

(v) $\Rightarrow$ (i) If $x,y \in L_0(\mathscr{M})$ are positive operators, then by (v), we have that

$$\begin{aligned} \mu_t(x-y) &= \mu_t(x+(-y)) \leqslant \mu_t((x_+ + (-y)_+)\oplus(x_- + (-y)_-)) \\ &= \mu_t((x+0)\oplus(0+y)) = \mu_t(x\oplus y), \qquad t>0. \quad \square \end{aligned}$$

THEOREM 4. *Let E be a symmetric Banach function space on $(0,\alpha)$. Then the following statements are equivalent:*

(i) *If $y \in E(\mathscr{M})$ is a self-adjoint operator, then*

$$\|y\|_E \leqslant \|y_+ \oplus y_-\|_E.$$

(ii) *If $x,y \in E(\mathscr{M})$ are positive operators, then*

$$\|x-y\|_E \leqslant \|x \oplus y\|_E.$$

(iii) *If $x,y \in E(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then*

$$2\|y\|_E \leqslant \|(x+y)\oplus(x-y)\|_E.$$

(iv) *If $x,y \in E(\mathscr{M})$ are self-adjoint operators, then*

$$\|x-y\|_E \leqslant \|(x_+ + y_+)\oplus(x_- + y_-)\|_E.$$

Proof. (i) $\Rightarrow$ (ii) If x,y are positive operators, then $x-y \leqslant x$ and $-(x-y) \leqslant y$. Let $e_1 = e_{[0,\infty)}(x-y)$ and $e_2 = e_{(-\infty,0)}(x-y)$. Then $(x-y)_+ = e_1(x-y)e_1 \leqslant e_1xe_1$, $(x-y)_- = e_2(x-y)e_2 \leqslant e_2ye_2$. Hence, by (i),

$$\begin{aligned}\|x-y\|_E &\leqslant \|(x-y)_+ \oplus (x-y)_-\|_E \leqslant \|e_1xe_1 \oplus e_2ye_2\|_E\\ &= \|(e_1\oplus e_2)(x\oplus y)(e_1\oplus e_2)\|_E \leqslant \|x\oplus y\|_E.\end{aligned}$$

(ii) $\Rightarrow$(iii) If x,y are self-adjoint operators such that $\pm y \leqslant x$, then $x-y \geqslant 0$ and $x+y \geqslant 0$. By (ii), we get (iii).

(iii) $\Rightarrow$ (iv) We use the method in the proof of Corollary 1 to prove (iv).

(iv) $\Rightarrow$ (i) If y is a self-adjoint operator, then by (iv), we have that

$$\|y\|_E = \|y-0\|_E \leqslant \|(y_+ + 0)\oplus(y_- + 0) = \|y_+ \oplus y_-\|_E. \qquad \square$$

Using a similar method in the proof of [5, Theorem 3.2], we obtain that

THEOREM 5. *Let $x,y,z,a,b \in L_0(\mathscr{M})$. If $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$\mu_t(a^*zb + b^*z^*a) \leqslant \mu_t((a^*xa + b^*yb)\oplus(a^*xa + b^*yb)), \qquad t>0.$$

Consequently,

$$\mu_t(z+z^*) \leqslant \mu_t((x+y)\oplus(x+y)), \qquad t>0.$$

Proof. Since

$$\begin{pmatrix} a^*xa+b^*z^*a+a^*zb+b^*yb & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}^* \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix} \geqslant 0$$

and

$$\begin{pmatrix} a^*xa-b^*z^*a-a^*zb+b^*yb & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} a & 0 \\ -b & 0 \end{pmatrix}^* \begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \begin{pmatrix} a & 0 \\ -b & 0 \end{pmatrix} \geqslant 0,$$

we have that $\pm(b^*z^*a+a^*zb) \leqslant a^*xa+b^*yb$. By Lemma 2, we get

$$\mu_t(b^*z^*a+a^*zb) \leqslant \mu_t((a^*xa+b^*yb)\oplus(a^*xa+b^*yb)), \qquad t>0.$$

Taking $a=1$ and $b=1$, we obtain the second result. □

THEOREM 6. *The following statements are equivalent:*

(i) *If $x,y \in L_0(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then*

$$\mu_t(y) \leqslant \mu_t(x\oplus x), \qquad t>0.$$

(ii) *If $x,y,z,a,b \in L_0(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$\mu_t(a^*zb+b^*z^*a) \leqslant \mu_t((a^*xa+b^*yb)\oplus(a^*xa+b^*yb)), \qquad t>0.$$

(iii) *If $x,y,z \in L_0(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$\mu_t(z+z^*) \leqslant \mu_t((x+y)\oplus(x+y)), \qquad t>0.$$

(iv) *If $x,y, \in L_0(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$\mu_t(z) \leqslant \mu_t(x\oplus y), \qquad t>0.$$

Proof. (i) $\Rightarrow$ (ii) and (ii) $\Rightarrow$ (iii) follow from the proof of Theorem 5.

(iii) $\Rightarrow$ (i) From the proof of Theorem 2, we know that $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \geqslant 0$. By (iii), we obtain (i).

(i) $\Rightarrow$ (iv) flows from the proof of Theorem 1.

(iv) $\Rightarrow$ (i) Since $\begin{pmatrix} x & y \\ y & x \end{pmatrix} \geqslant 0$, by (iv), we get (i). □

Similarly to Theorem 4, we use the method in the proof of Theorem 6 to obtain the following result.

THEOREM 7. *Let E be a symmetric Banach function space on $(0,\alpha)$. Then the following statements are equivalent:*

(i) *If $x,y \in E(\mathscr{M})$ are self-adjoint operators such that $\pm y \leqslant x$, then*

$$\|y\|_E \leqslant \|x \oplus x\|_E.$$

(ii) *If $x,y,z \in E(\mathscr{M})$, $a,b \in \mathscr{M}$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$\|a^*zb + b^*z^*a\| \leqslant \|(a^*xa + b^*yb) \oplus (a^*xa + b^*yb)\|.$$

(iii) *If $x,y,z \in E(\mathscr{M})$ and $\begin{pmatrix} x & z \\ z^* & y \end{pmatrix} \geqslant 0$, then*

$$\|z + z^*\| \leqslant \|(x+y) \oplus (x+y)\|.$$

Acknowledgement. We thank the referee for very useful comments, which improved the paper. This research was funded by the Science Committee of the Ministry of Science and Higher Education of the Republic of Kazakhstan, grant No. AP09259802 in the area “Research in the field of natural sciences”.

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(Received September 30, 2022)

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